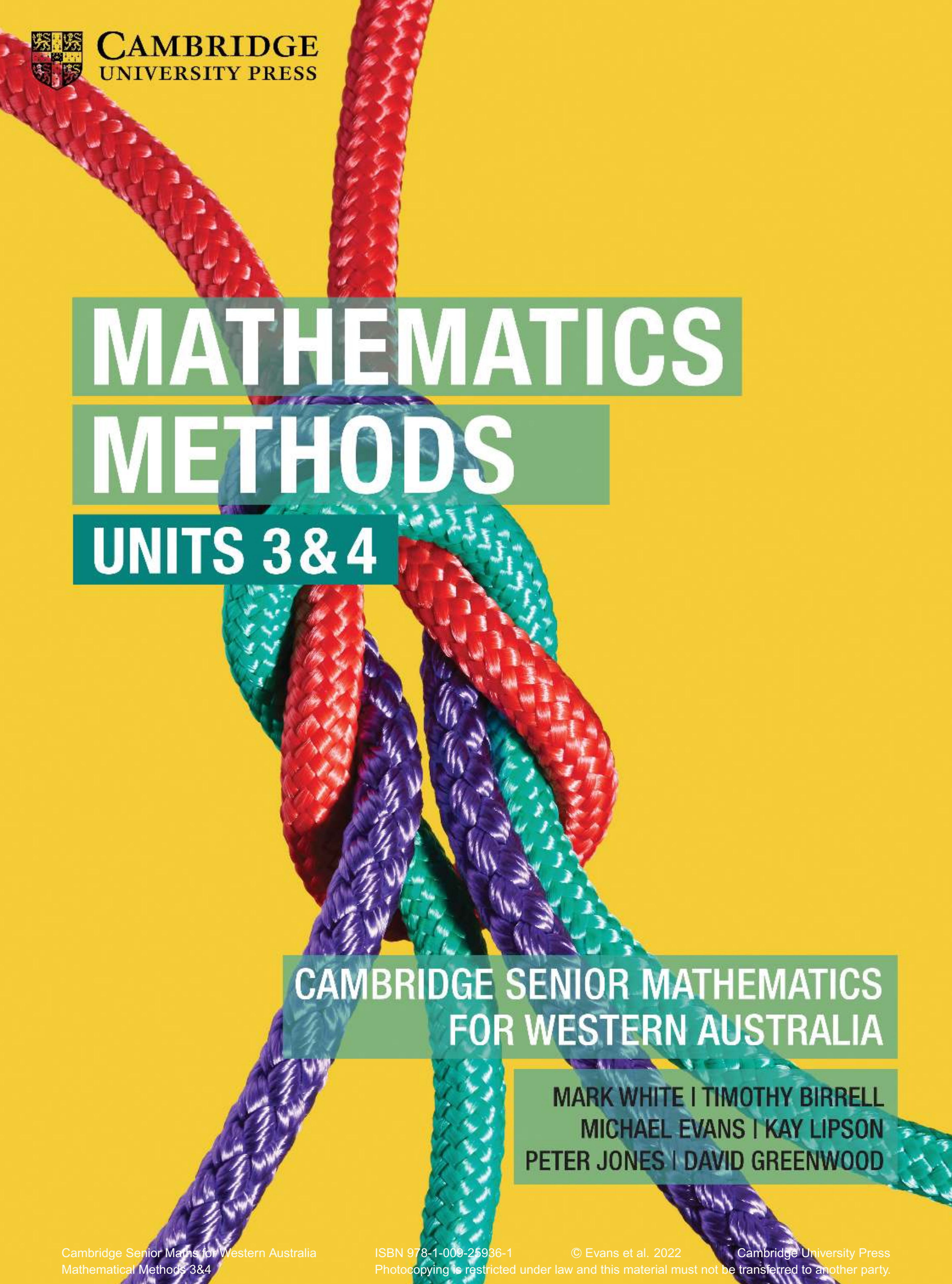

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UNITS 3 & 4

CAMBRIDGE SENIOR MATHEMATICS
FOR WESTERN AUSTRALIA

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Contents

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Introduction

Cambridge Mathematics Methods for Western Australia Units 3&4 is a new edition aligned specifically to the Western Australian Mathematics Methods Year 12 syllabus. Covering both Units 3 and 4 in one resource, it has been written with understanding as its chief aim and with ample practice offered through the worked examples and exercises.

Beginning with elementary functions, the textbook provides a study of exponential and logarithmic functions, algebra, calculus, probability and statistics and their applications in a variety of practical and theoretical contexts. Worked examples utilising CAS calculators are provided throughout, with screenshots and detailed user instructions for both ClassPad and TI-Nspire included for each CAS example.

Compared to the previous Australian Curriculum edition, this WA edition has undergone a number of revisions. Careful adjustments to notation and language have been made throughout to match that used in the WA syllabus and the Year 12 exam and in WA classrooms more generally. Sections on further coordinate geometry, matrix transformations, polynomial functions, further properties of trigonometric functions, normal approximation to the binomial distribution, and limits and continuity have all been removed. All multiple-choice questions that were formerly located in the chapter reviews and revision chapters have also been removed.

The book contains four revision chapters, including one that revises the entire course. These chapters provide technology-free and extended-response questions and are intended to help prepare students for examinations and other assessments, and the grading of the questions and the inclusion of challenging problems ensure that WA students have the opportunity to achieve at the highest standards.

The TI-Nspire calculator examples and instructions have been completed by Russell Brown and those for the Casio ClassPad have been completed by Maria Schaffner.

The integration of the features of the textbook and the new digital components of the package, powered by Cambridge HOTmaths, are illustrated on pages viii to xi.

About Cambridge HOTmaths

Cambridge HOTmaths is a comprehensive, award-winning mathematics learning system – an interactive online maths learning, teaching and assessment resource for students and teachers, for individuals or whole classes, for school and at home. Its digital engine or platform is used to host and power the Interactive Textbook and the Online Teaching Suite. All this is included in the price of the textbook.

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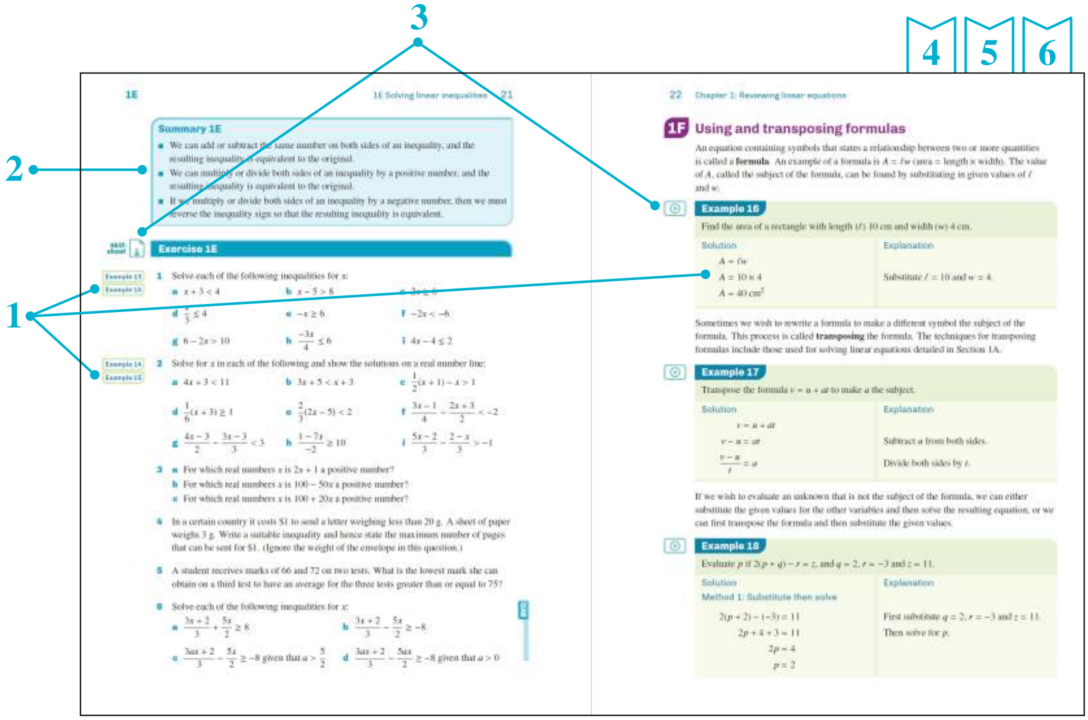
Overview

Overview of the print book

1. Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercise questions.
2. Section summaries provide important concepts in boxes for easy reference.
3. Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.
4. Chapter reviews contain a chapter summary and short-answer and extended-response questions.
5. Revision chapters provide comprehensive revision and preparation for assessment.
6. The glossary includes page numbers of the main explanation of each term.

Numbers refer to descriptions above.



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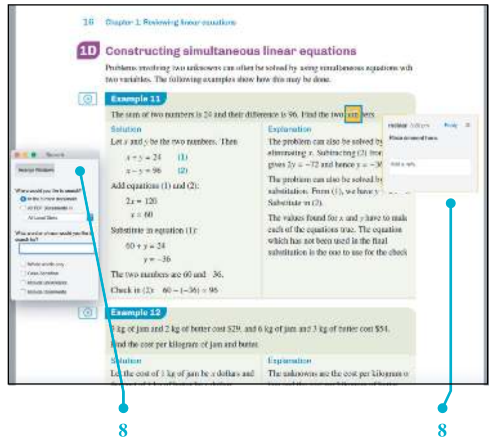
Overview of the downloadable PDF textbook

7 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

8 PDF annotation and search features are enabled.



Overview of the Interactive Textbook

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

9 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

10 **Workspaces** for all questions, which can be enabled or disabled by the teacher, allow students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing done on paper.

11 **Self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.

12 All worked examples have **video versions** to encourage independent learning.

13 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

14 An expanded and revised set of **Desmos interactives** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

15 The **Desmos graphics calculator, scientific calculator**, and **geometry tool** are also embedded for students to use for their own calculations and exploration.

16 **Quick quizzes** containing automarked multiple-choice questions have been thoroughly expanded and revised, enabling students to check their understanding.

17 **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

18 Messages from the teacher assign tasks and tests.

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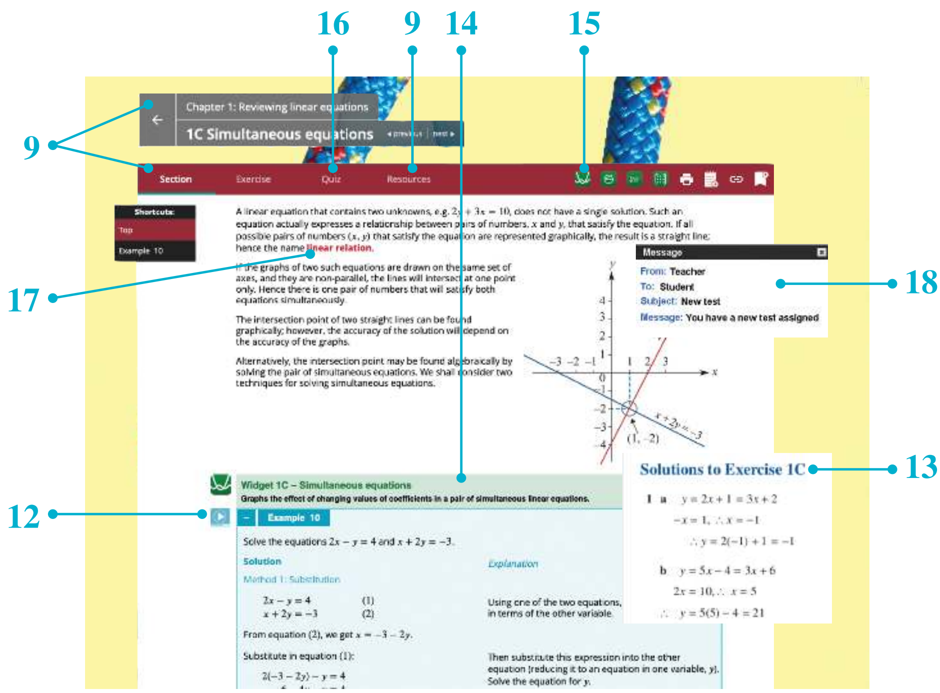
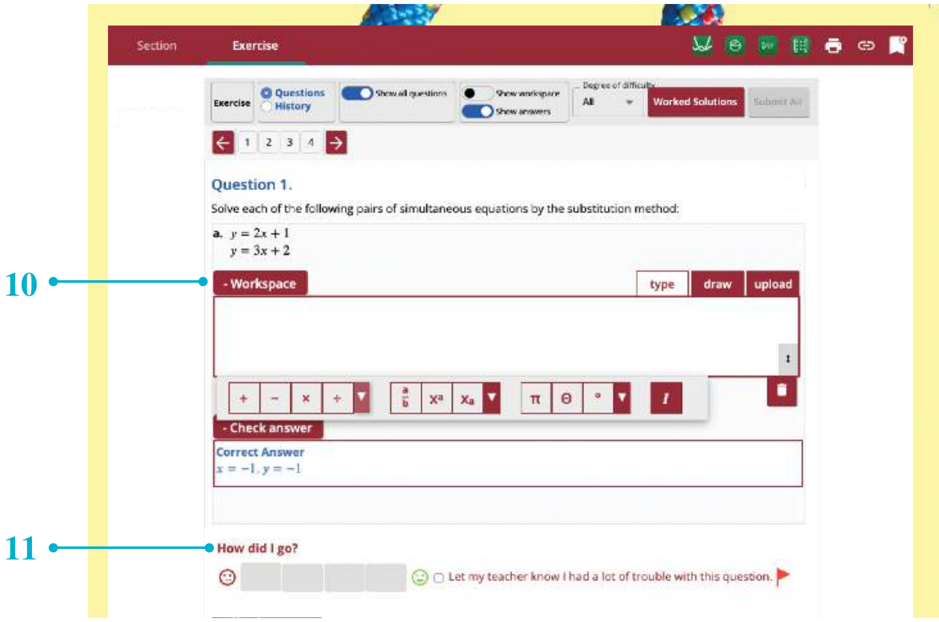
INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

A selection of features is shown. Numbers refer to the descriptions on pages xi–xii. HOTmaths platform features are updated regularly



WORKSPACES AND SELF-ASSESSMENT



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Overview of the Online Teaching Suite powered by the HOTmaths platform

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

19 The HOTmaths learning management system with class and student analytics and reports, and communication tools.

20 Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.

21 A HOTmaths-style test generator.

22 A suite of chapter tests and assignments.

23 Editable curriculum grids and teaching programs.

24 A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite includes a comprehensive bank of SCSA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- SCSA marking scheme
- All custom exams can be printed and completed under exam-like conditions or used as revision.

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1 Functions and relations

In this chapter

Syllabus references

Topics: This chapter revises the following topics from Unit 1: Functions; Language of events and sets

Subtopics: 1.1.23 – 1.1.28; 1.3.6 – 1.3.8

In this chapter we introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

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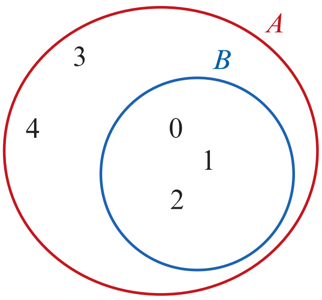
1A Set notation and sets of numbers

Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

- A **set** is a collection of objects. The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'.
- If x is **not an element** of A, we write $x \notin A$.
- A set B is called a **subset** of a set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A' or 'A contains B'.



For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then

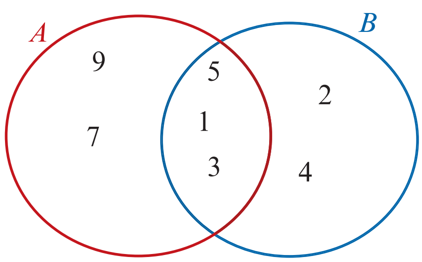
$$3 \in A, \quad 3 \notin B \quad \text{and} \quad B \subseteq A$$

as illustrated in the Venn diagram opposite.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.

For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:

$$A \cap B = \{1, 3, 5\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$$



Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$

b $A \cup B$

Solution	Explanation
a $A \cap B = \{3, 7\}$	The elements 3 and 7 are common to sets A and B.
b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$	The set $A \cup B$ contains all elements that belong to A or B (or both).

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Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \dots\}$ are called the **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

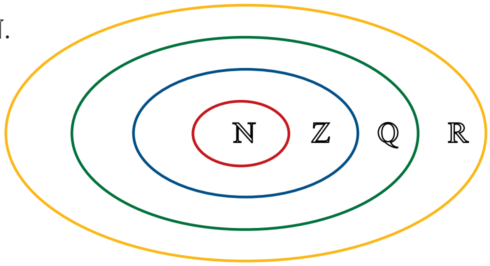
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of natural numbers will be denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

The set of all (real numbers) x such that ___ is denoted by $\{x \in \mathbb{R} : \_\_\_\}$. Thus, for example:

- $\{x \in \mathbb{R} : 0 < x < 1\}$ is the set of all real numbers strictly between 0 and 1
- $\{x \in \mathbb{R} : x \geq 3\}$ is the set of all real numbers greater than or equal to 3
- $\{x \in \mathbb{R} : x \neq 0\}$ is the set of all real numbers excluding 0
- $\{x \in \mathbb{Q} : x > 0\}$ is the set of all positive rational numbers
- $\{2n \in \mathbb{Z} : n = 0, 1, 2, \dots\}$ is the set of all non-negative even numbers
- $\{2n + 1 \in \mathbb{N} : n = 0, 1, 2, \dots\}$ is the set of all non-negative odd numbers.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \qquad [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$
$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\} \qquad [a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, \infty) = \{x \in \mathbb{R} : a < x\} \qquad [a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$
$$(-\infty, b) = \{x \in \mathbb{R} : x < b\} \qquad (-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$

Intervals may be represented by diagrams as shown in Example 2.

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Example 2

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution

a $[-2, 3]$

b $(-3, 4]$

c $(-\infty, 5]$

d $(-2, 4)$

e $(-3, \infty)$

Notes:

- The 'closed' circle (●) indicates that the number is included.
- The 'open' circle (○) indicates that the number is not included.

Summary 1A

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \varnothing$. The set $\varnothing$ is called the **empty set**.
- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x \in \mathbb{R} : a < x < b \} \qquad [a, b] = \{ x \in \mathbb{R} : a \le x \le b \}$$
$$(a, b] = \{ x \in \mathbb{R} : a < x \le b \} \qquad [a, b) = \{ x \in \mathbb{R} : a \le x < b \}$$
$$(a, \infty) = \{ x \in \mathbb{R} : a < x \} \qquad [a, \infty) = \{ x \in \mathbb{R} : a \le x \}$$
$$(-\infty, b) = \{ x \in \mathbb{R} : x < b \} \qquad (-\infty, b] = \{ x \in \mathbb{R} : x \le b \}$$

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Exercise 1A

Example 1

1 For $A = \{3, 8, 11, 18, 22, 23, 24\}$, $B = \{8, 11, 25, 30, 32\}$ and $C = \{1, 8, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$

d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

Example 2

2 Illustrate each of the following intervals on a number line:

a $[-2, 3)$ **b** $(-\infty, 4]$ **c** $[-3, -1]$ **d** $(-3, \infty)$ **e** $(-4, 3)$ **f** $(-1, 4]$

3 For $X = \{2, 3, 5, 7, 9, 11\}$, $Y = \{7, 9, 15, 19, 23\}$ and $Z = \{2, 7, 9, 15, 19\}$, find:

a $X \cap Y$ **b** $X \cap Y \cap Z$ **c** $X \cup Y$ **d** $X \cap Z$

e $[-2, 8] \cap X$ **f** $(-3, 8] \cap Y$ **g** $(2, \infty) \cap Y$ **h** $(3, \infty) \cup Y$

4 For $X = \{a, b, c, d, e\}$ and $Y = \{a, e, i, o, u\}$, find:

a $X \cap Y$ **b** $X \cup Y$ **c** $X \setminus Y$ **d** $Y \setminus X$

5 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

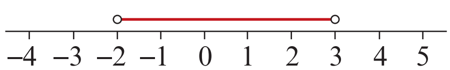
a $\{x \in \mathbb{R} : -3 \le x < 1\}$ **b** $\{x \in \mathbb{R} : -4 < x \le 5\}$

c $\{y \in \mathbb{R} : -\sqrt{2} < y < 0\}$ **d** $\left\{x \in \mathbb{R} : -\frac{1}{\sqrt{2}} < x < \sqrt{3}\right\}$

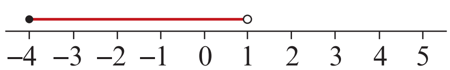
e $\{x \in \mathbb{R} : x < -3\}$ **f** $\{x \in \mathbb{R} : x \ge -2\}$

6 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc.:

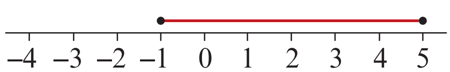
a



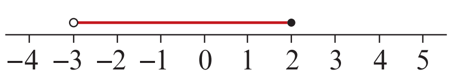
b



c



d



7 Illustrate each of the following intervals on a number line:

a $(-3, 2]$ **b** $(-4, 3)$ **c** $(-\infty, 3)$ **d** $[-4, -1]$ **e** $[-4, \infty)$ **f** $[-2, 5)$

8 For each of the following, use one number line on which to represent the sets:

a $[-3, 6]$, $[2, 4]$, $[-3, 6] \cap [2, 4]$ **b** $[-2, \infty)$, $(-\infty, 6]$, $[-2, \infty) \cap (-\infty, 6]$

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1B Identifying and describing relations and functions

Relations, domain and range

An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

A **relation** is a set of ordered pairs. The following are examples of relations:

a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$

b $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$

Every relation determines two sets:

- The set of all the first coordinates of the ordered pairs is called the **domain**.
- The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{ (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The **domain** is the set $X = \{1, 2, 3, 4\}$ and the **range** is the set $Y = \{2, 3, 4, 5\}$.

When the domain of a relation is not explicitly stated, it is understood to consist of all real numbers for which the defining rule has meaning. For example:

- $S = \{ (x, y) : y = x^2 \}$ is assumed to have domain $\mathbb{R}$
- $T = \{ (x, y) : y = \sqrt{x} \}$ is assumed to have domain $[0, \infty)$.

Example 3

Sketch the graph of each of the following relations and state the domain and range of each:

a $\{ (x, y) : y = x^2 \}$

b $\{ (x, y) : y \le x + 1 \}$

c $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

d $\{ (x, y) : x^2 + y^2 = 1 \}$

e $\{ (x, y) : 2x + 3y = 6,\ x \ge 0 \}$

f $\{ (x, y) : y = 2x - 1,\ x \in [-1, 2] \}$

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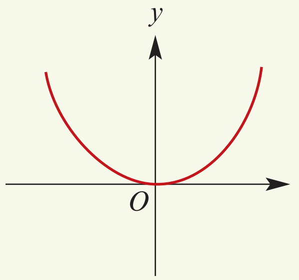
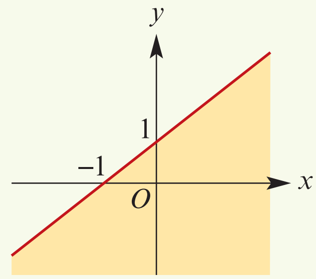
Solution

a



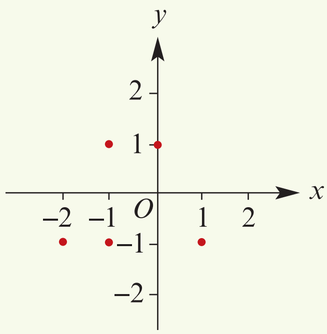
Domain = $\mathbb{R}$; Range = $[0, \infty)$

b



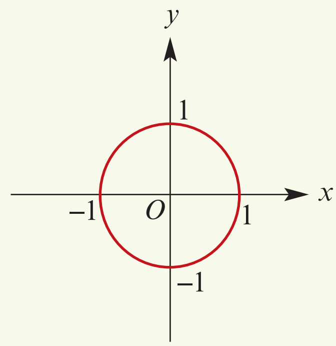
Domain = $\mathbb{R}$; Range = $\mathbb{R}$

c



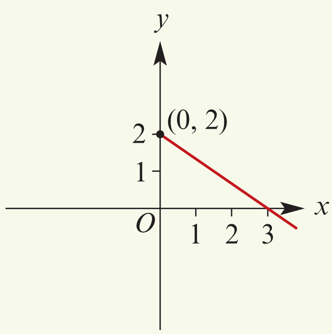
Domain = $\{-2, -1, 0, 1\}$
Range = $\{-1, 1\}$

d



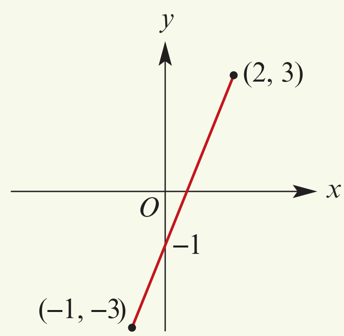
Domain = $[-1, 1]$; Range = $[-1, 1]$

e



Domain = $[0, \infty)$; Range = $(-\infty, 2]$

f



Domain = $[-1, 2]$; Range = $[-3, 3]$

Sometimes set notation is not used in the specification of a relation.

For the previous example:

a is written as $y = x^2$

b is written as $y \le x + 1$

e is written as $2x + 3y = 6, x \ge 0$

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Functions

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 4

Which of the following sets of ordered pairs defines a function?

a $S = \{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$ **b** $T = \{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

Solution

a S is a function because for each x-value there is only one y-value.

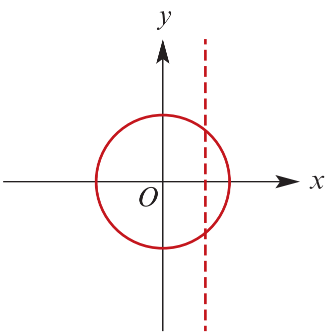
b T is not a function, because there is an x-value with two different y-values: the two ordered pairs $(-4, 1)$ and $(-4, -1)$ in T have the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

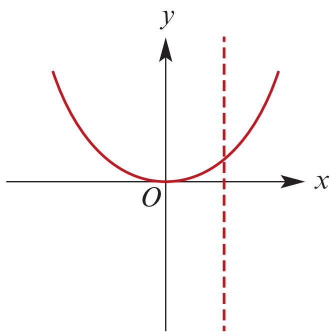
Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

For example:



$x^2 + y^2 = 1$ is not a function



$y = x^2$ is a function

Functions are usually denoted by lowercase letters such as f, g, h.

The definition of a function tells us that, for each x in the domain of f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

For $(x, y) \in f$, the element y is determined by x, and so we also use the notation $f(x)$, read as 'f of x', in place of y.

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This gives an alternative way of writing functions:

- For the function $\{(x, y) : y = x^2\}$, write $f(x) = x^2,\ x \in \mathbb{R}$.
- For the function $\{(x, y) : y = 2x - 1,\ x \in [0, 4]\}$, write $f(x) = 2x - 1,\ \{x \in \mathbb{R} : 0 \le x \le 4\}$.
- For the function $\left\{(x, y) : y = \dfrac{1}{x}\right\}$, write $f(x) = \dfrac{1}{x},\ \{x \in \mathbb{R} : x \neq 0\}$.

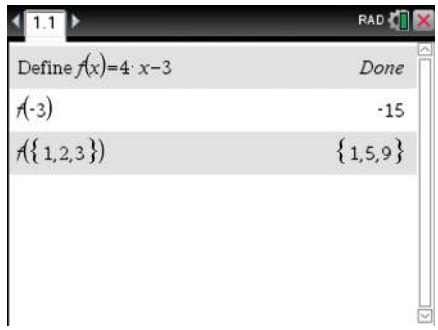
If the domain is $\mathbb{R}$, we often just write the rule: for example, $f(x) = x^2$.

With this notation for functions, the domain of f is written as **dom f** and range of f as **ran f**.

Using the TI-Nspire

Function notation can be used with a CAS calculator.

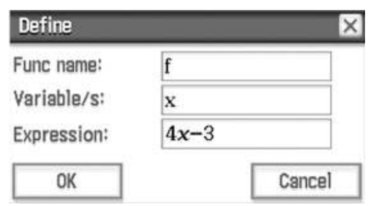
- Use (menu) > **Actions** > **Define** to define the function $f(x) = 4x - 3$.
- To find the value of $f(-3)$, type $f(-3)$ followed by (enter).
- To evaluate $f(1)$, $f(2)$ and $f(3)$, type $f(\{1, 2, 3\})$ followed by (enter).



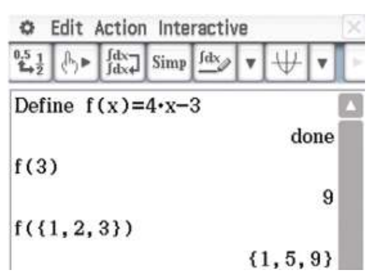
Using the Casio ClassPad

Function notation can be used with a CAS calculator.

- In Main $\sqrt{\alpha}$, select **Interactive** > **Define**.
- Enter the expression $4x - 3$ as shown and tap OK.



- Enter $f(3)$ in the entry line and tap (EXE).
- Enter $f(\{1, 2, 3\})$ to obtain the values of $f(1)$, $f(2)$ and $f(3)$.



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Example 5

For $f(x) = 2x^2 + x$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(x-1)$

Solution

a $f(3) = 2(3)^2 + 3$
$= 21$

b $f(-2) = 2(-2)^2 - 2$
$= 6$

c $f(x-1) = 2(x-1)^2 + (x-1)$
$= 2(x^2 - 2x + 1) + x - 1$
$= 2x^2 - 3x + 1$

Example 6

For $g(x) = 3x^2 + 1$:

a Find $g(-2)$ and $g(4)$.

b Express each of the following in terms of x:

i $g(-2x)$ **ii** $g(x-2)$ **iii** $g(x+2)$ **iv** $g(x^2)$

Solution

a $g(-2) = 3(-2)^2 + 1 = 13$ and $g(4) = 3(4)^2 + 1 = 49$

b **i** $g(-2x) = 3(-2x)^2 + 1$
$= 3 \times 4x^2 + 1$
$= 12x^2 + 1$

ii $g(x-2) = 3(x-2)^2 + 1$
$= 3(x^2 - 4x + 4) + 1$
$= 3x^2 - 12x + 13$

iii $g(x+2) = 3(x+2)^2 + 1$
$= 3(x^2 + 4x + 4) + 1$
$= 3x^2 + 12x + 13$

iv $g(x^2) = 3(x^2)^2 + 1$
$= 3x^4 + 1$

Example 7

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of t is $f(t) = t$?

c For what values of x is $f(x) \geq x$?

d Find the pre-image of 6.

Solution

a $f(2) = 2(2) - 4 = 0$
$f(-1) = 2(-1) - 4 = -6$
$f(t) = 2t - 4$

b $f(t) = t$
$2t - 4 = t$
$t - 4 = 0$
$\therefore \quad t = 4$

c $f(x) \geq x$
$2x - 4 \geq x$
$x - 4 \geq 0$
$\therefore \quad x \geq 4$

d $f(x) = 6$
$2x - 4 = 6$
$x = 5$
Thus 5 is the pre-image of 6.

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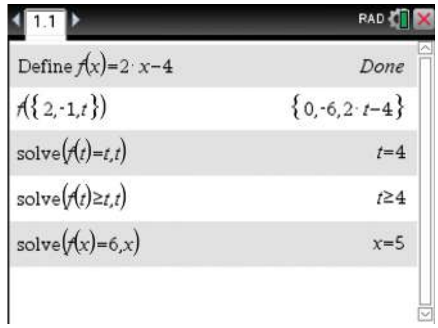
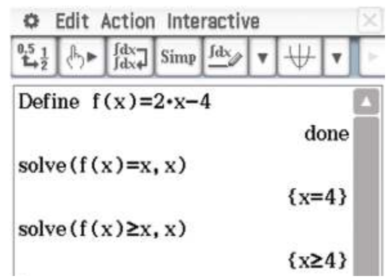
Using the TI-Nspire

- Use (menu) > **Actions** > **Define** to define the function and (menu) > **Algebra** > **Solve** to solve as shown.
- The symbol ≥ can be found using (ctrl) (=) or using (ctrl) (menu) > **Symbols**.



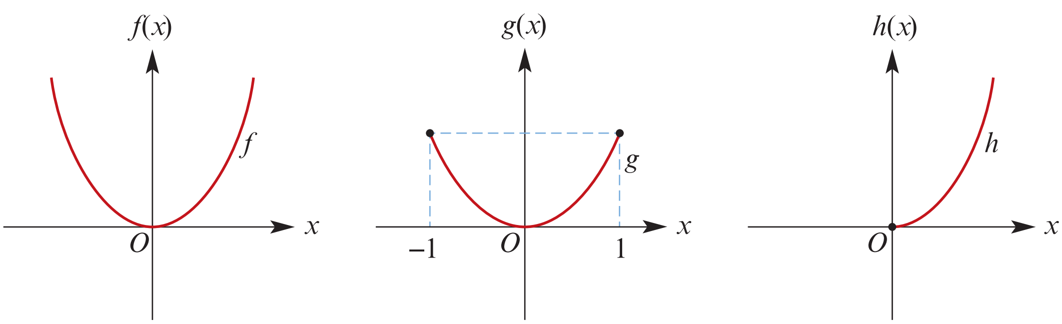
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, define the function $f(x) = 2x - 4$ using **Interactive** > **Define**.
- Now enter and highlight the equation $f(x) = x$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Ensure the variable is set as x and tap OK.
- To enter the inequality, find the symbol ≥ in the (Math3) keyboard.



Restriction of a function

Consider the following functions:



$f(x) = x^2, \; x \in \mathbb{R}$ $\qquad$ $g(x) = x^2, \; -1 \le x \le 1$ $\qquad$ $h(x) = x^2, \; x \in [0, \infty)$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they each have the same rule. They are different because they are defined for different domains.

We call g and h **restrictions** of f, since their domains are subsets of the domain of f.

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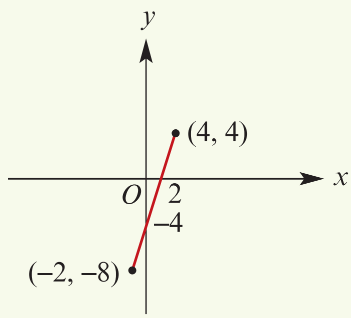
Example 8

For each of the following, sketch the graph and state the range:

a $f(x) = 2x - 4,\ \{x \in \mathbb{R} : -2 \le x \le 4\}$ **b** $g(x) = x^2,\ \{x \in \mathbb{R} : -1 < x \le 2\}$

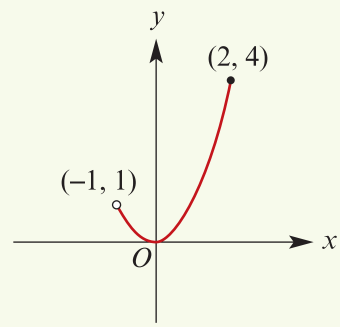
Solution

a



Range = $[-8, 4]$

b



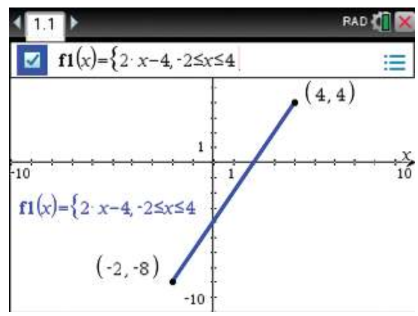
Range = $[0, 4]$

Using the TI-Nspire

Domain restrictions can be entered with the function if required.

For example: $f1(x) = 2x - 4 \mid -2 \le x \le 4$

Domain restrictions are entered using the 'with' symbol $\mid$, which is accessed using ctrl = or by using the **Symbols** palette ctrl 📖 and scrolling to the required symbol. The inequality symbols are also accessed from this palette.

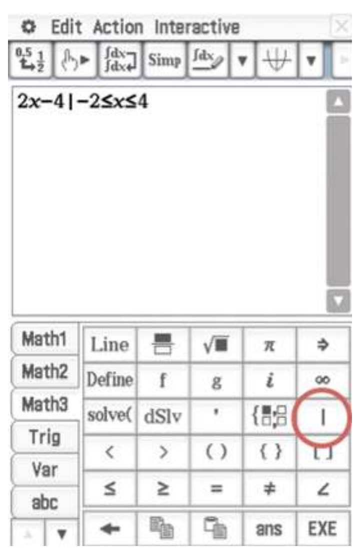


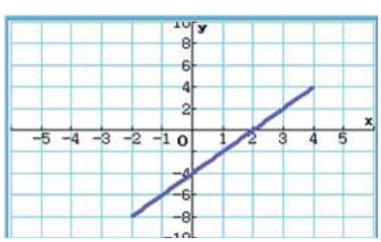
Using the Casio ClassPad

Domain restrictions can be entered with the function if required.

For example: $2x - 4 \mid -2 \le x \le 4$

Domain restrictions are entered using the 'with' symbol $\mid$, which is accessed from the Math3 palette in the soft keyboard. The inequality symbols are also accessed from Math3.





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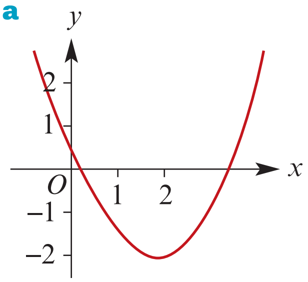
Summary 1B

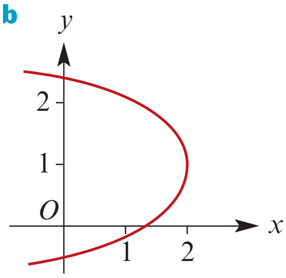
- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range: for example, $\{ (x, y) : y = x + 1,\ x \in [0, \infty) \}$.
- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- **Vertical-line test**: If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y. Since the y-value is determined by the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- Notation for defining functions: For example, we write $f(x) = 2x - 1$, $\{ x \in \mathbb{R} : 0 \le x \le 4 \}$ to define a function f with domain $[0, 4]$ and rule $f(x) = 2x - 1$.
- A **restriction** of a function has the same rule but a 'smaller' domain.

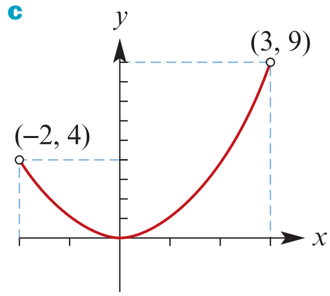
Skill-sheet

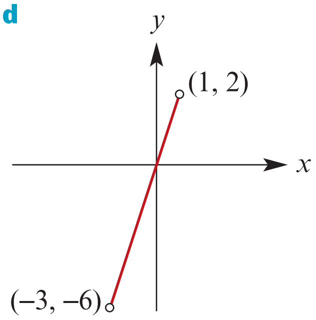
Exercise 1B

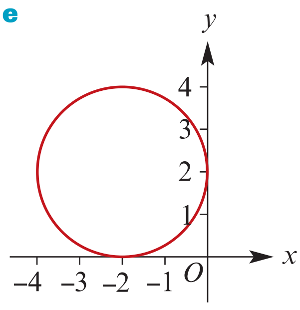
1 State the domain and range for the relations represented by each of the following graphs:

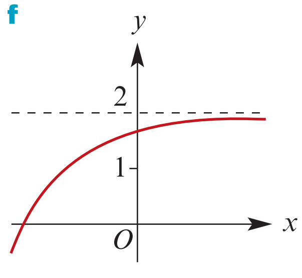
a


b


c


d


e


f


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Example 3

2 Sketch a graph of each of the following relations and state its domain and range:

a $\{(x, y) : y = x^2 + 1\}$

b $\{(x, y) : x^2 + y^2 = 9\}$

c $\{(x, y) : 3x + 12y = 24, \ x \geq 0\}$

d $y = \sqrt{2x}$

e $\{(x, y) : y = 5 - x, \ x \in [0, 5]\}$

f $y = x^2 + 2, \ x \in [0, 4]$

g $y = 3x - 2, \ x \in [-1, 2]$

h $y = 4 - x^2$

i $\{(x, y) : y \leq 1 - x\}$

Example 4

3 Which of the following relations are functions? State the domain and range for each.

a $\{(-1, 1), (-1, 2), (1, 2), (3, 4), (2, 3)\}$

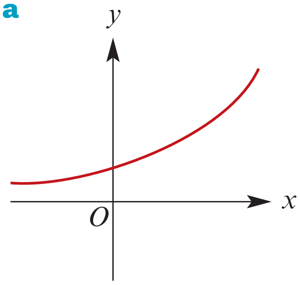
b $\{(-2, 0), (-1, -1), (0, 3), (1, 5), (2, -4)\}$

c $\{(-1, 1), (-1, 2), (-2, -2), (2, 4), (4, 6)\}$

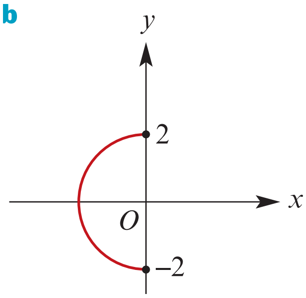
d $\{(-1, 4), (0, 4), (1, 4), (2, 4), (3, 4)\}$

4 Each of the following is the graph of a relation. Which are graphs of a function?

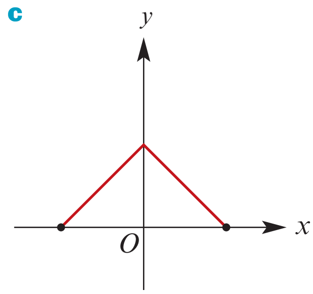
a



b



c



5 Which of the following relations are functions? State the domain and range for each.

a $\{(x, 4) : x \in \mathbb{R}\}$

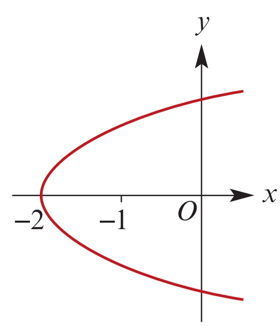
b $\{(2, y) : y \in \mathbb{Z}\}$

c $y = -2x + 4$

d $y \geq 3x + 2$

e $\{(x, y) : x^2 + y^2 = 16\}$

6 The graph of the relation $\{(x, y) : y^2 = x + 2\}$ is shown on the right. From this relation, form two functions and specify the range of each.



Example 5

7 Let $f(x) = 2x^2 + 4x$ and $g(x) = 2x^3 + 2x - 6$.

a Evaluate $f(-1)$, $f(2)$, $f(-3)$ and $f(2a)$.

b Evaluate $g(-1)$, $g(2)$, $g(3)$ and $g(a - 1)$.

Example 6

8 Consider the function $g(x) = 3x^2 - 2$.

a Find $g(-2)$ and $g(4)$.

b Express the following in terms of x:

i $g(-2x)$ **ii** $g(x - 2)$ **iii** $g(x + 2)$ **iv** $g(x^2)$

Example 7

9 Consider the function $f(x) = 2x - 3$. Find:

a the image of 3

b the pre-image of 11

c $\{x : f(x) = 4x\}$

d $\{x : f(x) > x\}$

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10 Consider the functions $g(x) = 6x + 7$ and $h(x) = 3x - 2$. Find:

a $\{ x : g(x) = h(x) \}$ **b** $\{ x : g(x) > h(x) \}$ **c** $\{ x : h(x) = 0 \}$

Example 8 **11** Sketch the graph of each of the following and state the range of each:

a $y = x + 1,\ x \in [2, \infty)$ **b** $y = -x + 1,\ x \in [2, \infty)$

c $y = 2x + 1,\ x \in [-4, \infty)$ **d** $y = 3x + 2,\ x \in (-\infty, 3)$

e $y = x + 1,\ x \in (-\infty, 3]$ **f** $y = 3x - 1,\ x \in [-2, 6]$

g $y = -3x - 1,\ x \in [-5, -1]$ **h** $y = 5x - 1,\ x \in (-2, 4)$

12 For $f(x) = 2x^2 - 6x + 1$ and $g(x) = 3 - 2x$:

a Evaluate $f(2)$, $f(-3)$ and $f(-2)$. **b** Evaluate $g(-2)$, $g(1)$ and $g(-3)$.

c Express the following in terms of a:

i $f(a)$ **ii** $f(a + 2)$ **iii** $g(-a)$ **iv** $g(2a)$

v $f(5 - a)$ **vi** $f(2a)$ **vii** $g(a) + f(a)$ **viii** $g(a) - f(a)$

13 For $f(x) = 3x^2 + x - 2$, find:

a $\{ x : f(x) = 0 \}$ **b** $\{ x : f(x) = x \}$ **c** $\{ x : f(x) = -2 \}$

d $\{ x : f(x) > 0 \}$ **e** $\{ x : f(x) > x \}$ **f** $\{ x : f(x) \le -2 \}$

14 For $f(x) = x^2 + x$, find:

a $f(-2)$ **b** $f(2)$

c $f(-a)$ in terms of a **d** $f(a) + f(-a)$ in terms of a

e $f(a) - f(-a)$ in terms of a **f** $f(a^2)$ in terms of a

15 For $g(x) = 3x - 2$, find:

a $\{ x : g(x) = 4 \}$ **b** $\{ x : g(x) > 4 \}$ **c** $\{ x : g(x) = a \}$

d $\{ x : g(-x) = 6 \}$ **e** $\{ x : g(2x) = 4 \}$ **f** $\left\{ x : \dfrac{1}{g(x)} = 6 \right\}$

16 Find the value of k for each of the following if $f(3) = 3$, where:

a $f(x) = kx - 1$ **b** $f(x) = x^2 - k$ **c** $f(x) = x^2 + kx + 1$

d $f(x) = \dfrac{k}{x}$ **e** $f(x) = kx^2$ **f** $f(x) = 1 - kx^2$

17 Find the values of x for which the given functions have the given value:

a $f(x) = 5x - 4,\ f(x) = 2$ **b** $f(x) = \dfrac{1}{x},\ f(x) = 5$

c $f(x) = \dfrac{1}{x^2},\ f(x) = 9$ **d** $f(x) = x + \dfrac{1}{x},\ f(x) = 2$

e $f(x) = (x + 1)(x - 2),\ f(x) = 0$

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1C Natural domains and types of functions

Natural domains

If the domain of a function is not specified, then the domain is the largest subset of $\mathbb{R}$ for which the rule is defined; this is called the **natural domain** or the **implied domain**.

Thus, for the function $f(x) = \sqrt{x}$, the natural domain is $[0, \infty)$. We write:

$$f(x) = \sqrt{x}, \ \{ x \in \mathbb{R} : x \geq 0 \}.$$

Example 9

Find the natural domain and the corresponding range for the functions with rules:

a $f(x) = 2x - 3$ **b** $f(x) = \dfrac{1}{(x-2)^2}$ **c** $f(x) = \sqrt{x+6}$ **d** $f(x) = \sqrt{4 - x^2}$

Solution

a $f(x) = 2x - 3$ is defined for all x. The natural domain is $\mathbb{R}$. The range is $\mathbb{R}$.

b $f(x) = \dfrac{1}{(x-2)^2}$ is defined for $x \neq 2$. The natural domain is $\{ x \in \mathbb{R} : x \neq 2 \}$. The range is $(0, \infty)$.

c $f(x) = \sqrt{x+6}$ is defined for $x + 6 \geq 0$, i.e. for $x \geq -6$.
Thus the natural domain is $[-6, \infty)$. The range is $[0, \infty)$.

d $f(x) = \sqrt{4 - x^2}$ is defined for $4 - x^2 \geq 0$, i.e. for $x^2 \leq 4$.
Thus the natural domain is $[-2, 2]$. The range is $[0, 2]$.

Example 10

Find the natural domain of the functions with the following rules:

a $f(x) = \dfrac{2}{2x-3}$ **b** $g(x) = \sqrt{5 - x}$

c $h(x) = \sqrt{x-5} + \sqrt{8-x}$ **d** $f(x) = \sqrt{x^2 - 7x + 12}$

Solution

a $f(x)$ is defined when $2x - 3 \neq 0$, i.e. when $x \neq \dfrac{3}{2}$. Thus the natural domain is $\left\{ x \in \mathbb{R} : x \neq \dfrac{3}{2} \right\}$.

b $g(x)$ is defined when $5 - x \geq 0$, i.e. when $x \leq 5$. Thus the natural domain is $(-\infty, 5]$.

c $h(x)$ is defined when $x - 5 \geq 0$ and $8 - x \geq 0$, i.e. when $x \geq 5$ and $x \leq 8$. Thus the natural domain is $[5, 8]$.

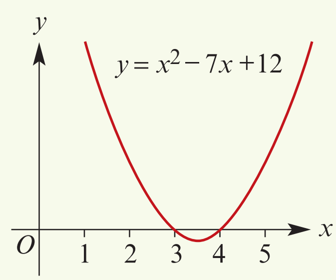
d $f(x)$ is defined when

$$x^2 - 7x + 12 \geq 0$$

which is equivalent to

$$(x-3)(x-4) \geq 0$$

Thus $f(x)$ is defined when $x \geq 4$ or $x \leq 3$.
The natural domain is $(-\infty, 3] \cup [4, \infty)$.



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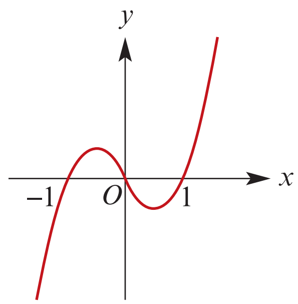
Odd and even functions

Odd functions

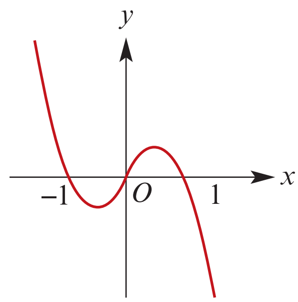
An **odd** function has the property that $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

For example, $f(x) = x^3 - x$ is an odd function, since

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -f(x) \end{aligned}$$



$y = f(x)$



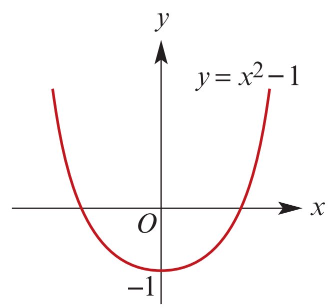
$y = f(-x)$

Even functions

An **even** function has the property that $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis.

For example, $f(x) = x^2 - 1$ is an even function, since

$$\begin{aligned} f(-x) &= (-x)^2 - 1 \\ &= x^2 - 1 \\ &= f(x) \end{aligned}$$



The properties of odd and even functions often facilitate the sketching of graphs.

Example 11

State whether each function is odd or even or neither:

a $f(x) = x^2 + 7$ **b** $f(x) = x^4 + x^2$ **c** $f(x) = -2x^3 + 7$

d $f(x) = \dfrac{1}{x}$ **e** $f(x) = \dfrac{1}{x-3}$ **f** $f(x) = x^5 + x^3 + x$

Solution

a $$\begin{aligned} f(-a) &= (-a)^2 + 7 \\ &= a^2 + 7 \\ &= f(a) \end{aligned}$$

The function is even.

b $$\begin{aligned} f(-a) &= (-a)^4 + (-a)^2 \\ &= a^4 + a^2 \\ &= f(a) \end{aligned}$$

The function is even.

c $f(-1) = -2(-1)^3 + 7 = 9$
but $f(1) = -2 + 7 = 5$
and $-f(1) = -5$

The function is neither even nor odd.

d $$\begin{aligned} f(-a) &= \frac{1}{-a} \\ &= -\frac{1}{a} \\ &= -f(a) \end{aligned}$$

The function is odd.

e $f(-1) = -\dfrac{1}{4}$
but $f(1) = -\dfrac{1}{2}$
and $-f(1) = \dfrac{1}{2}$

The function is neither even nor odd.

f $$\begin{aligned} &f(-a) \\ &= (-a)^5 + (-a)^3 + (-a) \\ &= -a^5 - a^3 - a \\ &= -f(a) \end{aligned}$$

The function is odd.

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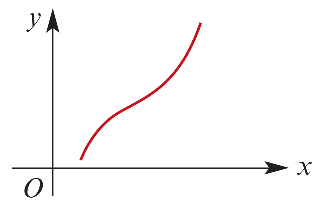
Increasing and decreasing functions

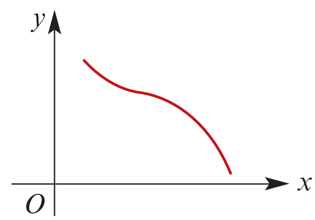
We say a function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f(x) = x^2$, $\{ x \in \mathbb{R} : x > 0 \}$ is strictly increasing.

We say a function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.



For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f(x) = x^2$, $\{ x \in \mathbb{R} : x < 0 \}$ is strictly decreasing.

Summary 1C

- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **natural domain** or the **implied domain** of the function.
- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Skill-sheet

Exercise 1C

Example 9

1 State the largest possible domain and range for the functions defined by each of the following rules:

a $y = 4 - x$ **b** $y = \sqrt{x}$ **c** $y = x^2 - 2$ **d** $y = \sqrt{16 - x^2}$

e $y = \dfrac{1}{x}$ **f** $y = 4 - 3x^2$ **g** $y = \sqrt{x - 3}$

2 Each of the following is the rule of a function. In each case, write down the natural domain and the range.

a $y = 3x + 2$ **b** $y = x^2 - 2$

c $f(x) = \sqrt{9 - x^2}$ **d** $g(x) = \dfrac{1}{x - 1}$

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Example 10

3 Find the natural domain for each of the following rules:

a $f(x) = \dfrac{1}{x-3}$ **b** $f(x) = \sqrt{x^2 - 3}$ **c** $g(x) = \sqrt{x^2 + 3}$

d $h(x) = \sqrt{x-4} + \sqrt{11-x}$ **e** $f(x) = \dfrac{x^2-1}{x+1}$ **f** $h(x) = \sqrt{x^2 - x - 2}$

g $f(x) = \dfrac{1}{(x+1)(x-2)}$ **h** $h(x) = \sqrt{\dfrac{x-1}{x+2}}$ **i** $f(x) = \sqrt{x - 3x^2}$

j $h(x) = \sqrt{25 - x^2}$ **k** $f(x) = \sqrt{x-3} + \sqrt{12-x}$

Example 11

4 State whether each of the following functions is odd, even or neither:

a $f(x) = x^4$ **b** $f(x) = x^5$ **c** $f(x) = x^4 - 3x$

d $f(x) = x^4 - 3x^2$ **e** $f(x) = x^5 - 2x^3$ **f** $f(x) = x^4 - 2x^5$

5 State whether each of the following functions is odd, even or neither:

a $f(x) = x^2 - 4$ **b** $f(x) = 2x^4 - x^2$ **c** $f(x) = -4x^3 + 7x$

d $f(x) = \dfrac{1}{2x}$ **e** $f(x) = \dfrac{1}{x+5}$ **f** $f(x) = 3 + 2x^2$

g $f(x) = x^2 - 5x$ **h** $f(x) = 3^x$ **i** $f(x) = x^4 + x^2 + 2$

1D Sums and products of functions

The domain of f is denoted by dom f and the domain of g by dom g. Let f and g be functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. The **sum**, $f + g$, and the **product**, fg, as functions on $\operatorname{dom} f \cap \operatorname{dom} g$ are defined by

$$(f+g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x)\,g(x)$$

The domain of both $f + g$ and fg is the intersection of the domains of f and g, i.e. the values of x for which both f and g are defined.

Example 12

If $f(x) = \sqrt{x-2}$ for all $x \geq 2$ and $g(x) = \sqrt{4-x}$ for all $x \leq 4$, find:

a $f + g$ **b** $(f+g)(3)$ **c** fg **d** $(fg)(3)$

Solution

Note that $\operatorname{dom} f \cap \operatorname{dom} g = [2, 4]$.

a $(f+g)(x) = f(x) + g(x)$

$= \sqrt{x-2} + \sqrt{4-x}$

$\operatorname{dom}(f+g) = [2, 4]$

b $(f+g)(3) = \sqrt{3-2} + \sqrt{4-3}$

$= 2$

c $(fg)(x) = f(x)\,g(x)$

$= \sqrt{(x-2)(4-x)}$

$\operatorname{dom}(fg) = [2, 4]$

d $(fg)(3) = \sqrt{(3-2)(4-3)}$

$= 1$

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Addition of ordinates

We have seen that, for two functions f and g, a new function $f + g$ can be defined by

$$(f+g)(x) = f(x) + g(x)$$
$$\text{dom}(f+g) = \text{dom}\, f \cap \text{dom}\, g$$

We now look at how to graph the new function $f + g$. This is a useful graphing technique and can be combined with other techniques such as finding axis intercepts, stationary points and asymptotes.

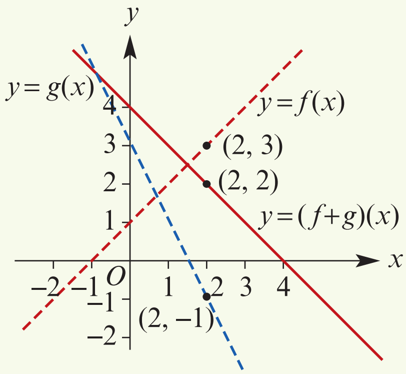
Example 13

Sketch the graphs of $f(x) = x + 1$ and $g(x) = 3 - 2x$ and hence the graph of $(f + g)(x)$.

Solution

For $f(x) = x + 1$ and $g(x) = 3 - 2x$, we have

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= (x+1) + (3-2x)\\ &= 4 - x\end{aligned}$$



For example:

$$\begin{aligned}(f+g)(2) &= f(2) + g(2)\\ &= 3 + (-1) = 2\end{aligned}$$

i.e. the ordinates are added.

Now check that the same principle applies for other points on the graphs. A table of values can be a useful aid to find points that lie on the graph of $y = (f + g)(x)$.

The table shows that $(-1, 5)$, $(0, 4)$, $\left(\frac{3}{2}, \frac{5}{2}\right)$ and $(2, 2)$ lie on the graph of $y = (f + g)(x)$.

x	$f(x)$	$g(x)$	$(f+g)(x)$
-1	0	5	5
0	1	3	4
$\frac{3}{2}$	$\frac{5}{2}$	0	$\frac{5}{2}$
2	3	-1	2

Example 14

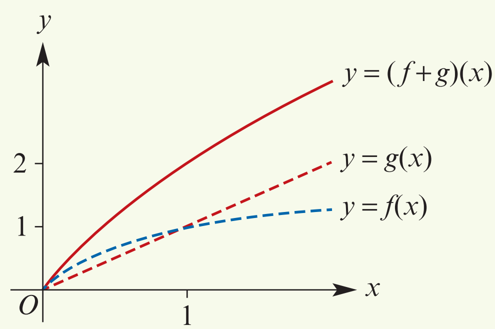
Sketch the graph of $y = (f + g)(x)$, where $f(x) = \sqrt{x}$ and $g(x) = x$.

Solution

The function with rule

$$(f+g)(x) = \sqrt{x} + x$$

is defined by the addition of the two functions f and g.



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Example 15

Sketch the graph of $y = (f - g)(x)$, where $f(x) = x^2$ and $g(x) = \sqrt{x}$.

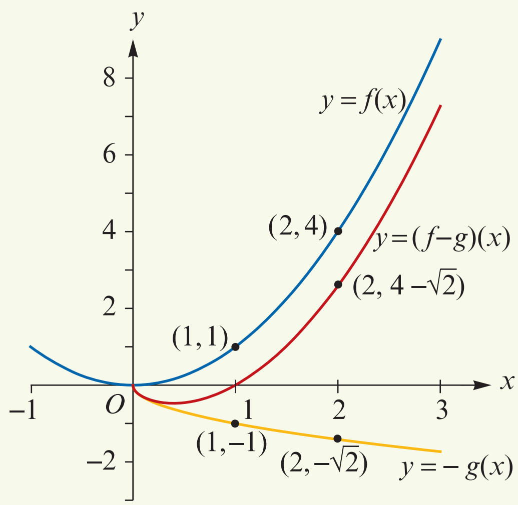
Solution

The function with rule

$$(f - g)(x) = x^2 - \sqrt{x}$$

is defined by the addition of the two functions f and $-g$.

The implied domain of $f - g$ is $[0, \infty)$.



Summary 1D

- **Sum of functions** $(f + g)(x) = f(x) + g(x)$, where $\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$
- **Difference of functions** $(f - g)(x) = f(x) - g(x)$, where $\text{dom}(f - g) = \text{dom}\, f \cap \text{dom}\, g$
- **Product of functions** $(f \cdot g)(x) = f(x) \cdot g(x)$, where $\text{dom}(f \cdot g) = \text{dom}\, f \cap \text{dom}\, g$
- **Addition of ordinates** This technique can be used to help sketch the graph of the sum of two functions. Key points to consider when sketching $y = (f + g)(x)$:
  - When $f(x) = 0$, $(f + g)(x) = g(x)$.
  - When $g(x) = 0$, $(f + g)(x) = f(x)$.
  - If $f(x)$ and $g(x)$ are positive, then $(f + g)(x) > g(x)$ and $(f + g)(x) > f(x)$.
  - If $f(x)$ and $g(x)$ are negative, then $(f + g)(x) < g(x)$ and $(f + g)(x) < f(x)$.
  - If $f(x)$ is positive and $g(x)$ is negative, then $g(x) < (f + g)(x) < f(x)$.
  - Look for values of x for which $f(x) + g(x) = 0$.

Exercise 1D

Example 12

1 For each of the following, find $(f + g)(x)$ and $(fg)(x)$ and state the domain for both $f + g$ and fg:

a $f(x) = 3x$ and $g(x) = x + 2$

b $f(x) = 1 - x^2$ for all $x \in [-2, 2]$ and $g(x) = x^2$ for all $x \in (0, \infty)$

c $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ for $x \in [1, \infty)$

d $f(x) = x^2$, $x \geq 0$ and $g(x) = \sqrt{4 - x}$, $0 \leq x \leq 4$

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2 Functions f, g, h and k are defined by:

i $f(x) = x^2 + 1, \ x \in \mathbb{R}$ **ii** $g(x) = x, \ x \in \mathbb{R}$

iii $h(x) = \dfrac{1}{x^2}, \ x \neq 0$ **iv** $k(x) = \dfrac{1}{x}, \ x \neq 0$

a State which of the above functions are odd and which are even.

b Give the rules for the functions $f + h$, fh, $g + k$, gk, $f + g$ and fg, stating which are odd and which are even.

Example 13 **3** Sketch the graphs of $f(x) = x + 2$ and $g(x) = 4 - 3x$ and hence the graph of $(f + g)(x)$.

Example 14 **4** Sketch the graph of $f(x) = \sqrt{x} + 2x$, $\{ x \in \mathbb{R} : x \geq 0 \}$ using addition of ordinates.

5 Sketch the graph of $f(x) = \sqrt{x+2} + x$, $\{ x \in \mathbb{R} : x \geq -2 \}$ using addition of ordinates.

6 Sketch the graph of $f(x) = -\sqrt{x} + x$, $\{ x \in \mathbb{R} : x \geq 0 \}$ using addition of ordinates.

7 Sketch the graph of $f(x) = \dfrac{1}{x} + \dfrac{1}{x^2}$, $\{ x \in \mathbb{R} : x \neq 0 \}$ using addition of ordinates.

8 For each of the following, sketch the graph of $f + g$:

a $f(x) = \sqrt{2 + x}$, $\{ x \in \mathbb{R} : x \geq -2 \}$ and $g(x) = -2x$

b $f(x) = \sqrt{2 - x}$, $\{ x \in \mathbb{R} : x \leq 2 \}$ and $g(x) = \sqrt{x + 2}$, $\{ x \in \mathbb{R} : x \geq -2 \}$

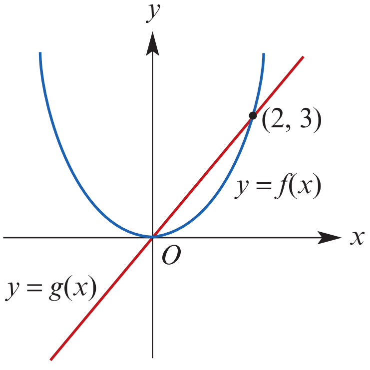
Example 15 **9** Sketch the graph of $y = (f - g)(x)$, where $f(x) = x^3$ and $g(x) = \sqrt{x}$.

10 Sketch the graph of $y = (f - g)(x)$, where $f(x) = 2x^2$ and $g(x) = 3\sqrt{x}$.

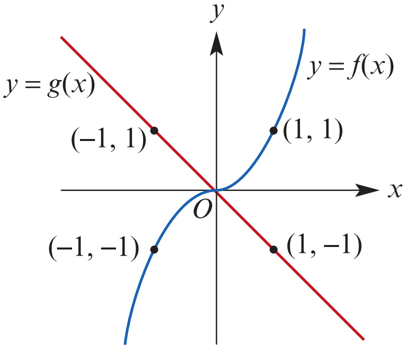
11 Sketch the graph of $f(x) = x^2$ and $g(x) = 3x + 2$ on the one set of axes and hence, using addition of ordinates, sketch the graph of $y = x^2 + 3x + 2$.

12 Copy and add the graph of $y = (f + g)(x)$ using addition of ordinates:

a



b



13 For each of the following, sketch the graph of $f + g$:

a $f(x) = x^2$ and $g(x) = 3$

b $f(x) = x^2 + 2x$ and $g(x) = \sqrt{x}$, $\{ x \in \mathbb{R} : x \geq 0\}$

c $f(x) = -x^2$ and $g(x) = \sqrt{x}$, $\{ x \in \mathbb{R} : x \geq 0\}$

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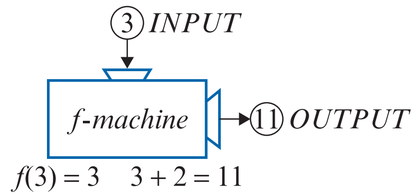
1E Composite functions

A function may be considered to be similar to a machine for which the input (domain) is processed to produce an output (range). For example, the diagram on the right represents an 'f-machine' where $f(x) = 3x + 2$.



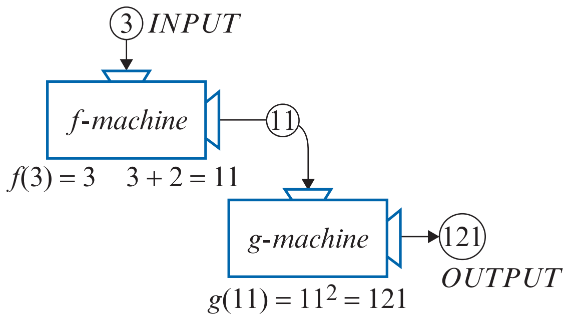
With many processes, more than one machine operation is required to produce an output.

Suppose an output is the result of one function being applied after another.

For example: $f(x) = 3x + 2$

followed by $g(x) = x^2$

This is illustrated on the right.



A new function h is formed. The rule for h is $h(x) = (3x + 2)^2$.

The diagram shows $f(3) = 11$ and then $g(11) = 121$. This may be written:

$$h(3) = g(f(3)) = g(11) = 121$$

The new function h is said to be the **composition** of g with f. This is written $h = g \circ f$ (read 'composition of f followed by g') and the rule for h is given by $h(x) = g(f(x))$.

In the example we have considered:

$$\begin{aligned} h(x) &= g(f(x)) \\ &= g(3x + 2) \\ &= (3x + 2)^2 \end{aligned}$$

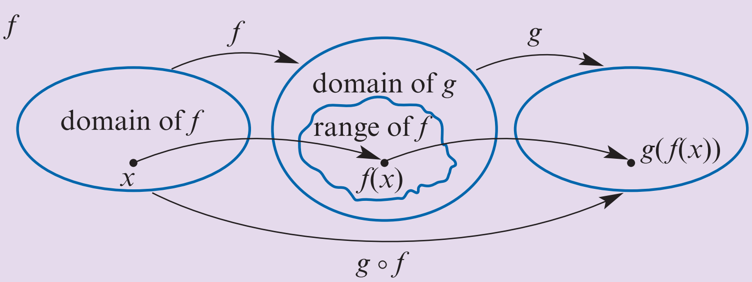
In general, for functions f and g such that

$$\operatorname{ran} f \subseteq \operatorname{dom} g$$

we define the **composite function** of g with f by

$$g \circ f(x) = g(f(x))$$

$$\operatorname{dom}(g \circ f) = \operatorname{dom} f$$



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Example 16

Find both $f \circ g$ and $g \circ f$, stating the domain and range of each, where:

$$f(x) = 2x - 1 \quad \text{and} \quad g(x) = 3x^2$$

Solution

To determine the existence of a composite function, it is useful to form a table of domains and ranges.

We see that $f \circ g$ is defined since $\operatorname{ran} g \subseteq \operatorname{dom} f$, and that $g \circ f$ is defined since $\operatorname{ran} f \subseteq \operatorname{dom} g$.

	Domain	Range
g	$\mathbb{R}$	$[0, \infty)$
f	$\mathbb{R}$	$\mathbb{R}$

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(3x^2) \\ &= 2(3x^2) - 1 \\ &= 6x^2 - 1 \end{aligned}$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(2x - 1) \\ &= 3(2x - 1)^2 \\ &= 12x^2 - 12x + 3 \end{aligned}$$

$\operatorname{dom}(f \circ g) = \operatorname{dom} g = \mathbb{R}$

$\operatorname{ran}(f \circ g) = [-1, \infty)$

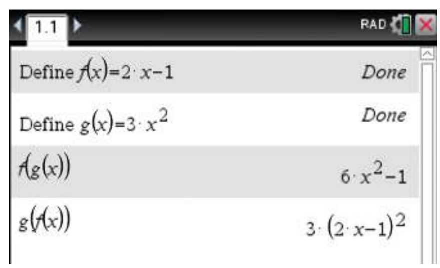
$\operatorname{dom}(g \circ f) = \operatorname{dom} f = \mathbb{R}$

$\operatorname{ran}(g \circ f) = [0, \infty)$

Note: It can be seen from this example that in general $f \circ g \neq g \circ f$.

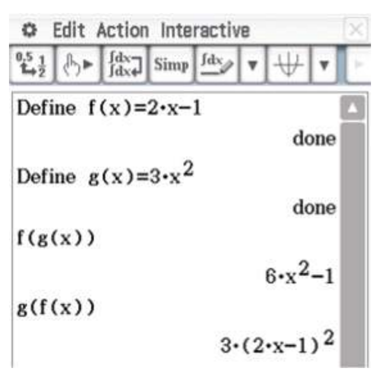
Using the TI-Nspire

- Define $f(x) = 2x - 1$ and $g(x) = 3x^2$.
- The rules for $f \circ g$ and $g \circ f$ can now be found using $f(g(x))$ and $g(f(x))$.



Using the Casio ClassPad

- Define $f(x) = 2x - 1$ and $g(x) = 3x^2$.
- The rules for $f \circ g$ and $g \circ f$ can now be found using $f(g(x))$ and $g(f(x))$.



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Example 17

For the functions $g(x) = 2x - 1$, $x \in \mathbb{R}$, and $f(x) = \sqrt{x}$, $x \geq 0$:

a State which of $f \circ g$ and $g \circ f$ is defined.

b For the composite function that is defined, state the domain and rule.

Solution

a Range of $f \subseteq$ domain of g

Range of $g \not\subseteq$ domain of f

Thus $g \circ f$ is defined, but $f \circ g$ is not defined.

	Domain	Range
g	$\mathbb{R}$	$\mathbb{R}$
f	$[0, \infty)$	$[0, \infty)$

b
$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(\sqrt{x}) \\ &= 2\sqrt{x} - 1 \end{aligned}$$

$\text{dom}(g \circ f) = \text{dom } f = [0, \infty)$

Example 18

For the functions $f(x) = x^2 - 1$, $x \in \mathbb{R}$, and $g(x) = \sqrt{x}$, $x \geq 0$:

a State why $g \circ f$ is not defined.

b Define a restriction f^* of f such that $g \circ f^*$ is defined, and find $g \circ f^*$.

Solution

a Range of $f \not\subseteq$ domain of g

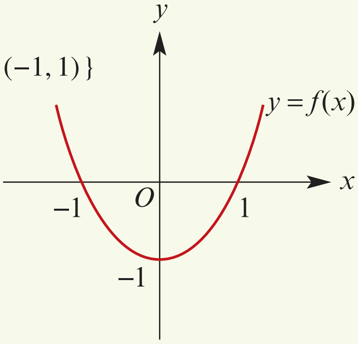
Thus $g \circ f$ is not defined.

	Domain	Range
f	$\mathbb{R}$	$[-1, \infty)$
g	$[0, \infty)$	$[0, \infty)$

b For $g \circ f^*$ to be defined, we need range of $f^* \subseteq$ domain of g, i.e. range of $f^* \subseteq [0, \infty)$.

For the range of f^* to be a subset of $[0, \infty)$, the domain of f must be restricted to a subset of

$$\{ x \in \mathbb{R} : x \leq -1 \} \cup \{ x \in \mathbb{R} : x \geq 1 \} = \{ x \in \mathbb{R} : x \notin (-1, 1) \}$$



So we define f^* by

$$f^*(x) = x^2 - 1, \ \{ x \in \mathbb{R} : x \notin (-1, 1) \}$$

Then
$$\begin{aligned} g \circ f^*(x) &= g(f^*(x)) \\ &= g(x^2 - 1) \\ &= \sqrt{x^2 - 1} \end{aligned}$$

$$\text{dom}(g \circ f^*) = \text{dom } f^* = \{ x \in \mathbb{R} : x \notin (-1, 1) \}$$

The composite function is $g \circ f^*(x) = \sqrt{x^2 - 1}$, $\{ x \in \mathbb{R} : x \notin (-1, 1) \}$

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Example 19

Express each of the following as the composition of two functions:

a $h(x) = (2x + 1)^2$ **b** $h(x) = \sqrt{x^2}$ **c** $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$

Solution

a $h(x) = (2x + 1)^2$

Choose $f(x) = 2x + 1$

and $g(x) = x^2$.

Then $h(x) = g \circ f(x)$.

b $h(x) = \sqrt{x^2}$

Choose $f(x) = x^2$

and $g(x) = \sqrt{x}$.

Then $h(x) = g \circ f(x)$.

c $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$

Choose $f(x) = x^2 - 2$

and $g(x) = x^n$.

Then $h(x) = g \circ f(x)$.

Note: These are not the only possible answers, but the 'natural' choices have been made.

Summary 1E

- If range of $f \subseteq$ domain of g, the composition $g \circ f$ is defined and

$$g \circ f(x) = g(f(x)) \quad \text{with } \operatorname{dom}(g \circ f) = \operatorname{dom} f$$

- If range of $g \subseteq$ domain of f, the composition $f \circ g$ is defined and

$$f \circ g(x) = f(g(x)) \quad \text{with } \operatorname{dom}(f \circ g) = \operatorname{dom} g$$

- In general, $f \circ g \neq g \circ f$.

Skill-sheet

Exercise 1E

Example 16

1 For each of the following, find $f(g(x))$ and $g(f(x))$:

a $f(x) = 2x - 1,\ g(x) = 2x$ **b** $f(x) = 4x + 1,\ g(x) = 2x + 1$

c $f(x) = 2x - 1,\ g(x) = 2x - 3$ **d** $f(x) = 2x - 1,\ g(x) = x^2$

e $f(x) = 2x^2 + 1,\ g(x) = x - 5$ **f** $f(x) = 2x + 1,\ g(x) = x^2$

2 For the functions $f(x) = 2x - 1$ and $h(x) = 3x + 2$, find:

a $f \circ h(x)$ **b** $h(f(x))$ **c** $f \circ h(2)$

d $h \circ f(2)$ **e** $f(h(3))$ **f** $h(f(-1))$

g $f \circ h(0)$

3 For the functions $f(x) = x^2 + 2x$ and $h(x) = 3x + 1$, find:

a $f \circ h(x)$ **b** $h \circ f(x)$ **c** $f \circ h(3)$

d $h \circ f(3)$ **e** $f \circ h(0)$ **f** $h \circ f(0)$

4 For the functions $h(x) = \dfrac{1}{x^2}$, $\{ x \in \mathbb{R} : x \neq 0 \}$ and $g(x) = 3x + 2$, $\{ x \in \mathbb{R} : x > 0 \}$, find:

a $h \circ g$ (state rule and domain) **b** $g \circ h$ (state rule and domain)

c $h \circ g(1)$ **d** $g \circ h(1)$

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Example 17

5 Consider the functions $f(x) = x^2 - 4$ and $g(x) = \sqrt{x}$, $\{x \in \mathbb{R} : x \geq 0\}$.

a State the ranges of f and g.

b Find $f \circ g$, stating its range.

c Explain why $g \circ f$ does not exist.

6 Let f and g be functions given by

$$f(x) = \frac{1}{2}\left(\frac{1}{x} + 1\right), \{x \in \mathbb{R} : x \neq 0\} \qquad g(x) = \frac{1}{2x - 1}, \left\{x \in \mathbb{R} : x \neq \frac{1}{2}\right\}$$

a Find $f \circ g$ and state its range.

b Find $g \circ f$ and state its range.

7 The functions f and g are defined by $f(x) = x^2 - 2$ and $g(x) = \sqrt{x}$, $\{x \in \mathbb{R} : x \geq 0\}$.

a Explain why $g \circ f$ does not exist.

b Find $f \circ g$ and sketch its graph.

Example 18

8 $f(x) = 3 - x$, $\{x \in \mathbb{R} : x \leq 3\}$ and $g(x) = x^2 - 1$

a Show that $f \circ g$ is not defined.

b Define a restriction g^* of g such that $f \circ g^*$ is defined and find $f \circ g^*$.

9 $f(x) = x^{-\frac{1}{2}}$, $\{x \in \mathbb{R} : x > 0\}$ and $g(x) = 3 - x$

a Show that $f \circ g$ is not defined.

b By suitably restricting the domain of g, obtain a function g_1 such that $f \circ g_1$ is defined.

Example 19

10 Express each of the following as the composition of two functions:

a $h(x) = (x^2 - 1)^4$

b $h(x) = \sqrt{x^4 + 3}$

c $h(x) = (x^2 - 2x)^n$ where $n \in \mathbb{N}$

d $h(x) = \dfrac{1}{2x + 3}$

e $h(x) = (x^2 - 2x)^3 - 2(x^2 - 2x)$

f $h(x) = 2(2x^2 + 1)^2 + 1$

11 Let $f(x) = x^2$ and let $g(x) = \sqrt{3 - x}$, $\{x \in \mathbb{R} : x \leq 3\}$. State with reasons whether:

a $f \circ g$ exists

b $g \circ f$ exists.

12 Let $f(x) = \sqrt{4 - x^2}$, $\{x \in S\}$, where S is the set of all real values of x for which $f(x)$ is defined. Let $g(x) = x^2 + 1$.

a Find S.

b Find the range of f and the range of g.

c State whether or not $f \circ g$ and $g \circ f$ are defined and give a reason for each assertion.

13 Let a be a positive number, let $f(x) = a - x$, $\{x \in \mathbb{R} : x \geq 2\}$ and let $g(x) = x^2 + a$, $\{x \in \mathbb{R} : x \leq 1\}$. Find all values of a for which both $f \circ g$ and $g \circ f$ exist.

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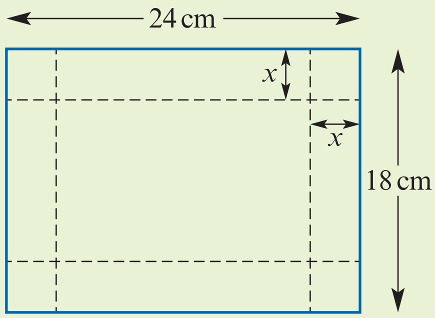
1F Applications of functions

In this section we use function notation in the solution of some problems.

Example 20

A rectangular piece of cardboard has dimensions 18 cm by 24 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps.

Find a function V which gives the volume of the box in terms of x, and state the domain of the function.



Solution

The dimensions of the box will be $24 - 2x$, $18 - 2x$ and x.

Thus the volume of the box is determined by the function

$$V(x) = (24 - 2x)(18 - 2x)x$$

For the box to be formed:

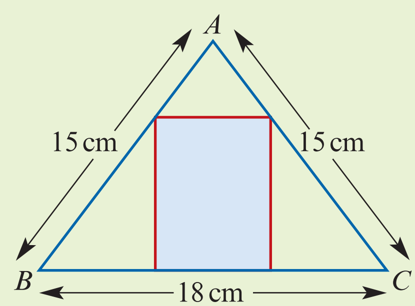
$$24 - 2x \geq 0 \quad \text{and} \quad 18 - 2x \geq 0 \quad \text{and} \quad x \geq 0$$

Therefore $x \leq 12$ and $x \leq 9$ and $x \geq 0$. The domain of V is $[0, 9]$.

Example 21

A rectangle is inscribed in an isosceles triangle with the dimensions as shown.

Find an area-of-the-rectangle function and state the domain.

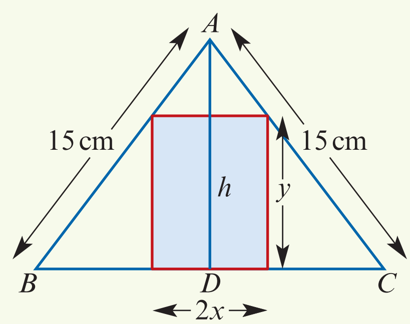


Solution

Let the height of the rectangle be y cm and the width $2x$ cm.

The height (h cm) of the triangle can be determined by Pythagoras' theorem:

$$h = \sqrt{15^2 - 9^2} = 12$$



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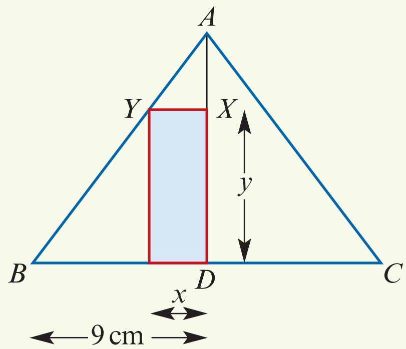
In the diagram opposite, the triangle AYX is similar to the triangle ABD. Therefore



$$\frac{x}{9} = \frac{12-y}{12}$$

$$\frac{12x}{9} = 12 - y$$

$$\therefore \quad y = 12 - \frac{12x}{9}$$

The area of the rectangle is $A = 2xy$, and so

$$A(x) = 2x\left(12 - \frac{12x}{9}\right)$$

$$= \frac{24x}{9}(9 - x)$$

For the rectangle to be formed, we need

$$x \geq 0 \quad \text{and} \quad 12 - \frac{12x}{9} \geq 0$$

$$\therefore \quad x \geq 0 \quad \text{and} \quad x \leq 9$$

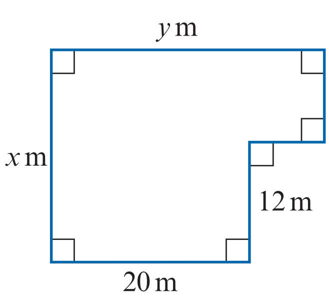
The domain is $[0, 9]$, and so the function is $A(x) = \frac{24x}{9}(9 - x)$, $\{ x \in \mathbb{R} : 0 \leq x \leq 9 \}$

Exercise 1F

Example 20

1 A rectangular piece of cardboard has dimensions 20 cm by 36 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps. Find a function V which gives the volume of the box in terms of x, and state the domain for the function.

2 The dimensions of an enclosure are shown. The perimeter of the enclosure is 160 m.



- **a** Find a rule for the area, A m^2, of the enclosure in terms of x.
- **b** State a suitable domain of the function $A(x)$.
- **c** Sketch the graph of A against x.
- **d** Find the maximum possible area of the enclosure and state the corresponding values of x and y.

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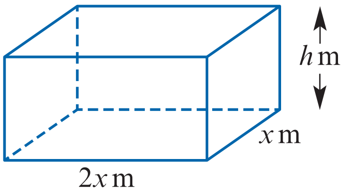
3 A cuboid tank is open at the top and the internal dimensions of its base are x m and $2x$ m. The height is h m. The volume of the tank is V m^3 and the volume is fixed. Let S m^2 denote the internal surface area of the tank.



a Find S in terms of:

i x and h

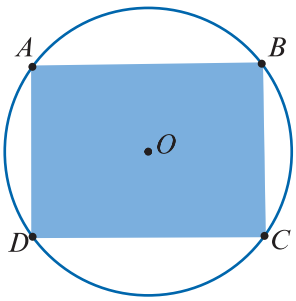
ii V and x

b State the natural domain for the function defined by the rule in part **a ii**.

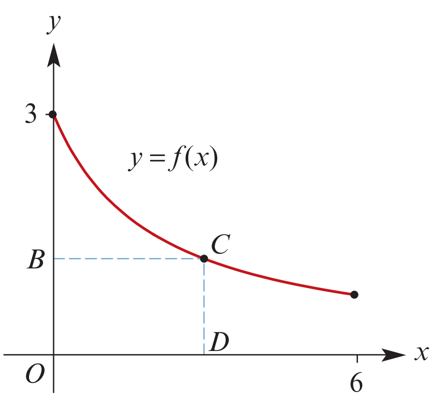
c If $2 \le x \le 15$, find the maximum value of S if $V = 1000\text{ m}^3$.

Example 21

4 A rectangle $ABCD$ is inscribed in a circle of radius a. The length of side AB is x. Find an area-of-the-rectangle function in terms of x and state the domain.



5 Let $f(x) = \dfrac{6}{x+2}$, $\{x \in \mathbb{R} : 0 \le x \le 6\}$.
Rectangle $OBCD$ is formed so that the coordinates of C are $(a, f(a))$.



a Find an expression for the area-of-rectangle function A.

b State the natural domain and range of A.

c State the maximum value of $A(x)$ for $x \in [0, 6]$.

d Sketch the graph of $y = A(x)$ for $x \in [0, 6]$.

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Chapter summary

Assign-ment

Nrich

Relations

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
- The **range** is the set of all the second coordinates of the ordered pairs in the relation.

Functions

- A **function** is a relation such that no two ordered pairs in the relation have the same first coordinate.
- For each x in the domain of a function f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f or the **value** of f at x and is denoted by $f(x)$.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **natural domain** or the **implied domain** of the function.
- For a function f, the domain is denoted by **dom** $\boldsymbol{f}$ and the range by **ran** $\boldsymbol{f}$.

Combining functions

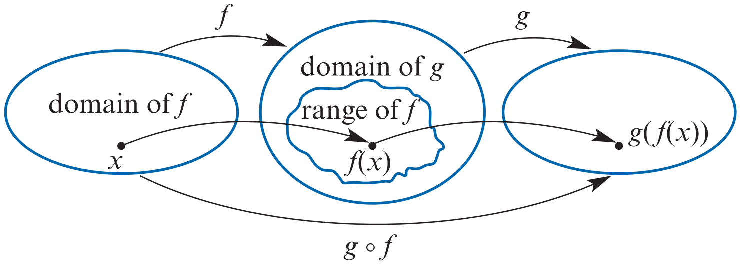
- Let f and g be functions such that $\text{dom}\, f \cap \text{dom}\, g \neq \emptyset$. Then the **sum**, $f + g$, and the **product**, fg, as functions on $\text{dom}\, f \cap \text{dom}\, g$ are defined by

  $$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x) \cdot g(x)$$

- The **composition** of function f followed by function g is denoted by $g \circ f$. The rule is given by

  $$g \circ f(x) = g(f(x))$$

  The domain of $g \circ f$ is the domain of f. The composition $g \circ f$ is defined only if the range of f is a subset of the domain of g.



Types of functions

- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

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Short-answer questions

1 Sketch the graph of each of the following relations and state the natural domain and range:

a $f(x) = x^2 + 1$ **b** $f(x) = 2x - 6$ **c** $\{ (x, y) : x^2 + y^2 = 25 \}$

d $\{ (x, y) : y \geq 2x + 1 \}$ **e** $\{ (x, y) : y < x - 3 \}$

2 For the function $g(x) = \dfrac{x + 3}{2}$, $\{ x \in \mathbb{R} : 0 \leq x \leq 5 \}$:

a Sketch the graph of $y = g(x)$.

b State the range of g.

c Find $\{ x : g(x) = 4 \}$.

3 For $g(x) = 5x + 1$, find:

a $\{ x : g(x) = 2 \}$ **b** $\{ x : g(x) \leq 2 \}$ **c** $\left\{ x : \dfrac{1}{g(x)} = 2 \right\}$

4 Find the natural domain for each of the following:

a $f(x) = \dfrac{1}{2x - 6}$ **b** $g(x) = \dfrac{1}{\sqrt{x^2 - 5}}$ **c** $h(x) = \dfrac{1}{(x - 1)(x + 2)}$

d $h(x) = \sqrt{25 - x^2}$ **e** $f(x) = \sqrt{x - 5} + \sqrt{15 - x}$ **f** $h(x) = \dfrac{1}{3x - 6}$

5 For $f(x) = (x + 2)^2$ and $g(x) = x - 3$, find $(f + g)(x)$ and $(fg)(x)$.

6 For $f(x) = (x - 1)^2$, $\{ x \in \mathbb{R} : 1 \leq x \leq 5 \}$ and $g(x) = 2x$, find $f + g$ and fg.

7 For $f(x) = 2x + 3$ and $g(x) = -x^2$, find:

a $(f + g)(x)$ **b** $(fg)(x)$ **c** $\{ x : (f + g)(x) = 0 \}$

8 For $f(x) = 2x + 3$ and $g(x) = -x^3$, find:

a $f \circ g(x)$ **b** $g \circ f(x)$ **c** $g \circ g(x)$

d $f \circ f(x)$ **e** $f \circ (f + g)(x)$ **f** $f \circ (f - g)(x)$

g $f \circ (f \cdot g)(x)$

9 Express each of the following as the composition of two functions:

a $h(x) = (x^3 - 1)^{\frac{1}{3}}$ **b** $h(x) = \dfrac{1}{x^2 + 1}$ **c** $h(x) = \dfrac{1}{x^2 - 1}$

10 If the function f has the rule $f(x) = \sqrt{x^2 - 16}$ and the function g has rule $g(x) = x + 5$, find the largest domain for g such that $f \circ g$ is defined.

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Extended-response questions

1 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 $64 per day + 25 cents per kilometre

Method 2 $89 per day with unlimited travel.

a Write a rule for each method if x kilometres per day are travelled and the cost in dollars is C_1 using method 1 and C_2 using method 2.

b Draw the graph of each, using the same axes.

c Determine, from the graph, the distance that must be travelled per day if method 2 is cheaper than method 1.

2 Express the total surface area, S, of a cube as a function of:

a the length x of an edge

b the volume V of the cube.

3 Express the area, A, of an equilateral triangle as a function of:

a the length s of each side

b the altitude h.

4 The base of a 3 m ladder leaning against a wall is x metres from the wall.

a Express the distance, d, from the top of the ladder to the ground as a function of x and sketch the graph of the function.

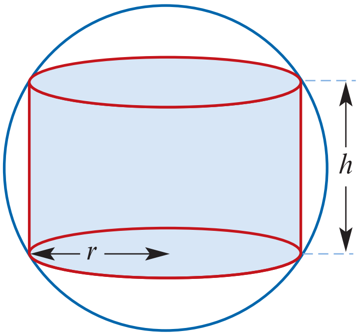
b State the domain and range of the function.

5 A car travels half the distance of a journey at an average speed of 80 km/h and half at an average speed of x km/h. Define a function, S, which gives the average speed for the total journey as a function of x.

6 A cylinder is inscribed in a sphere with a radius of length 6 cm.

a Define a function, V_1, which gives the volume of the cylinder as a function of its height, h. (State the rule and domain.)

b Define a function, V_2, which gives the volume of the cylinder as a function of the radius of the cylinder, r. (State the rule and domain.)



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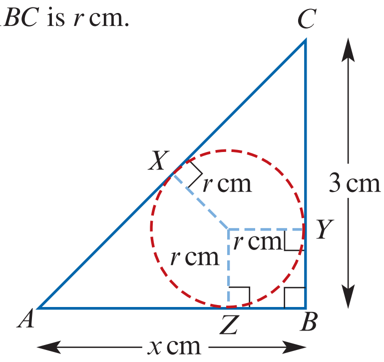
Review

7 The radius of the incircle of the right-angled triangle ABC is r cm.



a Find:

i YB in terms of r

ii ZB in terms of r

iii AZ in terms of r and x

iv CY

b Use the geometric results $CY = CX$ and $AX = AZ$ to find an expression for r in terms of x.

c **i** Find r when $x = 4$. **ii** Find x when $r = 0.5$.

d Use a CAS calculator to investigate the possible values r can take.

8 Let $f(x) = \dfrac{px + q}{x + r}$ where $x \in \{-r, r\}$.

a If $f(x) = f(-x)$ for all x, show that $f(x) = p$ for $x \in \{-r, r\}$.

b If $f(-x) = -f(x)$ for $x \neq 0$, find the rule for $f(x)$ in terms of q.

c If $p = 3$, $q = 8$ and $r = -3$, find the values of x for which $f(x) = x$.

9 **a** Let $f(x) = \dfrac{x + 1}{x - 1}$.

i Find $f(2)$, $f(f(2))$ and $f(f(f(2)))$. **ii** Find $f(f(x))$.

b Let $f(x) = \dfrac{x - 3}{x + 1}$. Find $f(f(x))$ and $f(f(f(x)))$.

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2 Exponential and logarithmic functions

In this chapter

Syllabus references

Topics: Logarithmic functions; Calculus of the natural logarithmic function

Subtopics: 4.1.1 – 4.1.10

An **exponential function** has a rule of the form $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1. Exponential functions were introduced in Mathematical Methods Year 11 and applied to practical situations such as radioactive decay and population growth.

In this chapter, we introduce the exponential function $f(x) = e^x$, which has many interesting properties. In particular, this function is its own derivative. That is, $f'(x) = f(x)$.

Here we define the number e as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

We will show that limits such as this arise in the consideration of compound interest.

A **logarithm function** has a rule of the form $f(x) = \log_a x$, where the base a is a positive real number other than 1. These functions are the 'reverse' of exponential functions, and have both historical and practical importance.

Logarithms were used to simplify calculations before the invention of electronic calculators. Today, we use logarithms in several well-known measurement scales, including the Richter scale (earthquakes), decibels (noise) and pH (acidity).

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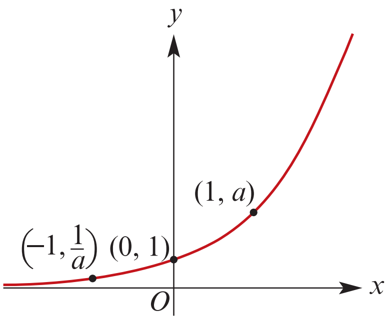
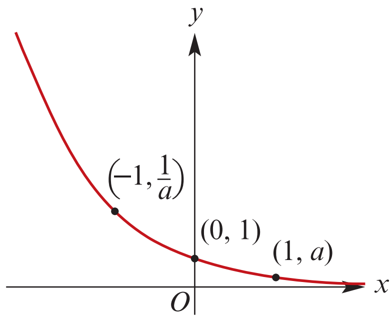
2A Exponential functions

The function $f(x) = a^x$, where $a \in (0, 1) \cup (1, \infty)$, is an **exponential function**. The shape of the graph depends on whether $a > 1$ or $0 < a < 1$.

Graph of $f(x) = a^x$ for $a > 1$



Graph of $f(x) = a^x$ for $0 < a < 1$



- Key values are $f(-1) = \dfrac{1}{a}$, $f(0) = 1$ and $f(1) = a$.
- The maximal domain is $\mathbb{R}$ and the range is $(0, \infty)$.
- The x-axis is a horizontal asymptote.

An exponential function with $a > 1$ is strictly increasing, and an exponential function with $0 < a < 1$ is strictly decreasing.

Graphing transformations of $f(x) = a^x$

Translations

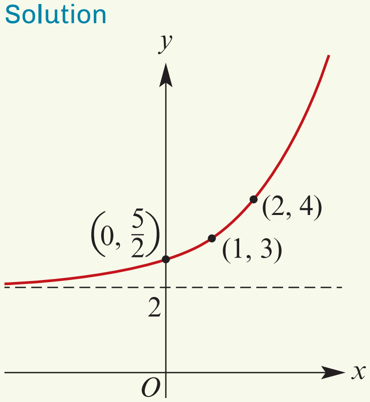
If the translation $(x, y) \to (x + h, y + k)$ is applied to the graph of $y = a^x$, then the image has equation $y = a^{x-h} + k$.

- The horizontal asymptote of the image has equation $y = k$.
- The range of the image is (k, ∞).

Example 1

Sketch the graph and state the range of $y = 2^{x-1} + 2$.

Solution



The range of the function is $(2, \infty)$.

Explanation

The graph of $y = 2^x$ is translated 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (x + 1, y + 2)$.

Translation of key points:

- $\left(-1, \dfrac{1}{2}\right) \to \left(0, \dfrac{5}{2}\right)$
- $(0, 1) \to (1, 3)$
- $(1, 2) \to (2, 4)$

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Reflections

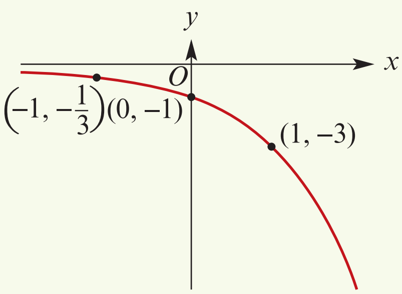
If a **reflection in the x-axis**, given by the mapping $(x, y) \to (x, -y)$, is applied to the graph of $y = a^x$, then the image has equation $y = -a^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(-\infty, 0)$.

Example 2

Sketch the graph of $y = -3^x$.

Solution



Explanation

The graph of $y = 3^x$ is reflected in the x-axis.

The mapping is $(x, y) \to (x, -y)$.

Reflection of key points:

- $\left(-1, \frac{1}{3}\right) \to \left(-1, -\frac{1}{3}\right)$
- $(0, 1) \to (0, -1)$
- $(1, 3) \to (1, -3)$

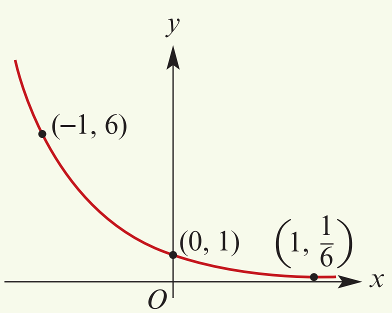
If a **reflection in the y-axis**, given by the mapping $(x, y) \to (-x, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{-x}$. This can also be written as $y = \frac{1}{a^x}$ or $y = \left(\frac{1}{a}\right)^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

Example 3

Sketch the graph of $y = 6^{-x}$.

Solution



Explanation

The graph of $y = 6^x$ is reflected in the y-axis.

The mapping is $(x, y) \to (-x, y)$.

Reflection of key points:

- $\left(-1, \frac{1}{6}\right) \to \left(1, \frac{1}{6}\right)$
- $(0, 1) \to (0, 1)$
- $(1, 6) \to (-1, 6)$

Dilations

For $k > 0$, if a **dilation of factor k parallel to the y-axis**, given by the mapping $(x, y) \to (x, ky)$, is applied to the graph of $y = a^x$, then the image has equation $y = ka^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

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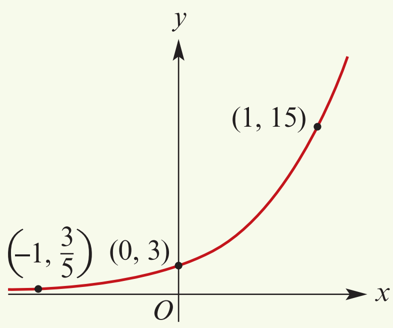
Example 4

Sketch the graph of each of the following:

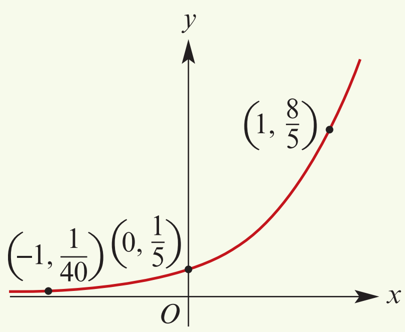
a $y = 3 \times 5^x$ **b** $y = 0.2 \times 8^x$

Solution

a



b



Explanation

The graph of $y = 5^x$ is dilated by factor 3 parallel to the y-axis.

The mapping is $(x, y) \to (x, 3y)$.

Dilation of key points:

- $\left(-1, \frac{1}{5}\right) \to \left(-1, \frac{3}{5}\right)$
- $(0, 1) \to (0, 3)$
- $(1, 5) \to (1, 15)$

The graph of $y = 8^x$ is dilated by factor $\frac{1}{5}$ parallel to the y-axis.

The mapping is $(x, y) \to \left(x, \frac{1}{5}y\right)$.

Dilation of key points:

- $\left(-1, \frac{1}{8}\right) \to \left(-1, \frac{1}{40}\right)$
- $(0, 1) \to \left(0, \frac{1}{5}\right)$
- $(1, 8) \to \left(1, \frac{8}{5}\right)$

For $k > 0$, if a **dilation of factor k parallel to the x-axis**, given by the mapping $(x, y) \to (kx, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{\frac{x}{k}}$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

Example 5

Sketch the graph of each of the following:

a $y = 9^{\frac{x}{2}}$

b $y = 3^{2x}$

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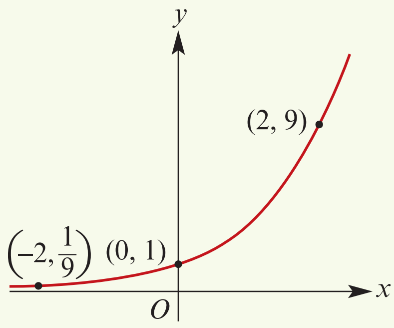
Solution

a



Explanation

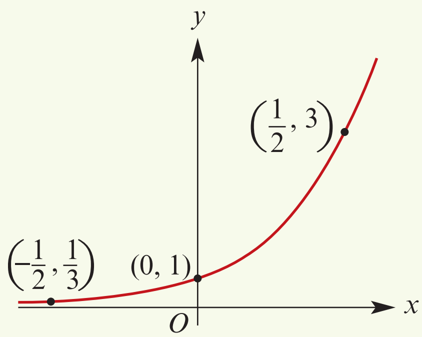
The graph of $y = 9^x$ is dilated by factor 2 parallel to the x-axis.

The mapping is $(x, y) \to (2x, y)$.

Dilation of key points:

- $\left(-1, \frac{1}{9}\right) \to \left(-2, \frac{1}{9}\right)$
- $(0, 1) \to (0, 1)$
- $(1, 9) \to (2, 9)$

b



The graph of $y = 3^x$ is dilated by factor $\frac{1}{2}$ parallel to the x-axis.

The mapping is $(x, y) \to \left(\frac{1}{2}x, y\right)$.

Dilation of key points:

- $\left(-1, \frac{1}{3}\right) \to \left(-\frac{1}{2}, \frac{1}{3}\right)$
- $(0, 1) \to (0, 1)$
- $(1, 3) \to \left(\frac{1}{2}, 3\right)$

Note: Since $9^{\frac{x}{2}} = \left(9^{\frac{1}{2}}\right)^x = 3^x$, the graph of $y = 9^{\frac{x}{2}}$ is the same as the graph of $y = 3^x$. Similarly, the graph of $y = 3^{2x}$ is the same as the graph of $y = 9^x$.

A translation parallel to the x-axis results in a dilation parallel to the y-axis. For example, if the graph of $y = 5^x$ is translated 3 units in the positive direction of the x-axis, then the image is the graph of $y = 5^{x-3}$, which can be written $y = 5^{-3} \times 5^x$. Hence, a translation of 3 units in the positive direction of the x-axis is equivalent to a dilation of factor 5^{-3} parallel to the y-axis.

Combinations of transformations

We have seen translations, reflections and dilations applied to exponential graphs. In the following example we consider combinations of these transformations.

Example 6

Sketch the graph and state the range of each of the following:

a $y = 2^{-x} + 3$

b $y = 4^{3x} - 1$

c $y = -10^{x-1} - 2$

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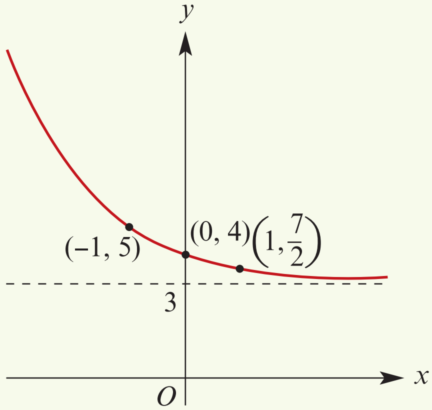
Solution

a



Graph of $y = 2^{-x} + 3$:

- The asymptote has equation $y = 3$.
- The y-axis intercept is $2^0 + 3 = 4$.
- The range of the function is $(3, \infty)$.

Explanation

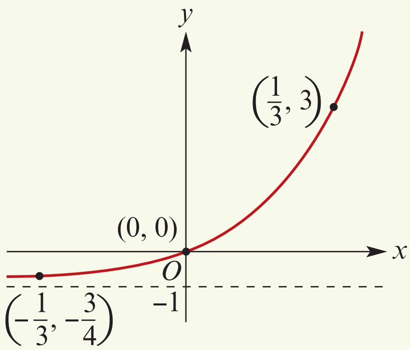
The graph of $y = 2^{-x} + 3$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis followed by a translation 3 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (-x, y + 3)$.

For example:

$$\left(-1, \frac{1}{2}\right) \to \left(1, \frac{7}{2}\right)$$
$$(0, 1) \to (0, 4)$$
$$(1, 2) \to (-1, 5)$$

b



Graph of $y = 4^{3x} - 1$:

- The asymptote has equation $y = -1$.
- The y-axis intercept is $4^0 - 1 = 0$.
- The range of the function is $(-1, \infty)$.

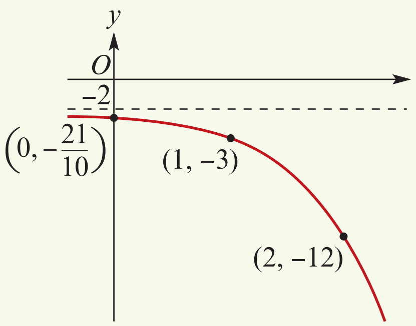
The graph of $y = 4^{3x} - 1$ is obtained from the graph of $y = 4^x$ by a dilation of factor $\frac{1}{3}$ parallel to the x-axis followed by a translation 1 unit in the negative direction of the y-axis.

The mapping is $(x, y) \to \left(\frac{1}{3}x, y - 1\right)$.

For example:

$$\left(-1, \frac{1}{4}\right) \to \left(-\frac{1}{3}, -\frac{3}{4}\right)$$
$$(0, 1) \to (0, 0)$$
$$(1, 4) \to \left(\frac{1}{3}, 3\right)$$

c



The graph of $y = -10^{x-1} - 2$ is obtained from the graph of $y = 10^x$ by a reflection in the x-axis followed by a translation 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

The mapping is $(x, y) \to (x + 1, -y - 2)$.

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Graph of $y = -10^{x-1} - 2$:

- The asymptote has equation $y = -2$.
- The y-axis intercept is $-10^{-1} - 2 = -\frac{21}{10}$.
- The range of the function is $(-\infty, -2)$.

For example:

$$\left(-1, \frac{1}{10}\right) \to \left(0, -\frac{21}{10}\right)$$

$$(0, 1) \to (1, -3)$$

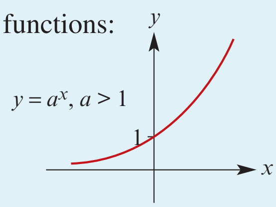
$$(1, 10) \to (2, -12)$$

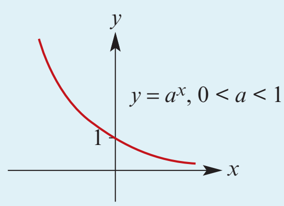
Note: We can use the method for determining transformations for each of the graphs in Example 6. Here we show the method for part **c**:

- Write the equation as $y' = -10^{x'-1} - 2$.
- Rearrange to $-y' - 2 = 10^{x'-1}$.
- We choose to write $y = -y' - 2$ and $x = x' - 1$.
- Hence $y' = -y - 2$ and $x' = x + 1$.

Summary 2A

- Graphs of exponential functions:





- For $a \in (0, 1) \cup (1, \infty)$, the graph of $y = a^x$ has the following properties:
  - The x-axis is an asymptote.
  - The y-axis intercept is 1.
  - The y-values are always positive.
  - There is no x-axis intercept.
- Transformations can be applied to exponential functions. For example, the graph of

  $$y = a^{b(x-h)} + k, \quad \text{where } b > 0$$

  can be obtained from the graph of $y = a^x$ by a dilation of factor $\frac{1}{b}$ parallel to the x-axis followed by the translation $(x, y) \to (x + h, y + k)$.

Exercise 2A

Example 1

1 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 2^{x+1} - 2$ **b** $y = 2^{x-3} - 1$ **c** $y = 2^{x+2} - 1$ **d** $y = 2^{x-2} + 2$

Example 2 Example 3

2 For each of the following, use the one set of axes to sketch the two graphs (labelling asymptotes):

a $y = 2^x$ and $y = 3^x$ **b** $y = 2^{-x}$ and $y = 3^{-x}$

c $y = 5^x$ and $y = -5^x$ **d** $y = 1.5^x$ and $y = -1.5^x$

Example 4 Example 5

3 For each of the following functions, sketch the graph and state the range:

a $y = 3 \times 2^x$ **b** $y = \frac{1}{2} \times 5^x$ **c** $y = 2^{3x}$ **d** $y = 2^{\frac{x}{3}}$

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Example 6

4 Sketch the graph (labelling the asymptote) and state the range of each of the following:

a $y = 3^{-x} + 2$ **b** $y = 2^{5x} - 4$ **c** $y = -10^{x-2} - 2$

5 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 3^x$ **b** $y = 3^x + 1$ **c** $y = 1 - 3^x$

d $y = \left(\frac{1}{3}\right)^x$ **e** $y = 3^{-x} + 2$ **f** $y = \left(\frac{1}{3}\right)^x - 1$

6 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = \left(\frac{1}{2}\right)^{x-2}$ **b** $y = \left(\frac{1}{2}\right)^x - 1$ **c** $y = \left(\frac{1}{2}\right)^{x-2} + 1$

7 For $f(x) = 2^x$, sketch the graph of each of the following, labelling asymptotes where appropriate:

a $y = f(x+1)$ **b** $y = f(x) + 1$ **c** $y = f(-x) + 2$ **d** $y = -f(x) - 1$

e $y = f(3x)$ **f** $y = f\left(\frac{x}{2}\right)$ **g** $y = 2f(x-1) + 1$ **h** $y = f(x-2)$

8 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 10^x - 1$ **b** $y = 10^{\frac{x}{10}} + 1$ **c** $y = 2 \times 10^x - 20$

d $y = 1 - 10^{-x}$ **e** $y = 10^{x+1} + 3$ **f** $y = 2 \times 10^{\frac{x}{10}} + 4$

9 A bank offers cash loans at 0.04% interest per day, compounded daily. A loan of \$10 000 is taken and the interest payable at the end of x days is given by $C_1 = 10\,000\,[(1.0004)^x - 1]$.

a Plot the graph of C_1 against x.

b Find the interest at the end of:

i 100 days **ii** 300 days.

c After how many days is the interest payable \$1000?

d A loan company offers \$10 000 and charges a fee of \$4.25 per day. The amount charged after x days is given by $C_2 = 4.25x$.

i Plot the graph of C_2 against x (using the same window as in part **a**).

ii Find the smallest value of x for which $C_2 < C_1$.

10 If you invest \$100 at an interest rate of 2% per day, compounded daily, then after x days the amount of money you have (in dollars) is given by $y = 100(1.02)^x$. For how many days would you have to invest to double your money?

11 **a** **i** Graph $y = 2^x$, $y = 3^x$ and $y = 5^x$ on the same set of axes.

ii For what values of x is $2^x > 3^x > 5^x$?

iii For what values of x is $2^x < 3^x < 5^x$?

iv For what values of x is $2^x = 3^x = 5^x$?

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b Repeat part **a** for $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$ and $y = \left(\frac{1}{5}\right)^x$.

c Use your answers to parts **a** and **b** to sketch the graph of $y = a^x$ for:

i $a > 1$ **ii** $a = 1$ **iii** $0 < a < 1$

2B The exponential function $f(x) = e^x$

In the previous section, we explored the family of exponential functions $f(x) = a^x$, where $a \in (0, 1) \cup (1, \infty)$. One particular member of this family is of great importance in mathematics.

This function has the rule $f(x) = e^x$, where e is Euler's number, named after the eighteenth century Swiss mathematician Leonhard Euler.

Euler's number is defined as follows.

Euler's number

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

To see what the value of e might be, we could try large values of n and use a calculator to evaluate $\left(1 + \frac{1}{n}\right)^n$, as shown in the table on the right.

n	$\left(1 + \frac{1}{n}\right)^n$
100	$(1.01)^{100} = 2.704\,813\ldots$
1000	$(1.001)^{1000} = 2.716\,923\ldots$
10 000	$(1.0001)^{10\,000} = 2.718\,145\ldots$
100 000	$(1.000\,01)^{100\,000} = 2.718\,268\ldots$
1 000 000	$(1.000\,001)^{1\,000\,000} = 2.718\,280\ldots$

As n is taken larger and larger, it can be seen that $\left(1 + \frac{1}{n}\right)^n$ approaches a limiting value ($\approx 2.718\,28$).

Like π, the number e is irrational:

$$e = 2.718\,281\,828\,459\,045\ldots$$

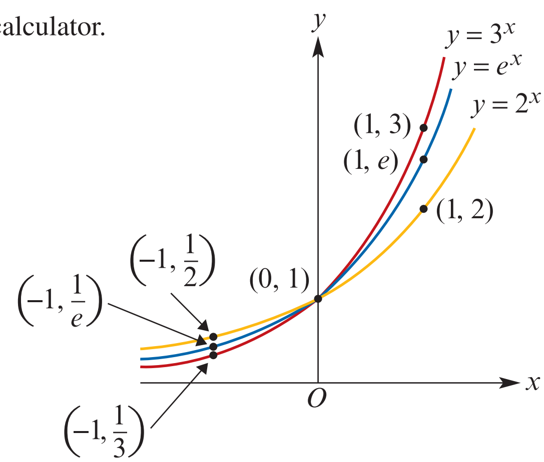
The function $f(x) = e^x$ is very important in mathematics. In Chapter 4 you will find that it has the remarkable property that $f'(x) = f(x)$. That is, the derivative of e^x is e^x.

Note: The function e^x can be found on your calculator.

Graphing $f(x) = e^x$

The graph of $y = e^x$ is as shown.

The graphs of $y = 2^x$ and $y = 3^x$ are shown on the same set of axes.



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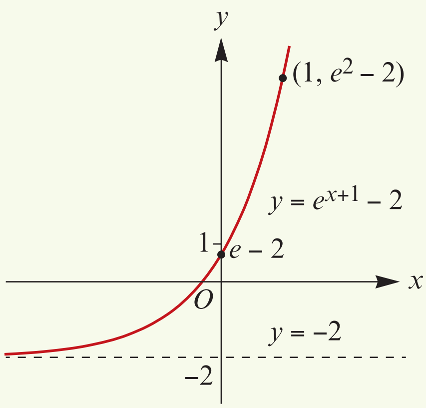
Example 7

Sketch the graph of $f(x) = e^{x+1} - 2$.

Solution



The asymptote has equation $y = -2$.
The y-axis intercept is $e - 2$.

Explanation

To find the transformation:

- Write the image as $y' + 2 = e^{x'+1}$.
- We can choose $y = y' + 2$ and $x = x' + 1$.
- Hence $y' = y - 2$ and $x' = x - 1$.

The mapping is

$$(x, y) \to (x - 1, y - 2)$$

which is a translation of 1 unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis.

Compound interest

Assume that you invest $\$P$ at an annual interest rate r. If the interest is compounded only once per year, then the balance of your investment after t years is given by $A = P(1 + r)^t$.

Now assume that the interest is compounded n times per year. The interest rate in each period is $\frac{r}{n}$. The balance at the end of one year is $P\left(1 + \frac{r}{n}\right)^n$, and the balance at the end of t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{rt}$$

We recognise that

$$\lim_{\frac{n}{r}\to\infty}\left(1 + \frac{r}{n}\right)^{\frac{n}{r}} = e$$

So, as $n \to \infty$, we can write $A = Pe^{rt}$.

For example, if \$1000 is invested for one year at 5%, the resulting amount is \$1050. However, if the interest is compounded 'continuously', then the amount is given by

$$A = Pe^{rt} = 1000 \times e^{0.05} = 1000 \times 1.051\,271\ldots \approx 1051.27$$

That is, the balance after one year is \$1051.27.

Summary 2B

Euler's number is the natural base for exponential functions:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$
$$= 2.718\,281\ldots$$

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Skill-sheet

Exercise 2B

Example 7

1 Sketch the graph of each of the following and state the range:

a $f(x) = e^x + 1$ **b** $f(x) = 1 - e^x$ **c** $f(x) = 1 - e^{-x}$
d $f(x) = e^{-2x}$ **e** $f(x) = e^{x-1} - 2$ **f** $f(x) = 2e^x$
g $h(x) = 2(1 + e^x)$ **h** $h(x) = 2(1 - e^{-x})$ **i** $g(x) = 2e^{-x} + 1$
j $h(x) = 2e^{x-1}$ **k** $f(x) = 3e^{x+1} - 2$ **l** $h(x) = 2 - 3e^x$

2 For each of the following, give a sequence of transformations that maps the graph of $y = e^x$ to the graph of $y = f_1(x)$:

a $f_1(x) = e^{x+2} - 3$ **b** $f_1(x) = 3e^{x+1} - 4$ **c** $f_1(x) = 5e^{2x+1}$
d $f_1(x) = 2 - e^{x-1}$ **e** $f_1(x) = 3 - 2e^{x+2}$ **f** $f_1(x) = 4e^{2x} - 1$

3 Find the rule of the image when the graph of $f(x) = e^x$ undergoes each of the following sequences of transformations:

a a dilation of factor 2 parallel to the y-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
b a dilation of factor 2 parallel to the y-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis
c a reflection in the x-axis, followed by a dilation of factor 2 parallel to the y-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
d a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 parallel to the y-axis.
e a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 parallel to the y-axis, followed by a reflection in the x-axis
f a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 parallel to the y-axis.

4 For each of the following, give a sequence of transformations that maps the graph of $y = f_1(x)$ to the graph of $y = e^x$:

a $f_1(x) = e^{x+2} - 3$ **b** $f_1(x) = 3e^{x+1} - 4$ **c** $f_1(x) = 5e^{2x+1}$
d $f_1(x) = 2 - e^{x-1}$ **e** $f_1(x) = 3 - 2e^{x+2}$ **f** $f_1(x) = 4e^{2x} - 1$

5 Solve each of the following equations using a calculator. Give answers correct to three decimal places.

a $e^x = x + 2$ **b** $e^{-x} = x + 2$ **c** $x^2 = e^x$ **d** $x^3 = e^x$

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6 **a** Using a calculator, plot the graph of $y = f(x)$ where $f(x) = e^x$.

b Using the same screen, plot the graphs of:

i $y = f(x-2)$ **ii** $y = f\left(\frac{x}{3}\right)$ **iii** $y = f(-x)$

2C Exponential equations

One method for solving exponential equations is to use the following property of exponential functions:

> $a^x = a^y$ implies $x = y$, for $a \in (0,1) \cup (1,\infty)$

Example 8

Find the value of x for which:

a $4^x = 256$ **b** $3^{x-1} = 81$

Solution

a
$$4^x = 256$$
$$4^x = 4^4$$
$$\therefore \quad x = 4$$

b
$$3^{x-1} = 81$$
$$3^{x-1} = 3^4$$
$$\therefore \quad x - 1 = 4$$
$$x = 5$$

When solving an exponential equation, you may also need to use the index laws.

Index laws

For all positive numbers a and b and all real numbers x and y:

- $a^x \times a^y = a^{x+y}$
- $a^x \div a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $a^{-x} = \frac{1}{a^x}$
- $a^x = \frac{1}{a^{-x}}$
- $a^0 = 1$

Note: More generally, each index law applies for real numbers a and b provided both sides of the equation are defined. For example: $a^m \times a^n = a^{m+n}$ for $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$.

Example 9

Find the value of x for which $5^{2x-4} = 25^{-x+2}$.

Solution	Explanation
$5^{2x-4} = 25^{-x+2}$	Express both sides of the equation as powers with base 5.
$= (5^2)^{-x+2}$	
$= 5^{-2x+4}$	
$\therefore \quad 2x - 4 = -2x + 4$	Use the fact that $5^a = 5^b$ implies $a = b$.
$4x = 8$	
$x = 2$	

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To solve the equations in the next example, we must recognise that they will become quadratic equations once we make a substitution.

Example 10

Solve for x:

a $9^x = 12 \times 3^x - 27$

b $3^{2x} = 27 - 6 \times 3^x$

Solution

a We can write the equation as

$$(3^x)^2 = 12 \times 3^x - 27$$

Let $y = 3^x$. The equation becomes

$$\begin{aligned} y^2 &= 12y - 27 \\ y^2 - 12y + 27 &= 0 \\ (y-3)(y-9) &= 0 \\ \therefore\ y = 3 \quad &\text{or} \quad y = 9 \\ 3^x = 3 \quad &\text{or} \quad 3^x = 3^2 \\ x = 1 \quad &\text{or} \quad x = 2 \end{aligned}$$

b We can write the equation as

$$(3^x)^2 = 27 - 6 \times 3^x$$

Let $y = 3^x$. The equation becomes

$$\begin{aligned} y^2 &= 27 - 6y \\ y^2 + 6y - 27 &= 0 \\ (y-3)(y+9) &= 0 \\ \therefore\ y = 3 \quad &\text{or} \quad y = -9 \\ 3^x = 3 \quad &\text{or} \quad 3^x = -9 \end{aligned}$$

The only solution is $x = 1$, since $3^x > 0$ for all x.

Summary 2C

- One method for solving an exponential equation, without using a calculator, is first to express both sides of the equation as powers with the same base and then to equate the indices (since $a^x = a^y$ implies $x = y$, for any base $a \in (0, 1) \cup (1, \infty)$).
  For example: $2^{x+1} = 8 \Leftrightarrow 2^{x+1} = 2^3 \Leftrightarrow x + 1 = 3 \Leftrightarrow x = 2$
- Equations such as $3^{2x} - 6 \times 3^x - 27 = 0$ can be solved by making a substitution. In this case, substitute $y = 3^x$ to obtain a quadratic equation in y.

Exercise 2C

1 Simplify the following expressions:

a $3x^2y^3 \times 2x^4y^6$ **b** $\dfrac{12x^8}{4x^2}$ **c** $18x^2y^3 \div (3x^4y)$

d $(4x^4y^2)^2 \div (2(x^2y)^4)$ **e** $(4x^0)^2$ **f** $15(x^5y^{-2})^4 \div (3(x^4y)^{-2})$

g $\dfrac{3(2x^2y^3)^4}{2x^3y^2}$ **h** $(8x^3y^6)^{\frac{1}{3}}$ **i** $\dfrac{x^2 + y^2}{x^{-2} + y^{-2}}$

Example 8

2 Solve for x in each of the following:

a $3^x = 81$ **b** $81^x = 9$ **c** $2^x = 256$

d $625^x = 5$ **e** $32^x = 8$ **f** $5^x = 125$

g $16^x = 1024$ **h** $2^{-x} = \dfrac{1}{64}$ **i** $5^{-x} = \dfrac{1}{625}$

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Example 9

3 Solve for n in each of the following:

a $5^{2n} \times 25^{2n-1} = 625$ **b** $4^{2n-2} = 1$ **c** $4^{2n-1} = \frac{1}{256}$

d $\frac{3^{n-2}}{9^{2-n}} = 27$ **e** $2^{2n-2} \times 4^{-3n} = 64$ **f** $2^{n-4} = 8^{4-n}$

g $27^{n-2} = 9^{3n+2}$ **h** $8^{6n+2} = 8^{4n-1}$ **i** $125^{4-n} = 5^{6-2n}$

j $2^{n-1} \times 4^{2n+1} = 16$ **k** $(27 \times 3^n)^n = 27^n \times 3^{\frac{1}{4}}$

Example 10

4 Solve for x:

a $3^{2x} - 2(3^x) - 3 = 0$ **b** $5^{2x} - 23(5^x) - 50 = 0$ **c** $5^{2x} - 10(5^x) + 25 = 0$

d $2^{2x} = 6(2^x) - 8$ **e** $8(3^x) - 6 = 2(3^{2x})$ **f** $2^{2x} - 20(2^x) = -64$

g $4^{2x} - 5(4^x) = -4$ **h** $3(3^{2x}) = 28(3^x) - 9$ **i** $7(7^{2x}) = 8(7^x) - 1$

2D Logarithms

Consider the statement

$$2^3 = 8$$

This may be written in an alternative form:

$$\log_2 8 = 3$$

which is read as 'the logarithm of 8 to the base 2 is equal to 3'.

For $a \in (0, 1) \cup (1, \infty)$, the **logarithm function** with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

Note: Since a^x is positive, the expression $\log_a y$ is only defined when y is positive.

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 11

Without the aid of a calculator, evaluate the following:

a $\log_2 32$ **b** $\log_3 81$

Solution

a Let $\log_2 32 = x$

Then $2^x = 32$

$2^x = 2^5$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

Then $3^x = 81$

$3^x = 3^4$

Therefore $x = 4$, giving $\log_3 81 = 4$.

Note: To find $\log_2 32$, we ask 'What power of 2 gives 32?'
To find $\log_3 81$, we ask 'What power of 3 gives 81?'

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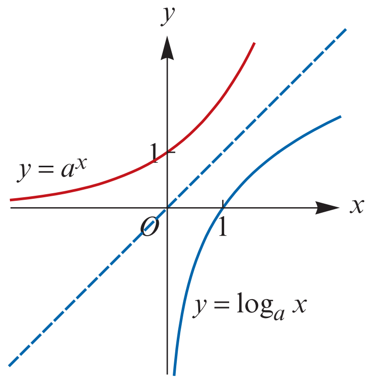
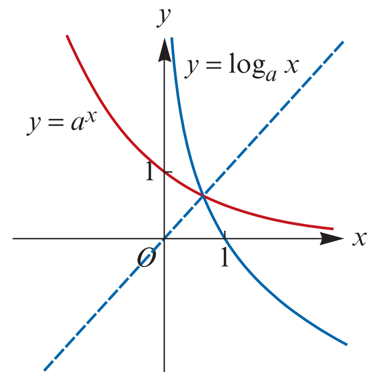
Inverse relationship between logarithms and exponentials

When the graph of $y = a^x$ is reflected in the line $y = x$, we obtain the graph of $x = a^y$. By the defining property of logarithms, this corresponds exactly to the graph of $y = \log_a x$.



$a > 1$



$0 < a < 1$

The logarithm function with base a is the 'reverse' of the exponential function with base a. This can be stated more precisely as follows:

Inverse relationship

Let $a \in (0, 1) \cup (1, \infty)$. The function with rule $y = \log_a x$, for $x > 0$, and the function with rule $y = a^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:

- $\log_a(a^x) = x$ for all x
- $a^{\log_a x} = x$ for all positive values of x

The natural logarithm

Earlier in the chapter we defined the number e and the important exponential function $y = e^x$. The corresponding logarithmic function is $y = \log_e x$.

The logarithm function with base e is known as the **natural logarithm** function and is commonly written as $y = \ln x$. That is,

$$\ln x = \log_e x$$

Natural logarithm

The natural logarithm is defined as follows:

$$e^x = y \quad \text{is equivalent to} \quad \ln y = x$$

The natural logarithm function $y = \ln x$, for $x > 0$, and the exponential function $y = e^x$ are inverse to each other in the following way:

- $\ln(e^x) = x$ for all x
- $e^{\ln x} = x$ for all positive values of x

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Laws of logarithms

The index laws are used to establish rules for computations with logarithms.

Law 1: Logarithm of a product

The logarithm of a product is the sum of their logarithms:

$$\log_a(mn) = \log_a m + \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then $a^x = m$ and $a^y = n$, and therefore

$$mn = a^x \times a^y = a^{x+y} \qquad \text{(using the first index law)}$$

Hence $\log_a(mn) = x + y = \log_a m + \log_a n$.

For example:

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5)\\ &= \log_{10} 1000 = 3\end{aligned}$$

Law 2: Logarithm of a quotient

The logarithm of a quotient is the difference of their logarithms:

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then as before $a^x = m$ and $a^y = n$, and therefore

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{(using the second index law)}$$

Hence $\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$.

For example:

$$\begin{aligned}\log_2 32 - \log_2 8 &= \log_2\left(\frac{32}{8}\right)\\ &= \log_2 4 = 2\end{aligned}$$

Law 3: Logarithm of a power

$$\log_a(m^p) = p\log_a m$$

Proof Let $\log_a m = x$. Then $a^x = m$, and therefore

$$m^p = (a^x)^p = a^{xp} \qquad \text{(using the third index law)}$$

Hence $\log_a(m^p) = xp = p\log_a m$.

For example:

$$\log_2 32 = \log_2(2^5) = 5$$

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Law 4: Logarithm of $\frac{1}{m}$

$$\log_a(m^{-1}) = -\log_a m$$

Proof Use logarithm law 3 with $p = -1$.

For example:

$$\log_a\left(\frac{1}{2}\right) = \log_a(2^{-1}) = -\log_a 2$$

Law 5

$$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1$$

Proof Since $a^0 = 1$, we have $\log_a 1 = 0$. Since $a^1 = a$, we have $\log_a a = 1$.

Example 12

Express each of the following as the logarithm of a single term:

a $\log_{10} 5 + 2\log_{10} 7$

b $2\ln 3 + \ln 16 - 2\ln\left(\frac{6}{5}\right)$

Solution

a $\log_{10} 5 + 2\log_{10} 7$

$= \log_{10} 5 + \log_{10}(7^2)$

$= \log_{10} 5 + \log_{10} 49$

$= \log_{10}(5 \times 49)$

$= \log_{10} 245$

b $2\ln 3 + \ln 16 - 2\ln\left(\frac{6}{5}\right)$

$= \ln(3^2) + \ln 16 - \ln\left(\frac{6}{5}\right)^2$

$= \ln 9 + \ln 16 - \ln\left(\frac{36}{25}\right)$

$= \ln\left(9 \times 16 \times \frac{25}{36}\right)$

$= \ln 100$

Note: For part **b**, remember that $\ln x = \log_e x$.

Logarithmic equations

Example 13

Solve each of the following equations for x:

a $\log_2 x = 5$

b $\log_2(2x - 1) = 4$

c $\ln(3x + 1) = 0$

Solution

a $\log_2 x = 5$

$x = 2^5$

$\therefore \quad x = 32$

b $\log_2(2x - 1) = 4$

$2x - 1 = 2^4$

$2x = 17$

$\therefore \quad x = \frac{17}{2}$

c $\ln(3x + 1) = 0$

$3x + 1 = e^0$

$3x = 1 - 1$

$\therefore \quad x = 0$

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Example 14

Solve each of the following equations for x:

a $\ln(x-1)+\ln(x+2)=\ln(6x-8)$

b $\log_2 x-\log_2(7-2x)=\log_2 6$

Solution

a

$$\begin{aligned}\ln(x-1)+\ln(x+2)&=\ln(6x-8)\\ \ln((x-1)(x+2))&=\ln(6x-8)\\ x^2+x-2&=6x-8\\ x^2-5x+6&=0\\ (x-3)(x-2)&=0\\ \therefore\quad x=3 \text{ or } x&=2\end{aligned}$$

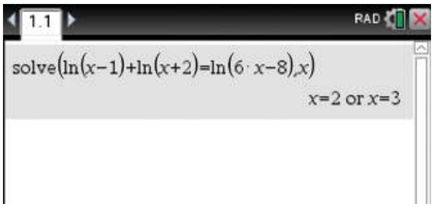
Note: The solutions must satisfy $x-1>0$, $x+2>0$ and $6x-8>0$. Therefore both of these solutions are allowable.

b

$$\begin{aligned}\log_2 x-\log_2(7-2x)&=\log_2 6\\ \log_2\left(\frac{x}{7-2x}\right)&=\log_2 6\\ \frac{x}{7-2x}&=6\\ x&=42-12x\\ 13x&=42\\ \therefore\quad x&=\frac{42}{13}\end{aligned}$$

Using the TI-Nspire

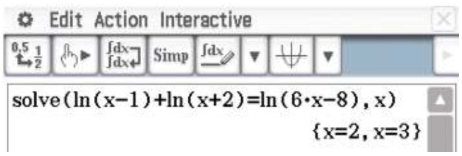
- Use **solve** from the **Algebra** menu as shown.
- The natural logarithm (base e) is available on the keypad by pressing ctrl e^x.



Note: Logarithms with other bases are obtained by pressing the **log** key (ctrl 10^x) and completing the template.

Using the Casio ClassPad

- For the natural logarithm (base e), use ln from the Math1 keyboard.
- Enter and highlight the equation $\ln(x-1)+\ln(x+2)=\ln(6x-8)$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Ensure the variable is set to x.



Note: For logarithms with other bases, tap $\log_{\blacksquare}\square$ and complete the template.

Example 15

Solve each of the following equations for x:

a $\ln(2x+1)-\ln(x-1)=4$

b $\ln(x-1)+\ln(x+1)=1$

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Solution

a $\ln(2x+1) - \ln(x-1) = 4$

$$\ln\left(\frac{2x+1}{x-1}\right) = 4$$

$$\frac{2x+1}{x-1} = e^4$$

$$2x+1 = e^4(x-1)$$

$$(2-e^4)x = -(e^4+1)$$

$$\therefore \quad x = \frac{e^4+1}{e^4-2}$$

b $\ln(x-1) + \ln(x+1) = 1$

$$\ln((x-1)(x+1)) = 1$$

$$\ln(x^2-1) = 1$$

$$x^2 - 1 = e$$

$$\therefore \quad x = \pm\sqrt{e+1}$$

But the original equation is not defined for $x = -\sqrt{e+1}$ and so the only solution is $x = \sqrt{e+1}$.

Example 16

Solve the equation $\log_x 27 = \frac{3}{2}$ for x.

Solution

$\log_x 27 = \frac{3}{2}$ is equivalent to

$$x^{\frac{3}{2}} = 27$$

$$(\sqrt{x})^3 = 3^3$$

$$\sqrt{x} = 3$$

$$\therefore \quad x = 9$$

Summary 2D

- For $a \in (0, 1) \cup (1, \infty)$, the logarithm function base a is defined as follows:

  $a^x = y$ is equivalent to $\log_a y = x$

- To evaluate $\log_a y$ ask the question: 'What power of a gives y?'
- Let $a \in (0, 1) \cup (1, \infty)$. The logarithmic function $y = \log_a x$, for $x > 0$, and the exponential function $y = a^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:
  - $\log_a(a^x) = x$ for all x
  - $a^{\log_a x} = x$ for all positive values of x
- **The natural logarithm**

  The function $y = \log_e x$ is commonly written as $y = \ln x$. Therefore

  $e^x = y$ is equivalent to $\ln y = x$

  The natural logarithm function $y = \ln x$, for $x > 0$, and the exponential function $y = e^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:
  - $\ln(e^x) = x$ for all x
  - $e^{\ln x} = x$ for all positive values of x
- **Laws of logarithms**

  1 $\log_a(mn) = \log_a m + \log_a n$

  2 $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

  3 $\log_a(m^p) = p\log_a m$

  4 $\log_a(m^{-1}) = -\log_a m$

  5 $\log_a 1 = 0$ and $\log_a a = 1$

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Skill-sheet

Exercise 2D

Example 11

1 Evaluate each of the following:

a $\log_{10} 1000$ **b** $\log_2\left(\frac{1}{16}\right)$ **c** $\log_{10} 0.001$

d $\log_2 64$ **e** $\log_{10} 1\,000\,000$ **f** $\log_2\left(\frac{1}{128}\right)$

Example 12

2 Express each of the following as the logarithm of a single term:

a $\log_{10} 2 + \log_{10} 3$ **b** $\log_{10} 32 - \log_{10} 8$ **c** $\ln 10 + \ln 100 + \ln 1000$

d $\ln\left(\frac{1}{2}\right) + \ln 14$ **e** $\ln\left(\frac{1}{3}\right) + \ln\left(\frac{1}{4}\right) + \ln\left(\frac{1}{5}\right)$ **f** $\ln(uv) + \ln(uv^2) + \ln(uv^3)$

g $2 \ln x + 5 \ln x$ **h** $\ln(x + y) + \ln(x - y) - \ln(x^2 - y^2)$

Example 13

3 Solve each of the following equations for x:

a $\log_{10} x = 2$ **b** $2 \log_2 x = 8$ **c** $\ln(x - 5) = 0$

d $\log_2 x = 6$ **e** $2 \ln(x + 5) = 6$ **f** $\ln(2x) = 0$

g $\ln(2x + 3) = 0$ **h** $\log_{10} x = -3$ **i** $2 \log_2(x - 4) = 10$

Example 14

4 Solve each of the following equations for x:

a $\log_{10} x = \log_{10} 3 + \log_{10} 5$ **b** $\ln x = \ln 15 - \ln 3$

c $\ln x = \frac{2}{3} \ln 8$ **d** $\ln x + \ln(2x - 1) = 0$

e $2 \ln x - \ln(x - 1) = \ln(x + 3)$

5 Express each of the following as the logarithm of a single term:

a $\log_{10} 9 + \log_{10} 3$ **b** $\log_2 24 - \log_2 6$ **c** $\frac{1}{2} \log_{10} a - \frac{1}{2} \log_{10} b$

d $1 + \log_{10} a - \frac{1}{3} \log_{10} b$ **e** $\frac{1}{2} \log_{10} 36 - \frac{1}{3} \log_{10} 27 - \frac{2}{3} \log_{10} 64$

6 Without using your calculator, evaluate each of the following:

a $\log_{10} 5 + \log_{10} 2$ **b** $\log_{10} 5 + 3 \log_{10} 2 - \log_{10} 4$

c $\log_2 \sqrt{2} + \log_2 1 + 2 \log_2 2$ **d** $2 \log_{10} 5 + 2 \log_{10} 2 + 1$

e $4 \log_{10} 2 - \log_{10} 16$

7 Simplify the following expressions:

a $\log_3\left(\frac{1}{3^x}\right)$ **b** $\log_2 x - 2 \log_2 y + \log_2(xy^2)$

Example 15

8 Solve each of the following equations for x:

a $\ln(x^2 - 2x + 8) = 2 \ln x$ **b** $\ln(5x) - \ln(3 - 2x) = 1$

9 Solve each of the following equations for x:

a $\ln x + \ln(3x + 1) = 1$ **b** $8e^{-x} - e^x = 2$

Example 16

10 Solve each of the following equations for x:

a $\log_x 81 = 4$ **b** $\log_x\left(\frac{1}{32}\right) = 5$

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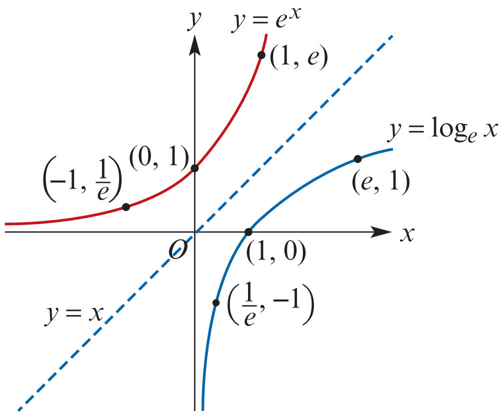
11 Solve $2\ln x + \ln 4 = \ln(9x - 2)$.

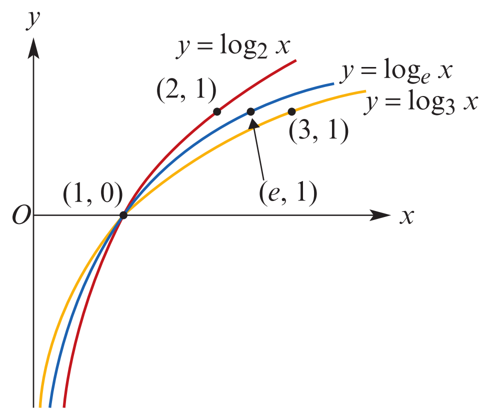
12 Given that $\log_a N = \frac{1}{2}(\log_a 24 - \log_a 0.375 - 6\log_a 3)$, find the value of N.

2E Graphing logarithmic functions

The graphs of $y = e^x$ and $y = \log_e x$ are shown on the one set of axes.



The graphs of $y = \log_2 x$, $y = \log_e x$ and $y = \log_3 x$ are shown on the one set of axes.



Note: Recall that we usually write the natural logarithm function $y = \log_e x$ as $y = \ln x$.

For each base $a \in (0, 1) \cup (1, \infty)$, the graph of $f(x) = \log_a x$ has the following features:

- Key values are $f\left(\frac{1}{a}\right) = -1$, $f(1) = 0$ and $f(a) = 1$.
- The natural domain is $(0, \infty)$ and the range is $\mathbb{R}$.
- The y-axis is a vertical asymptote.

A logarithmic function with $a > 1$ is strictly increasing, and a logarithmic function with $0 < a < 1$ is strictly decreasing.

Graphing transformations of $f(x) = \log_a x$

We now look at transformations applied to the graph of $f(x) = \log_a x$ where $a > 1$.

Example 17

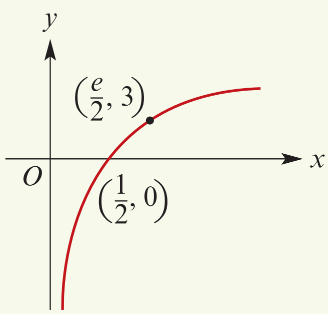
Sketch the graph of $y = 3\ln(2x)$.

Solution

This is obtained from the graph of $y = \ln x$ by a dilation of factor 3 parallel to the y-axis and a dilation of factor $\frac{1}{2}$ parallel to the x-axis.

The mapping is $(x, y) \to \left(\frac{1}{2}x, 3y\right)$.

- $(1, 0) \to \left(\frac{1}{2}, 0\right)$
- $(e, 1) \to \left(\frac{1}{2}e, 3\right)$



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Example 18

Sketch the graph and state the natural domain of each of the following:

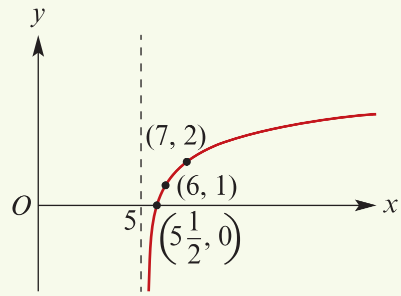
a $y = \log_2(x - 5) + 1$ **b** $y = -\log_3(x + 4)$

Solution

a The graph of $y = \log_2(x-5)+1$ is obtained from the graph of $y = \log_2 x$ by a translation of 5 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.

The mapping is $(x, y) \to (x + 5, y + 1)$.

- $(1, 0) \to (6, 1)$
- $(2, 1) \to (7, 2)$



The asymptote has equation $x = 5$.

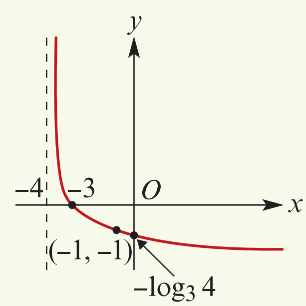
When $y = 0$,
$$\log_2(x - 5) + 1 = 0$$
$$\log_2(x - 5) = -1$$
$$x - 5 = 2^{-1}$$
$$\therefore \quad x = 5\frac{1}{2}$$

The domain of the function is $(5, \infty)$.

b The graph of $y = -\log_3(x + 4)$ is obtained from the graph of $y = \log_3 x$ by a reflection in the x-axis and a translation of 4 units in the negative direction of the x-axis.

The mapping is $(x, y) \to (x - 4, -y)$.

- $(1, 0) \to (-3, 0)$
- $(3, 1) \to (-1, -1)$



The asymptote has equation $x = -4$.

When $x = 0$,
$$y = -\log_3(0 + 4)$$
$$= -\log_3 4$$

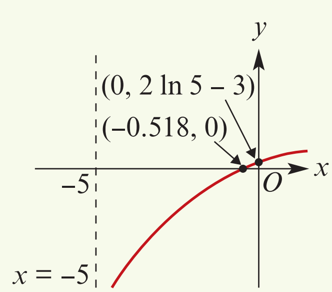
The domain of the function is $(-4, \infty)$.

Example 19

Sketch the graph of $y = 2\ln(x + 5) - 3$ and state the natural domain.

Solution

The graph of $y = 2\ln(x + 5) - 3$ is obtained from the graph of $y = \ln x$ by a dilation of factor 2 parallel to the y-axis followed by a translation of 5 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis.



The equation of the asymptote is $x = -5$.

The domain of the function is $(-5, \infty)$.

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Axis intercepts

When $x = 0$, $y = 2\ln(0 + 5) - 3$
$= 2\ln 5 - 3$

When $y = 0$, $2\ln(x + 5) - 3 = 0$

$$\ln(x + 5) = \frac{3}{2}$$

$$x + 5 = e^{\frac{3}{2}}$$

$$\therefore \quad x = e^{\frac{3}{2}} - 5$$

Exponential and logarithmic graphs with different bases

It is often useful to know how to go from one base to another.

To change the base of $\log_a x$ from a to b (where $a, b > 0$ and $a, b \neq 1$), we use the definition that $y = \log_a x$ implies $a^y = x$. Taking $\log_b$ of both sides:

$$\log_b(a^y) = \log_b x$$

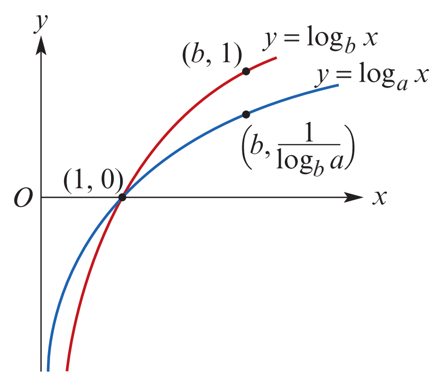
$$y \log_b a = \log_b x$$

$$y = \frac{\log_b x}{\log_b a}$$

Since $y = \log_a x$, this gives:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

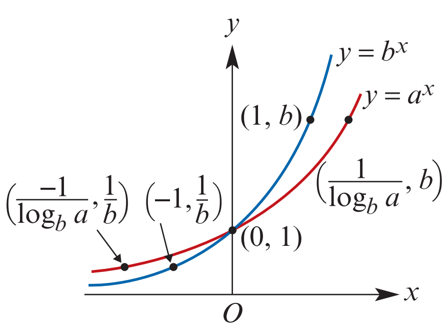
Hence the graph of $y = \log_a x$ can be obtained from the graph of $y = \log_b x$ by a dilation of factor $\dfrac{1}{\log_b a}$ parallel to the y-axis.



Using the inverse relationship between logarithms and exponentials, we can write $a = b^{\log_b a}$.
This gives:

$$a^x = b^{(\log_b a)x}$$

Hence the graph of $y = a^x$ can be obtained from the graph of $y = b^x$ by a dilation of factor $\dfrac{1}{\log_b a}$ parallel to the x-axis.



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Example 20

Find a transformation that takes the graph of $y = 2^x$ to the graph of $y = e^x$.

Solution

We can write $e = 2^{\log_2 e}$ and so

$$e^x = (2^{\log_2 e})^x$$
$$= 2^{(\log_2 e)x}$$

The graph of $y = e^x$ is the image of the graph of $y = 2^x$ under a dilation of factor $\dfrac{1}{\log_2 e}$ parallel to the x-axis.

Summary 2E

Let $a \in (0, 1) \cup (1, \infty)$.

- The function $y = \log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$.
- The graph of $y = \log_a x$ is the reflection of the graph of $y = a^x$ in the line $y = x$, and has the following properties:
  - The y-axis is an asymptote.
  - The x-axis intercept is 1.
  - The x-values are always positive.
  - There is no y-axis intercept.

Exercise 2E

Example 17

1 Sketch the graph of each of the following:

a $y = 2\ln(3x)$ **b** $y = 4\ln(5x)$ **c** $y = 2\ln(4x)$ **d** $y = 3\ln\left(\dfrac{x}{2}\right)$

Example 18

2 For each of the following functions, sketch the graph (labelling axis intercepts and asymptotes) and state the natural domain and range:

a $y = 2\ln(x - 3)$ **b** $y = \ln(x + 3) - 2$ **c** $y = 2\ln(x + 1) - 1$
d $y = 2 + \ln(3x - 2)$ **e** $y = -2\ln(x + 2)$ **f** $y = -2\ln(x - 2)$
g $y = 1 - \ln(x + 1)$ **h** $y = \ln(2 - x)$ **i** $y + 1 = \ln(4 - 3x)$

Example 19

3 Sketch the graph of each of the following. Label the axis intercepts and asymptotes. State the natural domain of each function.

a $y = \log_2(2x)$ **b** $y = \log_{10}(x - 5)$ **c** $y = -\log_{10} x$
d $y = \log_{10}(-x)$ **e** $y = \log_{10}(5 - x)$ **f** $y = 2\log_2(2x) + 2$
g $y = -2\log_2(3x)$ **h** $y = \log_{10}(-x - 5) + 2$ **i** $y = 4\log_2(-3x)$
j $y = 2\log_2(2 - x) - 6$ **k** $y = \ln(2x - 1)$ **l** $y = -\ln(3 - 2x)$

4 Solve each of the following equations using a calculator. Give answers correct to three decimal places.

a $-x + 2 = \ln x$ **b** $\dfrac{1}{3}\ln(2x + 1) = -\dfrac{1}{2}x + 1$

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5 **a** Using a calculator, plot the graph of $y = f(x)$ where $f(x) = \ln x$.

b Using the same screen, plot the graphs of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = f\left(\frac{x}{3}\right)$ **iv** $y = f(3x)$

Example 20

6 Find a transformation that takes the graph of $y = 3^x$ to the graph of $y = e^x$.

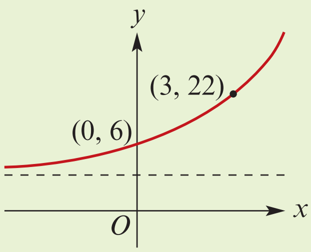
7 Find a transformation that takes the graph of $y = e^x$ to the graph of $y = 2^x$.

2F Determining rules for graphs of exponential and logarithmic functions

In this chapter, we consider rules for exponential and logarithmic functions.

Example 21

The rule for the function with the graph shown is of the form $y = ae^x + b$. Find the values of a and b.



Solution

When $x = 0$, $y = 6$ and when $x = 3$, $y = 22$:

$$6 = ae^0 + b \qquad (1)$$

$$22 = ae^3 + b \qquad (2)$$

Subtract (1) from (2):

$$16 = a(e^3 - e^0)$$

$$16 = a(e^3 - 1)$$

$$\therefore \quad a = \frac{16}{e^3 - 1}$$

From equation (1):

$$b = 6 - a$$

$$= 6 - \frac{16}{e^3 - 1}$$

$$= \frac{6e^3 - 22}{e^3 - 1}$$

The function has rule $y = \left(\frac{16}{e^3 - 1}\right)e^x + \frac{6e^3 - 22}{e^3 - 1}$.

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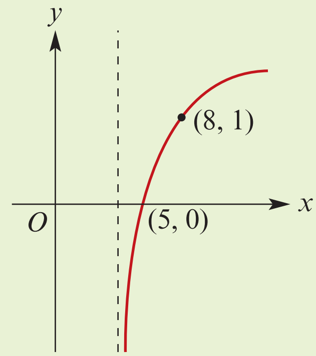
Example 22

The rule for the function with the graph shown is of the form $y = a\ln(x + b)$. Find the values of a and b.



Solution

$$0 = a\ln(5 + b) \qquad (1)$$
$$1 = a\ln(8 + b) \qquad (2)$$

From (1): $\ln(5 + b) = 0$

$$5 + b = e^0$$
$$\therefore \quad b = -4$$

Substitute in (2): $1 = a\ln 4$

$$\therefore \quad a = \frac{1}{\ln 4}$$

The rule is $y = \dfrac{1}{\ln 4}\ln(x - 4)$.

Explanation

Form two equations in a and b by substituting into the rule $y = a\ln(x + b)$:

- $y = 0$ when $x = 5$
- $y = 1$ when $x = 8$

Example 23

Given that $y = Ae^{bt}$ with $y = 6$ when $t = 1$ and $y = 8$ when $t = 2$, find A and b.

Solution

$$6 = Ae^{b} \qquad (1)$$
$$8 = Ae^{2b} \qquad (2)$$

Divide (2) by (1): $\dfrac{4}{3} = e^b$

$$\therefore \quad b = \ln\frac{4}{3}$$

Substitute in (1): $6 = Ae^{\ln\frac{4}{3}}$

$$6 = \frac{4}{3}A$$
$$\therefore \quad A = \frac{18}{4} = \frac{9}{2}$$

The rule is $y = \dfrac{9}{2}e^{\left(\ln\frac{4}{3}\right)t}$.

Explanation

Form two equations in A and b by substituting into the rule $y = Ae^{bt}$:

- $y = 6$ when $t = 1$
- $y = 8$ when $t = 2$

$e^{\ln\frac{4}{3}} = \dfrac{4}{3}$ since $e^{\ln a} = a$ for all $a > 0$

Note that $y = \dfrac{9}{2}\left(e^{\ln\frac{4}{3}}\right)^t = \dfrac{9}{2}\left(\dfrac{4}{3}\right)^t$

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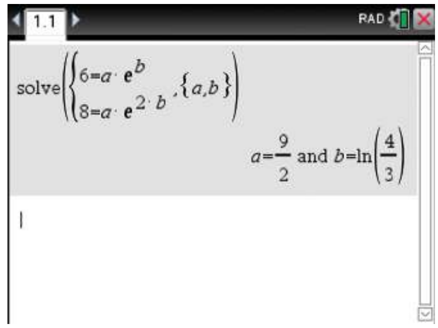
Using the TI-Nspire

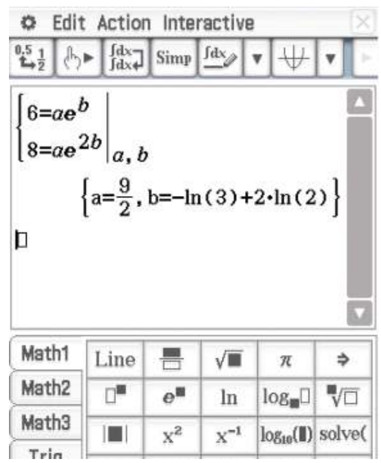
Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations** and complete as shown.

Note: Do not use the 'e' from the alpha keys; it will be treated as a variable.



Using the Casio ClassPad

- Select the simultaneous equations template.
- Enter the equations as shown: select $e^{\blacksquare}$ from the Math1 keyboard and select the parameters a, b from the Var keyboard.



Exercise 2F

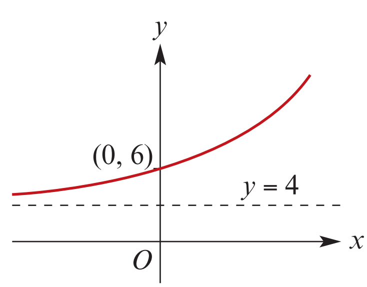
Example 21 **1** An exponential function has rule $y = a \times e^x + b$ and the points with coordinates $(0, 5)$ and $(4, 11)$ are on the graph of the function. Find the values of a and b.

Example 22 **2** A logarithmic function has rule $y = a\ln(x + b)$ and the points with coordinates $(5, 0)$ and $(10, 2)$ are on the graph of the function. Find the values of a and b.

3 The graph shown has rule

$$y = ae^x + b$$

Find the values of a and b.



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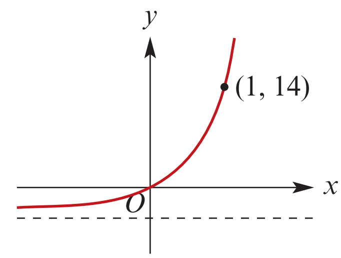
4 The rule for the function for which the graph is shown is of the form

$$y = ae^x + b$$

Find the values of a and b.



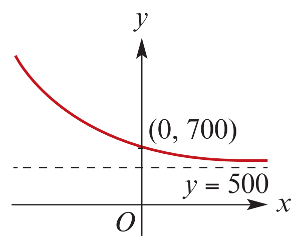
Example 23

5 Find the values of a and b such that the graph of $y = ae^{-bx}$ goes through the points $(3, 50)$ and $(6, 10)$.

6 The rule for the function f is of the form

$$f(x) = ae^{-x} + b$$

Find the values of a and b.

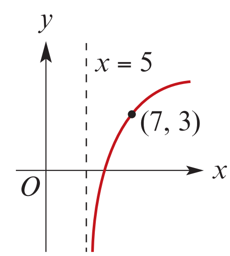


7 Find the values of a and b such that the graph of $y = a\log_2 x + b$ goes through the points $(8, 10)$ and $(32, 14)$.

8 The rule of the graph shown is of the form

$$y = a\log_2(x - b)$$

Find the values of a and b.



9 Find the values of a and b such that the graph of $y = ae^{bx}$ goes through the points $(3, 10)$ and $(6, 50)$.

10 Find the values of a and b such that the graph of $y = a\log_2(x - b)$ passes through the points $(5, 2)$ and $(7, 4)$.

11 The points $(3, 10)$ and $(5, 12)$ lie on the graph of $y = a\ln(x - b) + c$. The graph has a vertical asymptote with equation $x = 1$. Find the values of a, b and c.

12 The graph of the function with rule $f(x) = a\ln(-x) + b$ passes though the points $(-2, 6)$ and $(-4, 8)$. Find the values of a and b.

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2G Solution of equations using logarithms

Example 24

If $\log_2 6 = k\log_2 3 + 1$, find the value of k.

Solution

$$\begin{aligned}\log_2 6 &= k\log_2 3 + 1\\ &= \log_2(3^k) + \log_2 2\\ &= \log_2(2\times 3^k)\\ \therefore\quad 6 &= 2\times 3^k\\ 3 &= 3^k\\ k &= 1\end{aligned}$$

Example 25

Solve for x if $2^x = 11$, expressing the answer to two decimal places.

Solution

$$2^x = 11 \iff x = \log_2 11$$
$$= 3.45943\ldots$$

Therefore $x \approx 3.46$ correct to two decimal places.

Example 26

Solve $3^{2x-1} = 28$, expressing the answer to three decimal places.

Solution

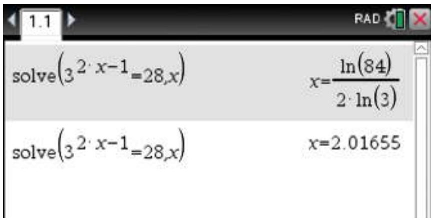
$$3^{2x-1} = 28 \iff 2x - 1 = \log_3 28$$

Thus $2x - 1 = \log_3 28$

$$\begin{aligned}2x &= \log_3 28 + 1\\ x &= \frac{1}{2}(\log_3 28 + 1)\\ &\approx 2.017 \qquad \text{correct to three decimal places}\end{aligned}$$

Using the TI-Nspire

- Use (menu) > **Algebra** > **Solve** and complete as shown.
- Convert to a decimal answer using (ctrl)(enter) or (menu) > **Actions** > **Convert to Decimal**.
- Round to three decimal places as required: $x = 2.017$.



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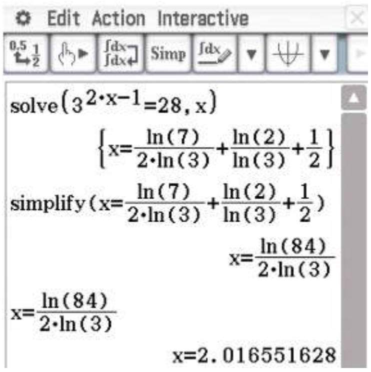
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $3^{2x-1} = 28$.
- Go to **Interactive** > **Equation/Inequality** > **solve** and tap OK.
- Copy and paste the answer into the next entry line and go to **Interactive** > **Transformation** > **simplify** to obtain a simplified exact answer.
- Highlight the answer and tap [0.5 ⇆ 1/2] to obtain the decimal approximation.



Example 27

Solve the inequality $0.7^x \geq 0.3$.

Solution

Taking $\log_{10}$ of both sides:

$$\log_{10}(0.7^x) \geq \log_{10} 0.3$$

$$x \log_{10} 0.7 \geq \log_{10} 0.3$$

$$\therefore \quad x \leq \frac{\log_{10} 0.3}{\log_{10} 0.7} \qquad \text{(direction of inequality reversed since } \log_{10} 0.7 < 0\text{)}$$

Alternatively, we can solve the inequality $0.7^x \geq 0.3$ directly as follows:

Note that $0 < 0.7 < 1$ and thus $y = 0.7^x$ is strictly decreasing. Therefore the inequality $0.7^x \geq 0.3$ holds for $x \leq \log_{0.7} 0.3$.

The inverse relationship between logarithms and exponentials can be used when solving and rearranging equations:

- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
- $a^{\log_a x} = x$ for all $x \in (0, \infty)$

Example 28

Rewrite the equation $y = 2\ln(x) + 3$ with x as the subject.

Solution

$$y = 2\ln(x) + 3$$

$$\frac{y-3}{2} = \ln x$$

$$\therefore \quad x = e^{\frac{y-3}{2}}$$

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Example 29

Rearrange the equation $x = 2\ln(y - 1) + 3$ to express y as a function of x.

Solution

$$x = 2\ln(y - 1) + 3$$

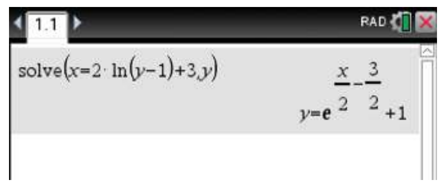
$$\frac{x-3}{2} = \ln(y - 1)$$

$$y - 1 = e^{\frac{x-3}{2}}$$

$$\therefore \quad y = e^{\frac{x-3}{2}} + 1$$

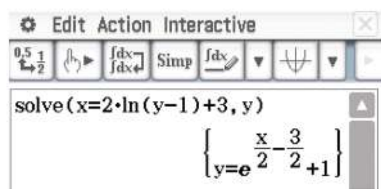
Using the TI-Nspire

Use **solve** from the **Algebra** menu as shown.



Using the Casio ClassPad

- Enter and highlight $x = 2\ln(y - 1) + 3$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to y.



Example 30

Rewrite the equation $P = Ae^{kt}$ with t as the subject.

Solution

$$P = Ae^{kt}$$

Take logarithms with base e of both sides:

$$\ln P = \ln(Ae^{kt})$$

$$= \ln A + \ln(e^{kt})$$

$$= \ln A + kt$$

$$\therefore \quad t = \frac{1}{k}(\ln P - \ln A)$$

$$= \frac{1}{k}\ln\left(\frac{P}{A}\right)$$

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Summary 2G

- If $a \in (0, 1) \cup (1, \infty)$ and $x \in \mathbb{R}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations and inequalities. For example:
  - $2^x = 5 \Leftrightarrow x = \log_2 5$
  - $2^x \geq 5 \Leftrightarrow x \geq \log_2 5$
  - $0.3^x = 5 \Leftrightarrow x = \log_{0.3} 5$
  - $0.3^x \geq 5 \Leftrightarrow x \leq \log_{0.3} 5$
- An exponential inequality may also be solved by taking $\log_a$ of both sides. For $a > 1$, the direction of the inequality stays the same (as $y = \log_a x$ is strictly increasing). For $0 < a < 1$, the direction of the inequality reverses (as $y = \log_a x$ is strictly decreasing).
- The inverse relationship between logarithms and exponentials is helpful when solving and rearranging equations:
  - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
  - $a^{\log_a x} = x$ for all $x \in (0, \infty)$

Exercise 2G

Example 24

1 **a** If $\log_2 8 = k \log_2 7 + 2$, find the value of k.
b If $\log_2 7 - x \log_2 7 = 4$, find the value of x.
c If $\ln 7 - x \ln 14 = 1$, find the value of x.

Example 25
Example 26

2 Use your calculator to solve each of the following equations, correct to two decimal places:

a $2^x = 6$ **b** $3^x = 0.7$ **c** $3^x = 11$
d $4^x = 5$ **e** $2^{-x} = 5$ **f** $0.2^x = 3$
g $5^x = 3^{x-1}$ **h** $8^x = 2005^{x+1}$ **i** $3^{x-1} = 8$
j $0.3^{x+2} = 0.7$ **k** $2^{x-1} = 3^{x+1}$ **l** $1.4^{x+2} = 25(0.9^x)$
m $5^x = 2^{2x-2}$ **n** $2^{\frac{1}{2}(x+2)} = 3^{x-1}$ **o** $2^{x+1} \times 3^{x-1} = 100$

3 Solve for x using a calculator. Express your answer correct to two decimal places.

a $2^x < 7$ **b** $3^x > 6$ **c** $0.2^x > 3$
d $3^{x-2} \leq 8$ **e** $0.2^x \leq 0.4$

4 Solve each of the following equations for x. Give exact answers.

a $2^x = 5$ **b** $3^{2x-1} = 8$ **c** $7^{3x+1} = 20$
d $3^x = 7$ **e** $3^x = 6$ **f** $5^x = 6$
g $3^{2x} - 3^{x+2} + 8 = 0$ **h** $5^{2x} - 4 \times 5^x - 5 = 0$

Example 27

5 Solve each of the following inequalities for x. Give exact answers.

a $7^x > 52$ **b** $3^{2x-1} < 40$ **c** $4^{3x+1} \geq 5$
d $3^{x-5} \leq 30$ **e** $3^x < 106$ **f** $5^x < 0.6$

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Example 28 **6** Rewrite the equation $y = 3\ln(x) - 4$ with x as the subject.

Example 29 **7** For each of the following functions $y = f(x)$, rearrange the rule to show that x can be expressed as a function of y:

a $y = \ln(2x)$ **b** $y = 3\ln(2x) + 1$
c $y = e^x + 2$ **d** $y = e^{x+2}$
e $y = \ln(2x + 1)$ **f** $y = 4\ln(3x + 2)$
g $y = \log_{10}(x + 1)$ **h** $y = 2e^{x-1}$

8 **a** Using a calculator, for each of the following plot the graphs of $y = f(x)$ and $y = g(x)$, together with the line $y = x$, on the one set of axes:

i $f(x) = \ln x$ and $g(x) = e^x$
ii $f(x) = 2\ln(x) + 3$ and $g(x) = e^{\frac{x-3}{2}}$
iii $f(x) = \log_{10} x$ and $g(x) = 10^x$

b Use your answers to part **a** to comment on the relationship in general between the graphs of $f(x) = a\log_b(x) + c$ and $g(x) = b^{\frac{x-c}{a}}$.

Example 30 **9** Rewrite the equation $P = Ae^{-kt} + b$ with t as the subject.

10 For each of the following formulas, make the pronumeral in brackets the subject:

a $y = 2\ln(x) + 5$ (x) **b** $P = Ae^{-6x}$ (x)
c $y = ax^n$ (n) **d** $y = 5 \times 10^x$ (x)
e $y = 5 - 3\ln(2x)$ (x) **f** $y = 6x^{2n}$ (n)
g $y = \ln(2x - 1)$ (x) **h** $y = 5(1 - e^{-x})$ (x)

11 **a** If $a\log_2 7 = 3 - \log_6 14$, find the value of a, correct to three significant figures.
b If $\log_3 18 = \log_{11} k$, find the value of k, correct to one decimal place.

12 Prove that if $\log_r p = q$ and $\log_q r = p$, then $\log_q p = pq$.

13 If $u = \log_9 x$, find in terms of u:

a x
b $\log_9(3x)$
c $\log_x 81$

14 Solve the equation $\log_5 x = 16\log_x 5$.

15 Given that $q^p = 25$, find $\log_5 q$ in terms of p.

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2H Exponential growth and decay

If the rate at which a quantity increases or decreases is proportional to its current value, then the quantity obeys the **law of exponential change**.

Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is the **rate constant**.

If $k > 0$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $k < 0$, the model represents **decay**:

- radioactive decay
- cooling of materials

An equivalent way to write this model is as $A = A_0 b^t$, where we take $b = e^k$. In this form, growth corresponds to $b > 1$ and decay corresponds to $b < 1$.

Cell growth

Suppose a particular type of bacteria cell divides into two new cells every T_D minutes. Let N_0 be the initial number of cells of this type. After t minutes the number of cells, N, is given by

$$N = N_0 2^{\frac{t}{T_D}}$$

where T_D is called the **generation time**.

Example 31

What is the generation time of a bacterial population that increases from 5000 cells to 100 000 cells in four hours of growth?

Solution

In this example, $N_0 = 5000$ and $N = 100\,000$ when $t = 240$.

Hence $100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$$20 = 2^{\frac{240}{T_D}}$$

Thus $T_D = \dfrac{240}{\log_2 20} \approx 55.53$ (correct to two decimal places).

The generation time is approximately 55.53 minutes.

Radioactive decay

Radioactive materials decay such that the amount of radioactive material, A, present at time t (in years) is given by

$$A = A_0 e^{-kt}$$

where A_0 is the initial amount and k is a positive constant that depends on the type of material. A radioactive substance is often described in terms of its **half-life**, which is the time required for half the material to decay.

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Example 32

After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Find the half-life of radium-226.

Solution

We use the formula $A = A_0 e^{-kt}$. When $t = 1000$, $A = 0.647A_0$. Thus

$$0.647A_0 = A_0 e^{-1000k}$$
$$0.647 = e^{-1000k}$$
$$-1000k = \ln 0.647$$
$$k = \frac{-\ln 0.647}{1000} \approx 0.000\,435$$

To find the half-life, we consider when $A = \frac{1}{2}A_0$:

$$A_0 e^{-kt} = \frac{1}{2}A_0$$
$$e^{-kt} = \frac{1}{2}$$
$$-kt = \ln\left(\frac{1}{2}\right)$$
$$t = -\frac{\ln(\frac{1}{2})}{k} \approx 1591.95$$

The half-life of radium-226 is approximately 1592 years.

Population growth

It is sometimes possible to model population growth through exponential models.

Example 33

The population of a town was 8000 at the beginning of 2007 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2016.

b In what year will the population be double that of 2014?

Solution

Let P be the population at time t years (measured from 1 January 2007). Then

$$P = 8000e^{kt}$$

At the end of 2014, $t = 8$ and $P = 15\,000$. Therefore

$$15\,000 = 8000e^{8k}$$
$$\frac{15}{8} = e^{8k}$$
$$k = \frac{1}{8}\ln\left(\frac{15}{8}\right) \approx 0.079$$

The rate of increase is 7.9% per annum.

Note: The approximation 0.079 was not used in the calculations which follow. The value for k was held in the calculator.

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a When $t = 10$, $P = 8000e^{10k}$

$\approx 17\,552.6049$

$\approx 17\,550$

The population is approximately 17 550.

b When does $P = 30\,000$? Consider the equation

$$30\,000 = 8000e^{kt}$$

$$\frac{30\,000}{8000} = e^{kt}$$

$$\frac{15}{4} = e^{kt}$$

$$\therefore \quad t = \frac{1}{k}\ln\left(\frac{15}{4}\right)$$

$$\approx 16.82$$

The population reaches 30 000 approximately 16.82 years after the beginning of 2007, i.e. during the year 2023.

Example 34

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let G_0 be the population of grey kangaroos at the start.

Then the number of grey kangaroos after n years is $G = G_0(1.11)^n$, and the number of red kangaroos after n years is $R = 10G_0(0.95)^n$.

When the proportions are reversed:

$$G = 10R$$

$$G_0(1.11)^n = 10 \times 10G_0(0.95)^n$$

$$(1.11)^n = 100(0.95)^n$$

Taking logarithms base e of both sides:

$$\ln((1.11)^n) = \ln(100(0.95)^n)$$

$$n\ln 1.11 = \ln 100 + n\ln 0.95$$

$$\therefore \quad n = \frac{\ln 100}{\ln 1.11 - \ln 0.95}$$

$$\approx 29.6$$

i.e. the proportions of the kangaroo populations will be reversed after 30 years.

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Summary 2H

There are many situations in which a varying quantity can be modelled by an exponential function. Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant. Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

Skill-sheet

Exercise 2H

Example 31

1 A population of 1000 E. coli bacteria doubles every 15 minutes.

a Determine the formula for the number of bacteria at time t minutes.

b How long will it take for the population to reach 10 000? (Give your answer to the nearest minute.)

2 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$ where d is the diameter (in cm) of the tree t years after the beginning of this period. The diameter is 52 cm after 1 year, and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

3 The number of people, N, who have a particular disease at time t years is given by $N = N_0 e^{kt}$.

a If the number is initially 20 000 and the number decreases by 20% each year, find:

i the value of N_0 **ii** the value of k.

b How long does it take until only 5000 people are infected?

Example 32

4 Polonium-210 is a radioactive substance. The decay of polonium-210 is described by the formula $M = M_0 e^{-kt}$, where M is the mass in grams of polonium-210 left after t days, and M_0 and k are constants. At $t = 0$, $M = 10$ g and at $t = 140$, $M = 5$ g.

a Find the values of M_0 and k.

b What will be the mass of the polonium-210 after 70 days?

c After how many days is the mass remaining 2 g?

5 A quantity A of radium at time t years is given by $A = A_0 e^{-kt}$, where k is a positive constant and A_0 is the amount of radium at time $t = 0$.

a Given that $A = \frac{1}{2}A_0$ when $t = 1690$ years, calculate k.

b After how many years does only 20% of the original amount remain? Give your answer to the nearest year.

6 The half-life of plutonium-239 is 24 000 years. If 20 grams are present now, how long will it take until only 20% of the original sample remains? (Give your answer to the nearest year.)

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7 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of ancient objects. A Babylonian cloth fragment now has 40% of the carbon-14 that it contained originally. How old is the fragment of cloth?

Example 33

8 The population of a town was 10 000 at the beginning of 2002 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2017.

b In what year will the population be double that of 2014?

Example 34

9 There are approximately five times as many magpies as currawongs in a certain area. If the population of currawongs increases at a rate of 12% per annum while that of the magpies decreases at 6% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

10 The pressure in the Earth's atmosphere decreases exponentially as you rise above the surface. The pressure in millibars at a height of h kilometres is given approximately by the function $P(h) = 1000 \times 10^{-0.05428h}$.

a Find the pressure at a height of 4 km. (Give your answer to the nearest millibar.)

b Find the height at which the pressure is 450 millibars. (Give your answer to the nearest metre.)

11 A biological culture contains 500 000 bacteria at 12 p.m. on Sunday. The culture increases by 10% every hour. At what time will the culture exceed 4 million bacteria?

12 When a liquid is placed into a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 e^{-kt}$. The temperature is initially 100°C and drops to 40°C in 5 minutes. Find the temperature of the liquid after 15 minutes.

13 The number of bacteria in a certain culture at time t weeks is given by the rule $N = N_0 e^{kt}$. If when $t = 2$, $N = 101$ and when $t = 4$, $N = 203$, calculate the values of N_0 and k.

14 Five kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount, S kg, of undissolved sugar remaining is given by $S = 5 \times e^{-kt}$.

a Calculate k given that $S = 3.2$ when $t = 2$.

b At what time will there be 1 kg of sugar remaining?

15 The number of bacteria, N, in a culture increases exponentially with time according to the rule $N = a \times b^t$, where time t is measured in hours. When observation started, there were 1000 bacteria, and 5 hours later there were 15 000 bacteria.

a Find the values of a and b.

b Find, to the nearest hour, when there were 5000 bacteria.

c Find, to the nearest hour, when the number of bacteria first exceeds 1 000 000.

d How many bacteria would there be 12 hours after the first observation?

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2I Logarithmic scales

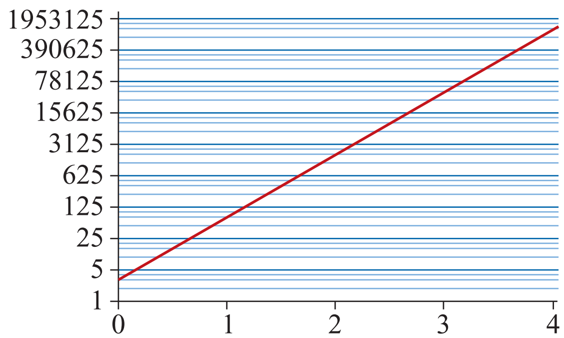
A **logarithmic scale** is a scale of measurement that uses the logarithm of a quantity. Familiar examples of logarithmic scales include the Richter scale (earthquakes), decibels (noise) and pH (acidity). In this section, we will show why such scales are useful and study these three examples.

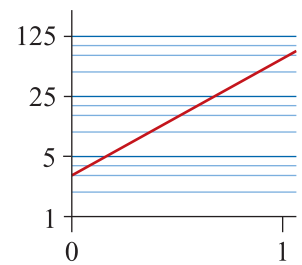
Charts with logarithmic scales

In the chart below we show an example for which the vertical axis has equally spaced increments that are labelled $1 = 5^0, 5 = 5^1, 25 = 5^2, 125 = 5^3, \dots$ instead of $0, 1, 2, 3, \dots$.

The major horizontal grid lines, which are equally spaced, are the integer powers of 5.

The straight line is the result of graphing $y = 3 \times 5^{2x}$ but with the vertical axis having a logarithmic scale (base 5).





The bottom-left corner of the chart is shown ‘blown up’ on the right.

- The horizontal grid lines between 1 and 5 represent 2, 3 and 4. Their positions are determined by $\log_5 2$, $\log_5 3$ and $\log_5 4$.
- The horizontal grid lines between 5 and 25 represent 10, 15 and 20. Their positions are determined by $\log_5 10$, $\log_5 15$ and $\log_5 20$.
- The horizontal grid lines between 25 and 125 represent 50, 75 and 100. Their positions are determined by $\log_5 50$, $\log_5 75$ and $\log_5 100$.

Notice that there is the same gap between the grid lines for 1 and 2, for 5 and 10, and for 25 and 50. This is because

$$\log_5 2 - \log_5 1 = \log_5 2 \qquad \log_5 10 - \log_5 5 = \log_5 2 \qquad \log_5 50 - \log_5 25 = \log_5 2$$

This is not what we are used to from working with linear scales.

- For a **linear scale**, the change between two values is determined by the difference between the values. That is, a change from 1 to 2 is the same as a change from 6 to 7. We use linear scales to measure temperature (degrees Celsius) and length (metres) and in standard Cartesian graphs.
- For a **logarithmic scale**, the change between two values is determined by the ratio of the values. That is, a change from 1 to 2 (ratio of 1:2) would be perceived as the same amount of increase as a change from 6 to 12 (also a ratio of 1:2).

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The logarithm rule

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

provides the connections between these types of scale.

Presentation of data on a logarithmic scale can be helpful when the data covers a large range of values. The use of the logarithms of the values rather than the actual values reduces a wide range to a more manageable size.

Graphing with logarithmic scales

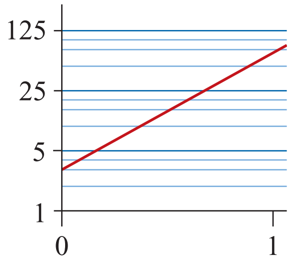
We will see how to sketch the graph of an exponential function using a logarithmic scale for the vertical axis and a linear scale for the horizontal axis.

Let $y = b \times a^{mx}$, where a, b and m are positive real numbers. Then taking logarithms base a of both sides gives

$$\log_a y = \log_a b + mx$$

So the graph of $\log_a y$ against x is a straight line with gradient m and with $(\log_a y)$-axis intercept at $(0, \log_a b)$.

In the case of the exponential function $y = 3 \times 5^{2x}$, the equation for the straight line becomes $\log_5 y = \log_5 3 + 2x$. On the chart, we have labelled the vertical axis with powers of 5 rather than the logarithm.



This straight line has gradient 2. The line through points $(0.5, 15)$ and $(1, 75)$ on this chart has actual gradient

$$\frac{\log_5 75 - \log_5 15}{1 - 0.5} = 2\log_5\left(\frac{75}{15}\right) = 2\log_5 5 = 2$$

Of course any two points on the line will give this result.

Example 35

Let $y = 4 \times 3^{2x}$, for $x \geq 0$.

a If $\log_3 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_3 y$ against x.

c Sketch the graph of $\log_3 y$ against x labelling your vertical axis with powers of 3.

Solution

a Take logarithms base 3 of both sides of the equation:

$$\log_3 y = \log_3(4 \times 3^{2x})$$
$$\log_3 y = \log_3 4 + \log_3(3^{2x})$$
$$\log_3 y = 2x + \log_3 4$$

Therefore $m = 2$ and $c = \log_3 4$.

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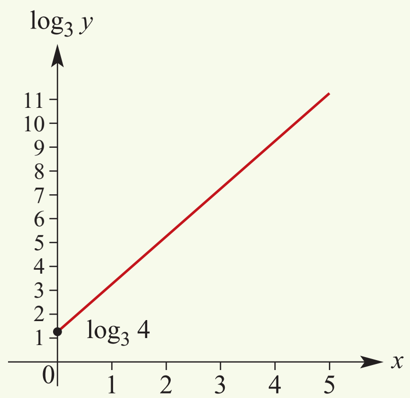
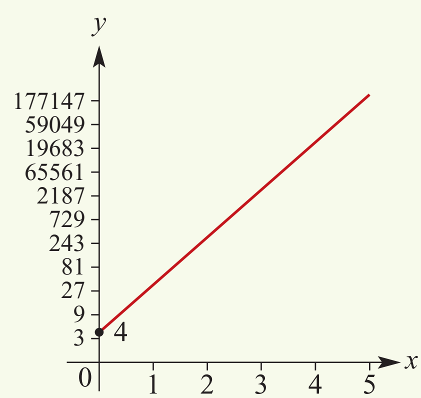
b



c



Note: For part **c**, using the logarithmic scale on the vertical axis means that the axis intercept is at $3^{\log_3 4} = 4$.

Applications

We now look at three uses of logarithmic scales.

Decibels

A **decibel** is defined as one-tenth of a bel, which is named after Alexander Graham Bell, the inventor of the telephone.

The decibel is a logarithmic scale for measuring 'loudness of noise'. The intensity of a sound in decibels can be defined by

$$dB = 10\log_{10}(P \times 10^{16})$$

where P is the power of the sound in watt/cm^2.

Example 36

A power mower generates a noise of 96 dB and a conversation in a restaurant generates noise of 60 dB. Find the power in watt/cm^2 for each of these.

Solution

- **Power mower**

  We have $96 = 10\log_{10}(P \times 10^{16})$.
  Hence

  $$\begin{aligned}\log_{10}(P \times 10^{16}) &= 9.6\\ P \times 10^{16} &= 10^{9.6}\\ P &= 10^{-6.4}\end{aligned}$$

  The power is $10^{-6.4}$ watt/cm^2.

- **Conversation**

  We have $60 = 10\log_{10}(P \times 10^{16})$.
  Hence

  $$\begin{aligned}\log_{10}(P \times 10^{16}) &= 6\\ P \times 10^{16} &= 10^{6}\\ P &= 10^{-10}\end{aligned}$$

  The power is 10^{-10} watt/cm^2.

Note: The maximum intensity which the ear can tolerate is about 10^{-4} watt/cm^2, which corresponds to a loudness level of about 120 dB.

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The Richter scale

Earthquake intensity is often reported on the **Richter scale**. The formula is

$$R = \log_{10}\left(\frac{a}{T}\right) + B$$

where a is the amplitude of the ground motion, T is the period of the seismic wave, and B is a term that allows for the weakening of the seismic wave with increasing distance from the epicentre of the earthquake.

Example 37

Assume that, for a particular earthquake, we have $a = 10$, $T = 1$ and $B = 6.8$. Find the earthquake's magnitude on the Richter scale.

Solution

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 7.8$$

Example 38

Early in the twentieth century an earthquake in San Francisco registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in South America that was four times stronger. We will take this to mean that the value of $\frac{a}{T}$ for South America is four times that for San Francisco. What was the magnitude of the earthquake in South America? Assume both were measured at the same distance from the epicentre, and so the constant B is the same for both.

Solution

For San Francisco:

$$8.3 = \log_{10}\left(\frac{a_1}{T_1}\right) + B \qquad (1)$$

Hence, for South America:

$$\begin{aligned} R &= \log_{10}\left(\frac{a_2}{T_2}\right) + B \\ &= \log_{10}\left(\frac{4a_1}{T_1}\right) + B \\ &= \log_{10} 4 + \log_{10}\left(\frac{a_1}{T_1}\right) + B \\ &= \log_{10} 4 + 8.3 && \text{using equation (1)} \\ &\approx 8.9 \end{aligned}$$

The magnitude was 8.9.

Note: Although the earthquake in South America was four times stronger, the magnitude on the Richter scale only increased by $\log_{10} 4 \approx 0.6$.

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The pH scale

The pH scale for measuring the acidity of a solution is logarithmic. The pH of a solution is determined by the concentration of hydronium ions, $[H_3O^+]$, in the solution. (Concentration is measured in moles per litre.) The definition is

$$pH = \log_{10}\left(\frac{1}{[H_3O^+]}\right) = -\log_{10}([H_3O^+])$$

Vinegar has pH 3, and bananas have pH in the interval $[4.5, 4.7]$. The scale ranges from approximately 0 for concentrated hydrochloric acid to 14 for concentrated sodium hydroxide, but values below 0 or above 14 are possible for extremely concentrated solutions.

Example 39

The pH of blood normally lies in the interval $[7.37, 7.44]$. Find the range for the concentration of hydronium ions.

Solution

For pH 7.37, we have

$$\begin{aligned} -\log_{10}([H_3O^+]) &= 7.37 \\ \log_{10}([H_3O^+]) &= -7.37 \\ [H_3O^+] &= 10^{-7.37} \text{ moles per litre} \end{aligned}$$

Now, for pH 7.44, we have

$$\begin{aligned} \log_{10}([H_3O^+]) &= -7.44 \\ [H_3O^+] &= 10^{-7.44} \text{ moles per litre} \end{aligned}$$

So the concentration of hydronium ions lies in the interval $[10^{-7.44}, 10^{-7.37}]$.

We can write this interval as $[3.63 \times 10^{-8}, 4.27 \times 10^{-8}]$ to three significant figures.

Summary 2I

- **Linear scale** The change between two values is determined by the difference between the values. That is, a change from 1 to 2 is the same as a change from 6 to 7. We use linear scales for temperature and length.
- **Logarithmic scale** The change between two values is determined by the ratio of the values. That is, a change from 1 to 2 (ratio of 1:2) is the same as a change from 6 to 12 (also a ratio of 1:2). We use logarithmic scales for noise and acidity.
- Consider the exponential function $y = b \times a^{mx}$, where a, b and m are positive real numbers. Taking logarithms base a of both sides gives

  $$\log_a y = \log_a b + mx$$

  So the graph of $\log_a y$ against x is a straight line with gradient m and with $(\log_a y)$-axis intercept at $(0, \log_a b)$.

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Exercise 2I

Example 35

1 Let $y = 3 \times 4^{2x}$, for $x \geq 0$.

a If $\log_4 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_4 y$ against x.

c Sketch the graph of y against x labelling your vertical axis with powers of 4.

2 Let $y = 2 \times 5^{3x}$, for $x \geq 0$.

a If $\log_5 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_5 y$ against x.

c Sketch the graph of y against x labelling your vertical axis with powers of 5.

Example 36

3 A busy street generates noise of 70 dB and a quiet car generates noise of 50 dB. Use the formula $dB = 10 \log_{10}(P \times 10^{16})$, where P is the power in watt/cm^2, to find the power in watt/cm^2 for each of these.

4 Use the formula $dB = 10 \log_{10}(P \times 10^{16})$ to answer the following:

a If P is increased by a factor of 2, what is the effect on dB?

b If P is increased by a factor of 10, what is the effect on dB?

c If dB is increased by a factor of 3, what is the effect on P?

d For what value of P is $dB = 0$?

e For what value of P is $dB = 100$?

5 If $dB_1 - dB_2 = \lambda$, find P_1 in terms of P_2.

Example 37

6 Find the magnitude on the Richter scale of an earthquake with $a = 10$, $T = 2$ and $B = 5$.

Example 38

7 An earthquake in Turkey registered 7.3 on the Richter scale. In the same year, an earthquake in Greece had a quarter of this strength. We will take this to mean that the value of $\frac{a}{T}$ for Greece is one-quarter that for Turkey. What was the magnitude of the earthquake in Greece? Assume both were measured at the same distance from the epicentre, and so the constant B is the same for both.

Example 39

8 The pH of a soft drink normally lies in the interval [2.0, 4.0]. Find the range for the concentration of hydronium ions.

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Chapter summary

Assign-
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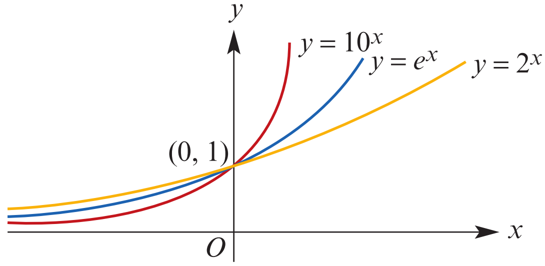
- Sketch graphs of the form $y = a^x$ and transformations of these graphs.



- **Index laws**

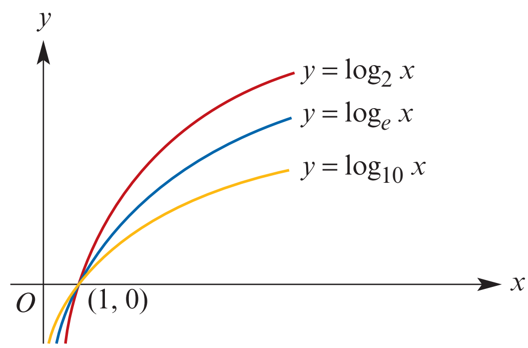
$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

- **Logarithms**

For $a \in (0, 1) \cup (1, \infty)$, the logarithm function with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

- Sketch graphs of the form $y = \log_a x$ and transformations of these graphs.



- **The natural logarithm**

The function $y = \log_e x$ is commonly written as $y = \ln x$.

- **Logarithm laws**

$$\log_a(mn) = \log_a m + \log_a n \qquad \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a\left(\frac{1}{n}\right) = -\log_a n \qquad \log_a(m^p) = p\log_a m$$

- **Change of base**

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{and} \quad a^x = b^{(\log_b a)x}$$

- **Inverse relationship** The functions $f(x) = a^x$ and $g(x) = \log_a x$, $\{ x \in \mathbb{R} : x > 0 \}$ are inverse to each other in the following way:
  - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
  - $a^{\log_a x} = x$ for all $x \in (0, \infty)$

- **Law of exponential change** If the rate at which the quantity A increases or decreases is proportional to its current value, then the value of A at time t is given by

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant. Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

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Review

Short-answer questions

1 Sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $f(x) = e^x - 2$ **b** $g(x) = 10^{-x} + 1$ **c** $h(x) = \frac{1}{2}(e^x - 1)$

d $f(x) = 2 - e^{-x}$ **e** $f(x) = \ln(2x + 1)$ **f** $h(x) = \ln(x - 1) + 1$

g $g(x) = -\ln(x - 1)$ **h** $f(x) = -\ln(1 - x)$

2 For each of the following, find y in terms of x:

a $\ln y = \ln x + 2$ **b** $\log_{10} y = \log_{10} x + 1$ **c** $\log_2 y = 3\log_2 x + 4$

d $\log_{10} y = -1 + 5\log_{10} x$ **e** $\ln y = 3 - \ln x$ **f** $\ln y = 2x - 3$

3 Solve each of the following equations for x, expressing your answers in terms of logarithms with base e:

a $3^x = 11$ **b** $2^x = 0.8$ **c** $2^x = 3^{x+1}$

4 Solve each of the following for x:

a $2^{2x} - 2^x - 2 = 0$ **b** $\ln(3x - 1) = 0$

c $\log_{10}(2x) + 1 = 0$ **d** $10^{2x} - 7 \times 10^x + 12 = 0$

5 The graph of the function with rule $y = 3\log_2(x + 1) + 2$ intersects the axes at the points $(a, 0)$ and $(0, b)$. Find the exact values of a and b.

6 The graph of $y = 5\log_{10}(x + 1)$ passes through the point $(k, 6)$. Find the value of k.

7 Find the exact value of x for which $4e^{3x} = 287$.

8 Find the value of x in terms of a, where $3\log_a x = 3 + \log_a 8$.

9 Given that $y = \log_3(x - 4)$, express x as a function of y.

10 The graph of the function with rule $f(x) = e^{2x} - 3ke^x + 5$ intersects the axes at $(0, 0)$ and $(a, 0)$ and has a horizontal asymptote at $y = b$. Find the exact values of a, b and k.

11 Given that $y = e^{3x} - 4$, express x as a function of y.

12 Show that, if $3^x = 4^y = 12^z$, then $z = \dfrac{xy}{x + y}$.

13 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

14 **a** Given that $\log_p 7 + \log_p k = 0$, find k.

b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, find q.

15 Let $f(x) = e^x + e^{-x}$ and $g(x) = e^x - e^{-x}$.

a Show that f is an even function. **b** Find $f(u) + f(-u)$.

c Find $f(u) - f(-u)$. **d** Find $[f(u)]^2 - 2$.

e Show that g is an odd function.

f Find $f(x) + g(x)$, $f(x) - g(x)$ and $f(x) \cdot g(x)$.

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Extended-response questions

1 A liquid cools from its original temperature of 90°C to a temperature of T°C in x minutes. Given that $T = 90(0.98)^x$, find:

a the value of T when $x = 10$

b the value of x when $T = 27$.

2 The population of a village at the beginning of the year 1800 was 240. The population increased so that, after a period of n years, the new population was $240(1.06)^n$. Find:

a the population at the beginning of 1820

b the year in which the population first reached 2500.

3 The value, $\$V$, of a particular car can be modelled by the equation $V = ke^{-\lambda t}$, where t years is the age of the car. The car's original price was \$22 497, and after 1 year it is valued at \$18 000.

a State the value of k and calculate λ, giving your answer to two decimal places.

b Find the value of the car when it is 3 years old.

4 The value, $\$M$, of a particular house during the period 1988 to 1994 can be modelled by the equation $M = Ae^{-pt}$, where t is the time in years after 1 January 1988. The value of the house on 1 January 1988 was \$65 000 and its value on 1 January 1989 was \$61 000.

a State the value of A and calculate the value of p, correct to two significant figures.

b What was the value of the house in 1993? Give your answer to the nearest \$100.

5 There are two species of insects living in a suburb: the *Asla bibla* and the *Cutus pius*. The number of *Asla bibla* alive at time t days after 1 January 2000 is given by

$$N_A(t) = 10\,000 + 1000t, \quad 0 \le t \le 15$$

The number of *Cutus pius* alive at time t days after 1 January 2000 is given by

$$N_C(t) = 8000 + 3 \times 2^t, \quad 0 \le t \le 15$$

a With a calculator, plot the graphs of $y = N_A(t)$ and $y = N_C(t)$ on the one screen.

b **i** Find the coordinates of the point of intersection of the two graphs.

ii At what time is $N_A(t) = N_C(t)$?

iii What is the number of each species of insect at this time?

c **i** Show that $N_A(t) = N_C(t)$ if and only if $t = 3\log_2 10 + \log_2\left(\frac{2+t}{3}\right)$.

ii Plot the graphs of $y = x$ and $y = 3\log_2 10 + \log_2\left(\frac{2+x}{3}\right)$ and find the coordinates of the point of intersection.

d It is found by observation that the model for *Cutus pius* does not quite work. It is known that the model for the population of *Asla bibla* is satisfactory. The form of the model for *Cutus pius* is $N_C(t) = 8000 + c \times 2^t$. Find the value of c, correct to two decimal places, if it is known that $N_A(15) = N_C(15)$.

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6 The number of a type of bacteria is modelled by the formula $n = A(1 - e^{-Bt})$, where n is the size of the population at time t hours, and A and B are positive constants.

a When $t = 2$, $n = 10\,000$ and when $t = 4$, $n = 15\,000$.

i Show that $2e^{-4B} - 3e^{-2B} + 1 = 0$.

ii Use the substitution $a = e^{-2B}$ to show that $2a^2 - 3a + 1 = 0$.

iii Solve this equation for a.

iv Find the exact value of B.

v Find the exact value of A.

b Sketch the graph of n against t.

c After how many hours is the population of bacteria 18 000?

7 The barometric pressure P (in centimetres of mercury) at a height h km above sea level is given by $P = 75(10^{-0.15h})$. Find:

a P when $h = 0$ **b** P when $h = 10$ **c** h when $P = 60$.

8 A radioactive substance is decaying such that the amount, A g, at time t years is given by the formula $A = A_0e^{kt}$. If when $t = 1$, $A = 60.7$ and when $t = 6$, $A = 5$, find the values of the constants A_0 and k.

9 In a chemical reaction the amount, x g, of a substance that has reacted is given by $x = 8(1 - e^{-0.2t})$, where t is the time in minutes from the beginning of the reaction.

a Sketch the graph of x against t.

b Find the amount of substance that has reacted after:

i 0 minutes **ii** 2 minutes **iii** 10 minutes.

c Find the time when exactly 7 g of the substance has reacted.

10 Newton's law of cooling for an object in a medium of constant temperature states

$$T - T_s = (T_0 - T_s)\, e^{-kt}$$

where:

- T is the temperature (in °C) of the object at time t (in minutes)
- T_s is the temperature of the surrounding medium
- T_0 is the initial temperature of the object.

An egg at 96°C is placed to cool in a sink of water at 15°C. After 5 minutes the egg's temperature is 40°C. (Assume that the temperature of the water does not change.)

a Find the value of k.

b Find the temperature of the egg when $t = 10$.

c How long does it take for the egg to reach a temperature of 30°C?

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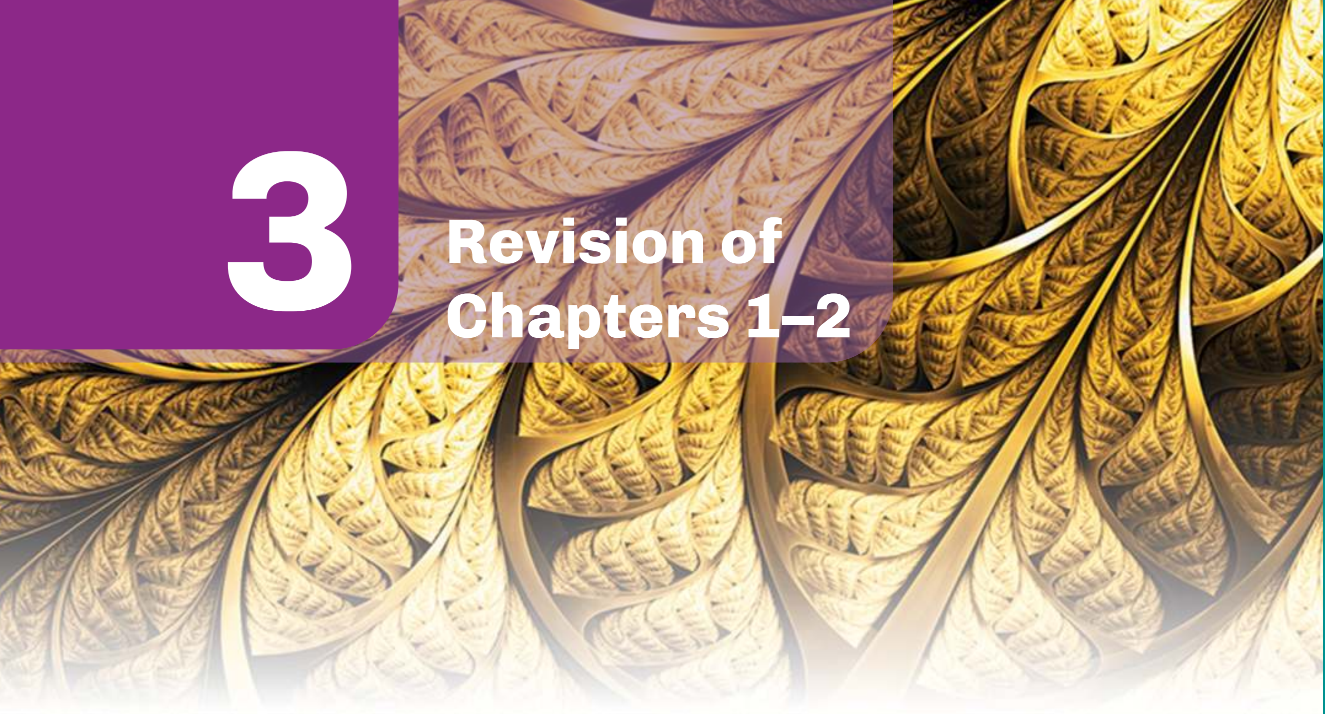

3 Revision of Chapters 1–2

3A Short-answer questions

1 State the natural domain and range of each of the following:

a $f(x) = \dfrac{1}{x} + 2$

b $f(x) = 3 - 2\sqrt{3x - 2}$

c $f(x) = \dfrac{4}{(x-2)^2} + 3$

d $h(x) = 4 - \dfrac{3}{x - 2}$

e $f(x) = \sqrt{x - 2} - 5$

2 Simplify $2\log_{10} 5 + 3\log_{10} 2 - \log_{10} 20$.

3 Find x in terms of a if $3\log_a x = 3 + \log_a 12$.

4 Solve $2 \times 2^{-x} = 1024$.

5 Solve the equation $4e^{2x} = 9$ for x.

6 **a** The graph of the function f with rule $f(x) = 2\ln(x + 2)$ intersects the axes at the points $(a, 0)$ and $(0, b)$. Find the exact values of a and b.

b Hence sketch the graph of $y = f(x)$.

7 Solve the equation $2^{4x} - 5 \times 2^{2x} + 4 = 0$ for x.

8 A function has rule $y = Ae^{kt}$. Given that $y = 4$ when $t = 1$ and that $y = 10$ when $t = 2$, find the values of A and k.

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3B Extended-response questions

1 The population of a country is found to be growing *continuously*, resulting in an increase of 2.96% every year after 1 January 1950. The population t years after 1 January 1950 is given by the formula

$$p(t) = (150 \times 10^6)e^{kt}$$

a Find the value of k.

b Find the population on 1 January 1950.

c Find the population on 1 January 2000.

d After how many years would the population be 300×10^6?

2 A large urn was filled with water. It was turned on, and the water was heated until its temperature reached 95°C. This occurred at exactly 2 p.m., at which time the urn was turned off and the water began to cool. The temperature of the room where the urn was located remained constant at 15°C.

Commencing at 2 p.m. and finishing at midnight, Jenny measured the temperature of the water every hour on the hour for the next 10 hours and recorded the results.
At 4 p.m., Jenny recorded the temperature of the water as 55°C. She found that the temperature, T°C, of the water could be described by the equation

$$T = Ae^{-kt} + 15, \quad \text{for } 0 \le t \le 10$$

where t is the number of hours after 2 p.m.

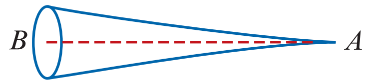
a Find the values of A and k.

b Find the temperature of the water at midnight.

c At what time did Jenny first record a temperature less than 24°C?

d Sketch the graph of T against t.

3 The diagram shows a conical glass fibre. The circular cross-sectional area at end B is 0.02 mm^2.



The cross-sectional area diminishes by a factor of $(0.92)^{\frac{1}{10}}$ per metre length of the fibre. The total length is 5 m.

a Write down a rule for the cross-sectional area of the fibre at a distance x m from B.

b What is the cross-sectional area of the fibre at a point one-third of its length from B?

c The fibre is constructed such that the strength increases in the direction B to A. At a distance of x m from B, the strength is given by the rule $S = (0.92)^{10-3x}$. If the load the fibre will take at each point before breaking is given by

$$\text{load} = \text{strength} \times \text{cross-sectional area}$$

write down an expression, in terms of x, for the load the fibre will stand at a distance of x m from B.

d A piece of glass fibre that will have to carry loads of up to $0.02 \times (0.92)^{2.5}$ units is needed. How much of the 5 m fibre could be used with confidence for this purpose?

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4 On an overnight interstate train, an electrical fault affected the illumination in two carriages, A and B. Before the fault occurred, the illumination in carriage A was I units and that in carriage B was $0.66\,I$ units. Every time the train stopped, the illumination in carriage A reduced by 17% and that in carriage B by 11%.

- **a** Write down exponential expressions for the expected illumination in each carriage after the train had stopped for the nth time.
- **b** At some time after the fault occurred, the illumination in both carriages was approximately the same. At how many stations did the train stop before this occurred?

5 A population of insects is determined by a rule of the form

$$n = \frac{c}{1 + ae^{-bt}}, \quad t \geq 0$$

where n is the number of insects alive at time t days.

- **a** Consider the population for $c = 5790$, $a = 4$ and $b = 0.03$.
  - **i** Find the equation of the horizontal asymptote by considering values of n as t becomes large.
  - **ii** Find n when $t = 0$.
  - **iii** Sketch the graph of the function.
  - **iv** Find the exact value of t for which $n = 4000$.
- **b**
  - **i** Use your calculator to find values of a, b and c such that the population growth yields the table on the right.
  - **ii** Sketch the graph for this population.

t	1	10	100
n	1500	2000	5000

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4 Differentiation

In this chapter

Syllabus references

Topics: Exponential functions; Trigonometric functions; Differentiation rules; Calculus of the natural logarithmic function

Subtopics: 3.1.3 – 3.1.9, 4.1.11, 4.1.13, 4.1.14

It is believed that calculus was discovered independently in the late seventeenth century by two great mathematicians: Isaac Newton and Gottfried Leibniz. Like most scientific breakthroughs, the discovery of calculus did not arise out of a vacuum. In fact, many mathematicians and philosophers going back to ancient times made discoveries relating to calculus.

In this chapter, we review some of the important ideas and results that have been introduced in earlier studies of calculus. We introduce the chain rule, the product rule and the quotient rule, along with the differentiation of exponential, logarithmic and trigonometric functions.

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4A The derivative

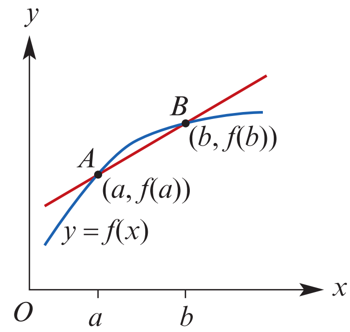
We begin this chapter by recalling the definition of average rate of change from Mathematical Methods Units 1 & 2.

Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through the two points $A(a, f(a))$ and $B(b, f(b))$.

That is:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



Example 1

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

$f(1) = (1)^2 - 2(1) + 5 = 4$

$f(5) = (5)^2 - 2(5) + 5 = 20$

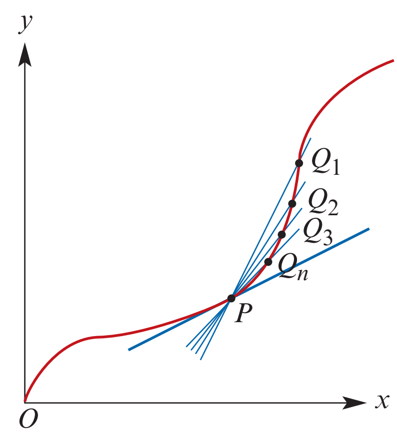
$$\text{Average rate of change} = \frac{20 - 4}{5 - 1} = 4$$

The tangent to a curve at a point

We first recall that a **chord** of a curve is a line segment joining points P and Q on the curve. A **secant** is a line through points P and Q on the curve.

The **instantaneous rate of change** at P can be defined by considering what happens when we look at a sequence of secants $PQ_1, PQ_2, PQ_3, \ldots, PQ_n, \ldots$, where the points Q_i get closer and closer to P.

Here we first focus our attention on the gradient of the tangent at P.



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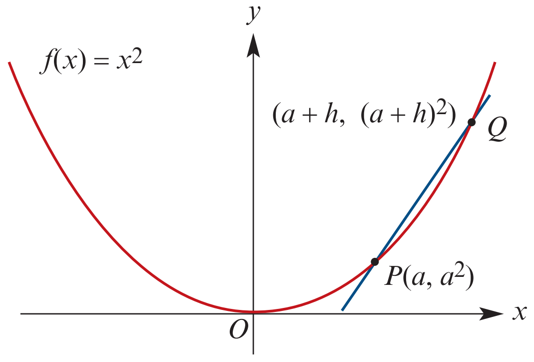
Consider the function $f(x) = x^2$.

The gradient of the secant PQ shown on the graph is

$$\text{gradient of } PQ = \frac{(a+h)^2 - a^2}{a+h-a}$$

$$= \frac{a^2 + 2ah + h^2 - a^2}{h}$$

$$= 2a + h$$



The limit of $2a + h$ as h approaches 0 is $2a$, and so the gradient of the tangent at P is said to be $2a$.

Note: This also can be interpreted as the instantaneous rate of change of f at $(a, f(a))$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

Limit notation

The notation for the limit of $2x + h$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h)$$

The derivative of a function with rule $f(x)$ may be found by:

1. finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2. finding the limit of this expression as h approaches 0.

Example 2

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, find the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\text{Gradient of } PQ = \frac{(2+h)^3 - 8}{2+h-2}$$

$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$

$$= \frac{12h + 6h^2 + h^3}{h}$$

$$= 12 + 6h + h^2$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

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The following example provides practice in determining limits.

Example 3

Find:

a $\lim\limits_{h \to 0} (22x^2 + 20xh + h)$

b $\lim\limits_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$

c $\lim\limits_{h \to 0} 12x$

d $\lim\limits_{h \to 0} 4$

Solution

a $\lim\limits_{h \to 0} (22x^2 + 20xh + h) = 22x^2$

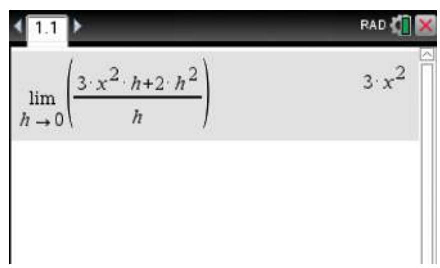
b $$\lim_{h \to 0} \frac{3x^2h + 2h^2}{h} = \lim_{h \to 0} (3x^2 + 2h)$$
$$= 3x^2$$

c $\lim\limits_{h \to 0} 12x = 12x$

d $\lim\limits_{h \to 0} 4 = 4$

Using the TI-Nspire

To calculate a limit, use menu > **Calculus** > **Limit** and complete as shown.



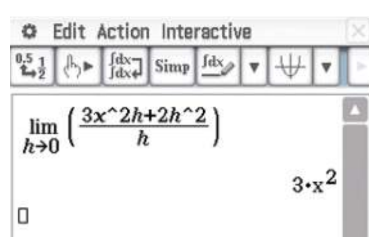
Note: The limit template can also be accessed from the 2D-template palette.
When you insert the limit template, you will notice a superscript field (small box) on the template – generally this will be left empty.

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $\dfrac{3x^2h + 2h^2}{h}$

Note: Use h from the Var keyboard.

- Select lim from the Math2 keyboard and tap EXE.
- Enter h and 0 in the spaces provided as shown.



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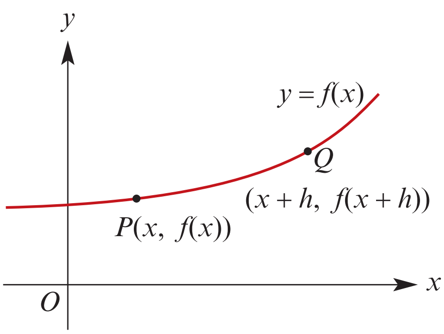
Definition of the derivative

In general, consider the graph $y = f(x)$ of a function $f: \mathbb{R} \to \mathbb{R}$.

Gradient of secant $PQ = \dfrac{f(x+h) - f(x)}{x+h-x}$

$$= \frac{f(x+h) - f(x)}{h}$$



The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.

Derivative of a function

The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain.

Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 4

Find $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for each of the following:

a $f(x) = 3x^2 + 2x + 2$

b $f(x) = 2 - x^3$

Solution

a
$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h}$$
$$= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 2 - 3x^2 - 2x - 2}{h}$$
$$= \frac{6xh + 3h^2 + 2h}{h}$$
$$= 6x + 3h + 2$$

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Therefore

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2$$

b $$\frac{f(x+h) - f(x)}{h} = \frac{2 - (x+h)^3 - (2 - x^3)}{h}$$

$$= \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - 2 + x^3}{h}$$

$$= \frac{-3x^2h - 3xh^2 - h^3}{h}$$

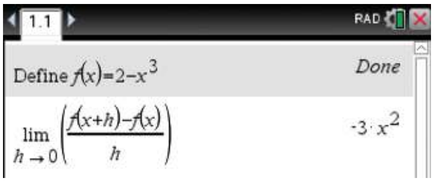
$$= -3x^2 - 3xh - h^2$$

Therefore

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-3x^2 - 3xh - h^2) = -3x^2$$

Using the TI-Nspire

- Define $f(x) = 2 - x^3$.
- Use (menu) > **Calculus** > **Limit** or the 2D-template palette (template key), and complete as shown.



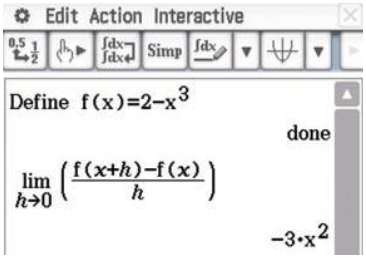
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $2 - x^3$. Select **Interactive** > **Define** and tap OK.
- Now enter and highlight the expression

$$\frac{f(x+h) - f(x)}{h}$$

Note: Select f from the (abc) keyboard and x, h from the (Var) keyboard.

- Select (lim template) from the (Math2) keyboard and tap (EXE).
- Enter h and 0 in the spaces provided as shown.



Summary 4A

- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

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Exercise 4A

Example 1

1 Find the average rate of change of the function with rule $f(x) = -x^2 + 2x + 1$ as x changes from -1 to 4.

2 Find the average rate of change of the function with rule $f(x) = 6 - x^3$ as x changes from -1 to 1.

Example 2

3 For the curve with equation $y = x^2 + 5x$:

a Find the gradient of the secant though points P and Q, where P is the point $(2, 14)$ and Q is the point $(2 + h, (2 + h)^2 + 5(2 + h))$.

b From the result of part **a**, find the gradient of the tangent to the curve at the point $(2, 14)$.

Example 3

4 Find:

a $\lim\limits_{h\to 0} \dfrac{4x^2h^2 + xh + h}{h}$

b $\lim\limits_{h\to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

c $\lim\limits_{h\to 0} (40 - 50h)$

d $\lim\limits_{h\to 0} 5h$

e $\lim\limits_{h\to 0} 5$

f $\lim\limits_{h\to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

g $\lim\limits_{h\to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

h $\lim\limits_{h\to 0} 3x$

i $\lim\limits_{h\to 0} \dfrac{3x^3h - 5x^2h^2 + xh}{h}$

j $\lim\limits_{h\to 0} (6x - 7h)$

5 For the curve with equation $y = x^3 - x$:

a Find the gradient of the chord PQ, where P is the point $(1, 0)$ and Q is the point $(1 + h, (1 + h)^3 - (1 + h))$.

b From the result of part **a**, find the gradient of the tangent to the curve at the point $(1, 0)$.

6 If $f(x) = x^2 - 2$, simplify $\dfrac{f(x+h) - f(x)}{h}$. Hence find the derivative of $x^2 - 2$.

7 Let P and Q be points on the curve $y = x^2 + 2x + 5$ at which $x = 2$ and $x = 2 + h$ respectively. Express the gradient of the line PQ in terms of h, and hence find the gradient of the tangent to the curve $y = x^2 + 2x + 5$ at $x = 2$.

Example 4

8 For each of the following, find $f'(x)$ by finding $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$:

a $f(x) = 5x^2$

b $f(x) = 3x + 2$

c $f(x) = 5$

d $f(x) = 3x^2 + 4x + 3$

e $f(x) = 5x^3 - 5$

f $f(x) = 5x^2 - 6x$

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4B Rules for differentiation

The derivative of x^n where n is a positive integer

Differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result:

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$

Proof We can prove this result using the binomial theorem.

Let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n \\ &= x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n - x^n \\ &= {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} x h^{n-2} + h^{n-1} \end{aligned}$$

Thus
$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} x h^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

The derivative of a polynomial function

The following results are very useful when finding the derivative of a polynomial function.

- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

You will meet rules for the derivatives of products and quotients later in this chapter.

The process of finding the derivative function is called **differentiation**.

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Example 5

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $f(x) = x^5 - 2x^3 + 2$

Then $f'(x) = 5x^4 - 2(3x^2) + 0$

$= 5x^4 - 6x^2$

Explanation

We use the following results:

- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 6

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Solution

Let $f(x) = 3x^3 - 6x^2 + 1$

Then $f'(x) = 3(3x^2) - 6(2x) + 0$

$= 9x^2 - 12x$

$\therefore \quad f'(1) = 9 - 12 = -3$

Using the TI-Nspire

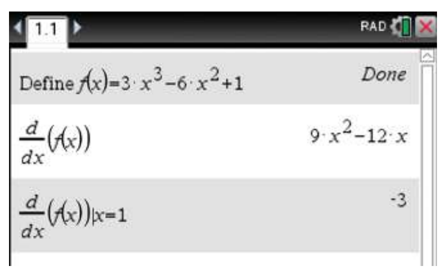
For Example 5:

- Use (menu) > **Calculus** > **Derivative** and complete as shown.

Note: The derivative template can also be accessed from the 2D-template palette (template key). Alternatively, using (shift) (–) will paste the derivative template to the screen.

For Example 6:

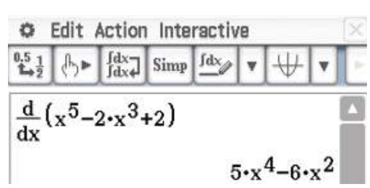
- Define $f(x) = 3x^3 - 6x^2 + 1$.
- Use (menu) > **Calculus** > **Derivative** to differentiate as shown.
- To evaluate the derivative at $x = 1$, use (menu) > **Calculus** > **Derivative at a Point**.



Using the Casio ClassPad

For Example 5:

- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^5 - 2x^3 + 2$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.



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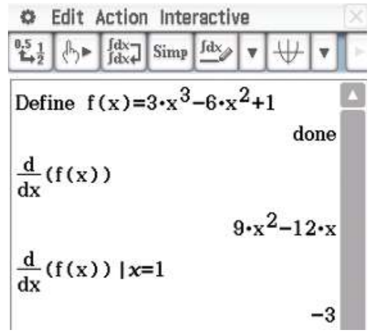
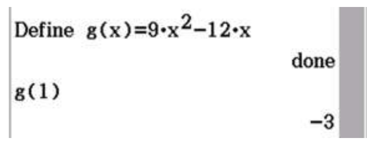
For Example 6:

- In $\sqrt[\text{Main}]{\alpha}$, enter and highlight the expression $3x^3 - 6x^2 + 1$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK; this will give the derivative only.
- To find the value of the derivative at $x = 1$, tap the stylus at the end of the entry line. Select | from the Math3 keyboard and type $x = 1$. Then tap EXE.



- Alternatively, define the derivative as $g(x)$ and find $g(1)$.



Finding the gradient of a tangent line

We discussed the tangent line at a point on a graph in Section 4A. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 7

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, find the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

Leibniz notation

An alternative notation for the derivative is the following:

> If $y = x^3$, then the derivative can be denoted by $\frac{dy}{dx}$, and so we write $\frac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\frac{dy}{dx}$.

Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\frac{dz}{dt}$.

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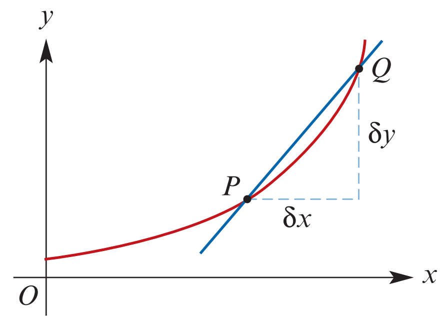
Warning: In Leibniz notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for finding the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.



Example 8

a If $y = t^2$, find $\frac{dy}{dt}$. **b** If $x = t^3 + t$, find $\frac{dx}{dt}$. **c** If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a
$$y = t^2$$
$$\frac{dy}{dt} = 2t$$

b
$$x = t^3 + t$$
$$\frac{dx}{dt} = 3t^2 + 1$$

c
$$z = \frac{1}{3}x^3 + x^2$$
$$\frac{dz}{dx} = x^2 + 2x$$

Example 9

a For $y = (x+3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a First write $y = (x+3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$
$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t + 2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$
$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$
$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d
$$y = 2x^3 - 1$$
$$\therefore \quad \frac{dy}{dx} = 6x^2$$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'find $\frac{d}{dx}(2x^2 - 4x)$'.

In general: $\frac{d}{dx}(f(x)) = f'(x)$.

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Example 10

Find:

a $\dfrac{d}{dx}(5x - 4x^3)$ **b** $\dfrac{d}{dz}(5z^2 - 4z)$ **c** $\dfrac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\dfrac{d}{dx}(5x - 4x^3)$

$= 5 - 12x^2$

b $\dfrac{d}{dz}(5z^2 - 4z)$

$= 10z - 4$

c $\dfrac{d}{dz}(6z^3 - 4z^2)$

$= 18z^2 - 8z$

Example 11

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient $= 8$ **b** $y = x^2 - 4x + 2$, gradient $= 0$

c $y = 4 - x^3$, gradient $= -6$

Solution

a $y = x^3$ implies $\dfrac{dy}{dx} = 3x^2$

$\therefore \quad 3x^2 = 8$

$\therefore \quad x = \pm\sqrt{\dfrac{8}{3}} = \dfrac{\pm 2\sqrt{6}}{3}$

The points are $\left(\dfrac{2\sqrt{6}}{3}, \dfrac{16\sqrt{6}}{9}\right)$ and $\left(\dfrac{-2\sqrt{6}}{3}, \dfrac{-16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\dfrac{dy}{dx} = 2x - 4$

$\therefore \quad 2x - 4 = 0$

$\therefore \quad x = 2$

The only point is $(2, -2)$.

c $y = 4 - x^3$ implies $\dfrac{dy}{dx} = -3x^2$

$\therefore \quad -3x^2 = -6$

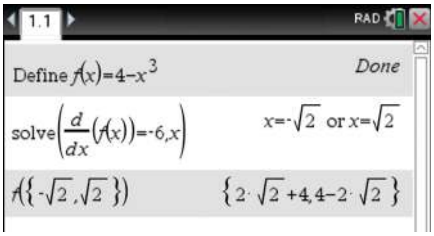
$\therefore \quad x^2 = 2$

$\therefore \quad x = \pm\sqrt{2}$

The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

Using the TI-Nspire

- Define $f(x) = 4 - x^3$.
- Solve the equation $\dfrac{d}{dx}(f(x)) = -6$.
- Substitute in $f(x)$ to find the y-coordinates.



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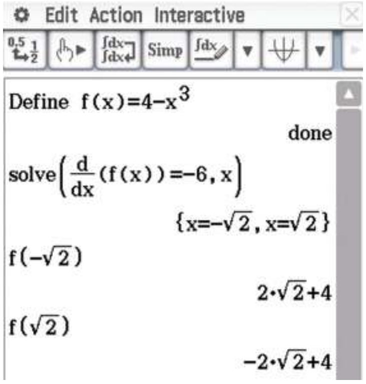
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $4 - x^3$.
- Go to **Interactive** > **Define** and tap OK.
- In the next entry line, type and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.
- Type $= -6$ after $\frac{d}{dx}(f(x))$. Highlight the equation and use **Interactive** > **Equation/Inequality** > **solve**.
- Enter $f(-\sqrt{2})$ and $f(\sqrt{2})$ to find the required y-values.



An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If α is the angle a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\alpha$. That is, $m = \tan\alpha$.

For example, if $\alpha = 135°$, then $\tan\alpha = -1$ and so the gradient is -1.

Example 12

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\frac{dy}{dx} = 2x - 7$

$2x - 7 = 1 \quad (\text{as } \tan 45° = 1)$

$2x = 8$

$\therefore \quad x = 4$

$y = 4^2 - 7 \times 4 + 8 = -4$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$2x - 7 = -2$

$2x = 5$

$\therefore \quad x = \frac{5}{2}$

The coordinates are $\left(\frac{5}{2}, -\frac{13}{4}\right)$.

Increasing and decreasing functions

We have discussed strictly increasing and strictly decreasing functions in Chapter 1.

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

We have the following very important results.

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If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing.
(Think of the tangents at each point – they each have positive gradient.)

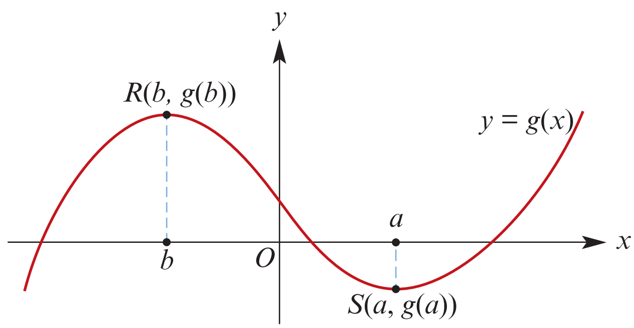
If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing.
(Think of the tangents at each point – they each have negative gradient.)

Warning: The function $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply* $f'(x) > 0$.

Sign of the derivative

Gradients of tangents can, of course, be negative or zero. They are not always positive.

At a point $(a, g(a))$ on the graph of $y = g(x)$, the gradient of the tangent is $g'(a)$.



Some features of the graph shown are:

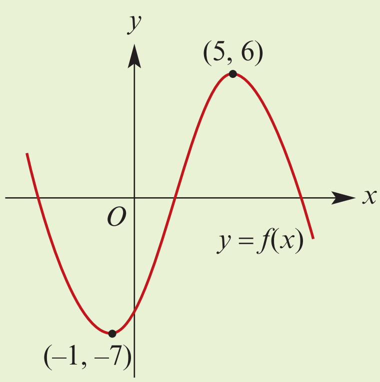
- For $x < b$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.
- For $x = b$, the gradient of the tangent is zero, i.e. $g'(b) = 0$.
- For $b < x < a$, the gradient of any tangent is negative, i.e. $g'(x) < 0$.
- For $x = a$, the gradient of the tangent is zero, i.e. $g'(a) = 0$.
- For $x > a$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.

Note: This function g is strictly decreasing on the open interval (b, a), but it is also strictly decreasing on the closed interval $[b, a]$. Similarly, the function g is strictly increasing on the intervals $[a, \infty)$ and $(-\infty, b]$.

Example 13

For the graph of $f(x)$, find:

a $\{x : f'(x) > 0\}$
b $\{x : f'(x) < 0\}$
c $\{x : f'(x) = 0\}$



Solution

a $\{x : f'(x) > 0\} = \{x : -1 < x < 5\} = (-1, 5)$
b $\{x : f'(x) < 0\} = \{x : x < -1\} \cup \{x : x > 5\} = (-\infty, -1) \cup (5, \infty)$
c $\{x : f'(x) = 0\} = \{-1, 5\}$

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Summary 4B

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \dots$
- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
- Angle of inclination of tangent
  - A straight line, the tangent, is associated with each point on a smooth curve.
  - If α is the angle that a straight line makes with the positive direction of the x-axis, then the gradient of the line is given by $m = \tan\alpha$.
- Increasing and decreasing functions
  - A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
  - A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
  - If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
  - If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

Skill-sheet

Exercise 4B

Example 5

1 For each of the following, find the derivative with respect to x:

a x^5 **b** $4x^7$ **c** $6x$
d $5x^2 - 4x + 3$ **e** $4x^3 + 6x^2 + 2x - 4$ **f** $5x^4 + 3x^3$
g $-2x^2 + 4x + 6$ **h** $6x^3 - 2x^2 + 4x - 6$

Example 6

2 For each of the following, find the derivative of $f(x)$ and thus find $f'(1)$:

a $f(x) = 2x^3 - 5x^2 + 1$ **b** $f(x) = -2x^3 - x^2 - 1$
c $f(x) = x^4 - 2x^3 + 1$ **d** $f(x) = x^5 - 3x^3 + 2$

Example 7

3 **a** For the curve determined by the rule $f(x) = 2x^3 - 5x^2 + 2$, find the gradient of the tangent line to the curve at the point $(1, -1)$.

b For the curve determined by the rule $f(x) = -2x^3 - 3x^2 + 2$, find the gradient of the tangent line to the curve at the point $(2, -26)$.

Example 8

4 **a** If $y = t^3$, find $\dfrac{dy}{dt}$.

b If $x = t^3 - t^2$, find $\dfrac{dx}{dt}$.

c If $z = \dfrac{1}{4}x^4 + 3x^3$, find $\dfrac{dz}{dx}$.

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Example 9

5 For each of the following, find $\dfrac{dy}{dx}$:

a $y = -2x$

b $y = 7$

c $y = 5x^3 - 3x^2 + 2x + 1$

d $y = \dfrac{2}{5}(x^3 - 4x + 6)$

e $y = (2x + 1)(x - 3)$

f $y = 3x(2x - 4)$

g $y = \dfrac{10x^7 + 2x^2}{x^2}, \ x \neq 0$

h $y = \dfrac{9x^4 + 3x^2}{x}, \ x \neq 0$

Example 10

6 Find:

a $\dfrac{d}{dx}(2x^2 - 5x^3)$

b $\dfrac{d}{dz}(-2z^2 - 6z)$

c $\dfrac{d}{dz}(6z^3 - 4z^2 + 3)$

d $\dfrac{d}{dx}(-2x - 5x^3)$

e $\dfrac{d}{dz}(-2z^2 - 6z + 7)$

f $\dfrac{d}{dz}(-z^3 - 4z^2 + 3)$

Example 11

7 Find the coordinates of the points on the curves given by the following equations at which the gradient has the given value:

a $y = 2x^2 - 4x + 1$, gradient $= -6$

b $y = 4x^3$, gradient $= 48$

c $y = x(5 - x)$, gradient $= 1$

d $y = x^3 - 3x^2$, gradient $= 0$

Example 12

8 Find the coordinates of the points on the curve with equation $y = 2x^2 - 3x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 2x + 8$.

9 Find the value of x such that the tangent line to the curve $f(x) = x^2 - x$ at $(x, f(x))$:

a makes an angle of 45° with the positive direction of the x-axis

b makes an angle of 135° with the positive direction of the x-axis

c makes an angle of 60° with the positive direction of the x-axis

d makes an angle of 30° with the positive direction of the x-axis

e makes an angle of 120° with the positive direction of the x-axis.

10 For each of the following, find the angle that the tangent line to the curve $y = f(x)$ makes with the positive direction of the x-axis at the given point:

a $y = x^2 + 3x, \ (1, 4)$

b $y = -x^2 + 2x, \ (1, 1)$

c $y = x^3 + x, \ (0, 0)$

d $y = -x^3 - x, \ (0, 0)$

e $y = x^4 - x^2, \ (1, 0)$

f $y = x^4 - x^2, \ (-1, 0)$

11 **a** Differentiate $y = (2x - 1)^2$ with respect to x.

b For $y = \dfrac{x^3 + 2x^2}{x}$, $x \neq 0$, find $\dfrac{dy}{dx}$.

c Given that $y = 2x^3 - 6x^2 + 18x$, find $\dfrac{dy}{dx}$. Hence show that $\dfrac{dy}{dx} > 0$ for all x.

d Given that $y = \dfrac{x^3}{3} - x^2 + x$, find $\dfrac{dy}{dx}$. Hence show that $\dfrac{dy}{dx} \geq 0$ for all x.

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12 At the points on the following curves corresponding to the given values of x, find the y-coordinate and the gradient:

a $y = x^2 + 2x + 1,\ x = 3$
b $y = x^2 - x - 1,\ x = 0$
c $y = 2x^2 - 4x,\ x = -1$
d $y = (2x + 1)(3x - 1)(x + 2),\ x = 4$
e $y = (2x + 5)(3 - 5x)(x + 1),\ x = 1$
f $y = (2x - 5)^2,\ x = 2\frac{1}{2}$

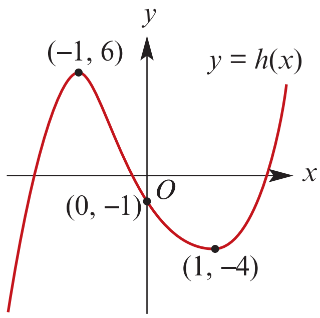
13 For the function $f(x) = 3(x - 1)^2$, find the value(s) of x for which:

a $f(x) = 0$
b $f'(x) = 0$
c $f'(x) > 0$
d $f'(x) < 0$
e $f'(x) = 10$
f $f(x) = 27$

Example 13

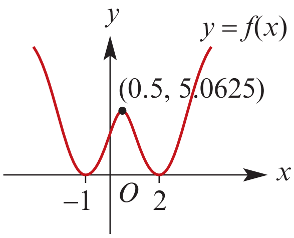
14 For the graph of $y = h(x)$ illustrated, find:

a $\{x : h'(x) > 0\}$
b $\{x : h'(x) < 0\}$
c $\{x : h'(x) = 0\}$



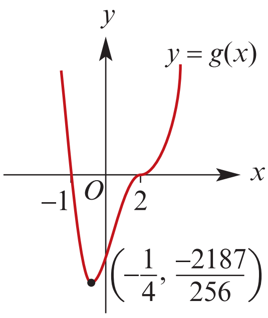
15 For the graph of $y = f(x)$ shown, find:

a $\{x : f'(x) > 0\}$
b $\{x : f'(x) < 0\}$
c $\{x : f'(x) = 0\}$



16 For the graph of $y = g(x)$ shown, find:

a $\{x : g'(x) > 0\}$
b $\{x : g'(x) < 0\}$
c $\{x : g'(x) = 0\}$



17 Find the coordinates of the points on the parabola $y = x^2 - 4x - 8$ at which:

a the gradient is zero
b the tangent is parallel to $y = 2x + 6$
c the tangent is parallel to $3x + 2y = 8$.

18 **a** Show that $f(x) = x^3$ is a strictly increasing function for $\mathbb{R}$ by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$ and, if $0 > b$, then $f(0) > f(b)$.

b Show that $f(x) = -x^3$ is a strictly decreasing function for $\mathbb{R}$.

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19 **a** Show that $f(x) = x^2, \{ x \in \mathbb{R} : x \geq 0 \}$ is a strictly increasing function.
b Show that $f(x) = x^2, \{ x \in \mathbb{R} : x \leq 0 \}$ is a strictly decreasing function.

20 For the function $f(x) = x^2 - x - 12$, show that the largest interval for which f is strictly increasing is $\left[\frac{1}{2}, \infty\right)$.

21 For each of the following, find the largest interval for which the function is strictly decreasing:
a $y = x^2 + 2x$ **b** $y = -x^2 + 4x$ **c** $y = 2x^2 + 3$ **d** $y = -2x^2 + 6x$

4C Differentiating x^n where n is a negative integer

In the previous sections we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. In particular, we will consider functions which involve linear combinations of powers of x, where the indices may be negative integers.

e.g. $f(x) = x^{-1}, \{ x \in \mathbb{R} : x \neq 0 \}$
$f(x) = 2x + x^{-1}, \{ x \in \mathbb{R} : x \neq 0 \}$
$f(x) = x + 3 + x^{-2}, \{ x \in \mathbb{R} : x \neq 0 \}$

Note: We have reintroduced function notation to emphasise the need to consider domains.

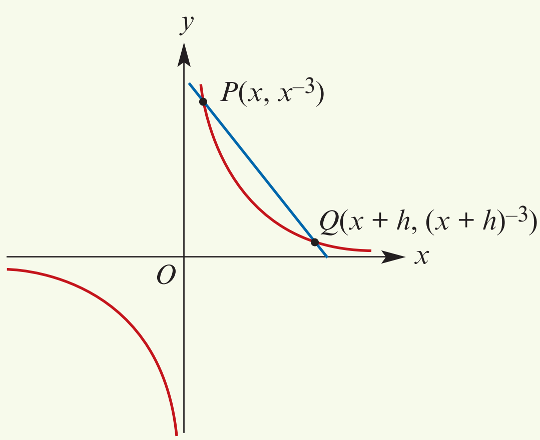
Example 14

Let $f(x) = x^{-3}, \{ x \in \mathbb{R} : x \neq 0 \}$. Find $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

$$\frac{(x+h)^{-3} - x^{-3}}{h}$$
$$= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

We are now in a position to state the generalisation of the result we found in Section 4B. This result can be proved by again using the binomial theorem.

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For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\{ x \in \mathbb{R} : x \neq 0 \}$.

Example 15

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2$, $x \neq 0$.

Solution

If $\quad f(x) = x^4 - 2x^{-3} + x^{-1} + 2 \qquad$ (for $x \neq 0$)

then $\quad f'(x) = 4x^3 - 2(-3x^{-4}) + (-x^{-2}) + 0$

$\qquad = 4x^3 + 6x^{-4} - x^{-2} \qquad$ (for $x \neq 0$)

Example 16

Find the derivative f' of $f(x) = 3x^2 - 6x^{-2} + 1$, $\{ x \in \mathbb{R} : x \neq 0 \}$.

Solution

$f'(x) = 3(2x) - 6(-2x^{-3}) + 0$

$\qquad = 6x + 12x^{-3}$

with domain $\{ x \in \mathbb{R} : x \neq 0 \}$

Example 17

Find the gradient of the tangent to the curve determined by the function $f(x) = x^2 + \dfrac{1}{x}$, $\{ x \in \mathbb{R} : x \neq 0 \}$ at the point $(1, 2)$.

Solution

$f'(x) = 2x + (-x^{-2})$

$\qquad = 2x - x^{-2}$

with domain $\{ x \in \mathbb{R} : x \neq 0 \}$

Therefore $f'(1) = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 18

Show that the derivative of the function $f(x) = x^{-3}$, $\{ x \in \mathbb{R} : x \neq 0 \}$ is always negative.

Solution

$f'(x) = -3x^{-4} = \dfrac{-3}{x^4}$

with domain $\{ x \in \mathbb{R} : x \neq 0 \}$

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

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Summary 4C

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Exercise 4C

1 **a** Sketch the graph of $f(x) = \dfrac{2}{x^2}$, $\{x \in \mathbb{R} : x \neq 0\}$.

b Let P be the point $(1, 2)$ and Q the point $(1 + h, f(1 + h))$. Find the gradient of the secant PQ.

c Hence find the gradient of the tangent to the curve $f(x) = \dfrac{2}{x^2}$ at $(1, 2)$.

Example 14

2 **a** Let $f(x) = \dfrac{1}{x-3}$, $\{x \in \mathbb{R} : x \neq 3\}$. Find $f'(x)$ by first principles.

b Let $f(x) = \dfrac{1}{x+2}$, $\{x \in \mathbb{R} : x \neq -2\}$. Find $f'(x)$ by first principles.

3 Let $f(x) = x^{-4}$, $\{x \in \mathbb{R} : x \neq 0\}$. Find $f'(x)$ by first principles.
Hint: Remember that $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

Example 15
Example 16

4 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\dfrac{5}{x^3} + 6x^2$ **c** $\dfrac{-5}{x^3} + \dfrac{4}{x^2} + 1$

d $6x^{-3} + 3x^{-2}$ **e** $\dfrac{4x^2 + 2x}{x^2}$

5 Find the derivative of each of the following:

a $\dfrac{2z^2 - 4z}{z^2}$, $z \neq 0$ **b** $\dfrac{6 + z}{z^3}$, $z \neq 0$ **c** $16 - z^{-3}$, $z \neq 0$

d $\dfrac{4z + z^3 - z^4}{z^2}$, $z \neq 0$ **e** $\dfrac{6z^2 - 2z}{z^4}$, $z \neq 0$ **f** $\dfrac{6}{x} - 3x^2$, $x \neq 0$

Example 17

6 Find the gradient of the tangent to each of the following curves at the stated point:

a $y = x^{-2} + x^3$, $x \neq 0$, at $\left(2, 8\frac{1}{4}\right)$ **b** $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $\left(4, \dfrac{1}{2}\right)$

c $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3})$, $x \neq 0$, at $(1, 1)$

Example 18

7 Show that the derivative of the function $f(x) = -2x^{-5}$, $\{x \in \mathbb{R} : x \neq 0\}$ is always positive.

8 Find the x-coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

9 Given that the curve $y = ax^2 + \dfrac{b}{x}$ has a gradient of -5 at the point $(2, -2)$, find the values of a and b.

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10 Find the gradient of the curve $y = \dfrac{2x-4}{x^2}$ at the point where the curve crosses the x-axis.

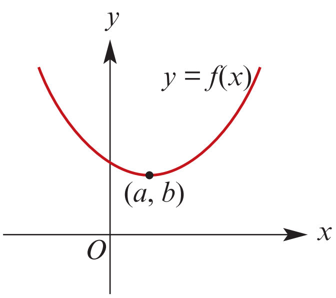
11 The gradient of the curve $y = \dfrac{a}{x} + bx^2$ at the point $(3, 6)$ is 7. Find the values of a and b.

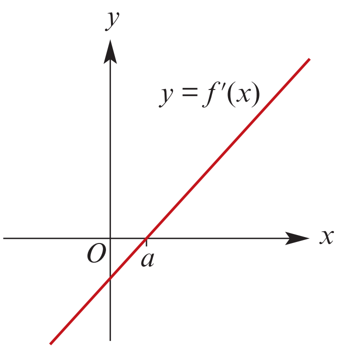
12 For the curve with equation $y = \dfrac{5}{3}x + kx^2 - \dfrac{8}{9}x^3$, calculate the possible values of k such that the tangents at the points with x-coordinates 1 and $-\dfrac{1}{2}$ are perpendicular.

4D The graph of the derivative function

First consider the quadratic function with rule $y = f(x)$ shown in the graph on the left. The vertex is at the point with coordinates (a, b).

- For $x < a$, $f'(x) < 0$.
- For $x = a$, $f'(x) = 0$.
- For $x > a$, $f'(x) > 0$.



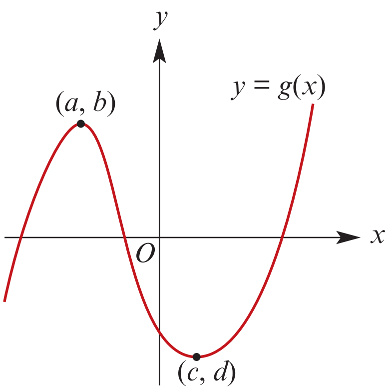


The graph of the derivative function with rule $y = f'(x)$ is therefore as shown on the right.

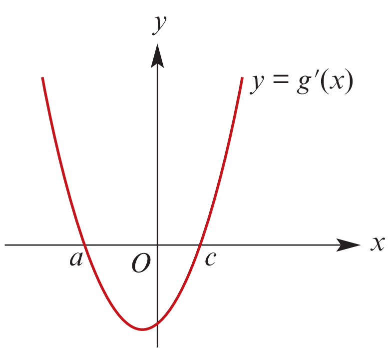
The derivative f' is known to be linear as f is quadratic.

Now consider the cubic function with rule $y = g(x)$ shown in the graph.

- For $x < a$, $g'(x) > 0$.
- For $x = a$, $g'(x) = 0$.
- For $a < x < c$, $g'(x) < 0$.
- For $x = c$, $g'(x) = 0$.
- For $x > c$, $g'(x) > 0$.



The graph of the derivative function with rule $y = g'(x)$ is therefore as shown to the right. The derivative g' is known to be quadratic as g is cubic.



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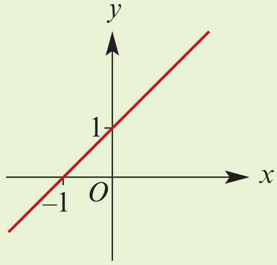
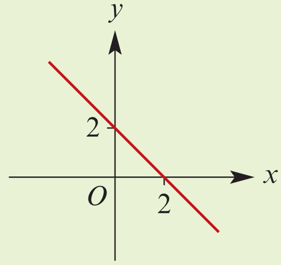
Example 19

Sketch the graph of the derivative function for each of the functions of the graphs shown:

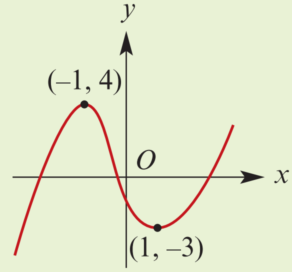
a



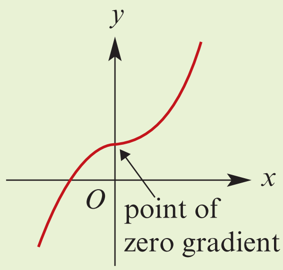
b



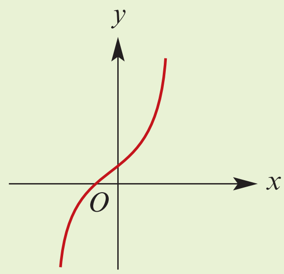
c



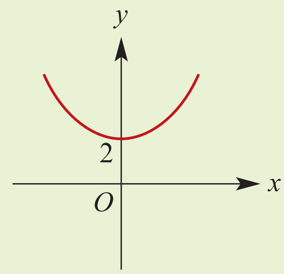
d



e



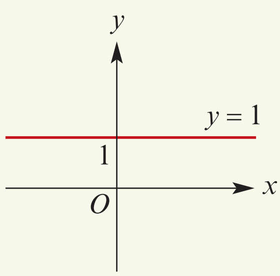
f



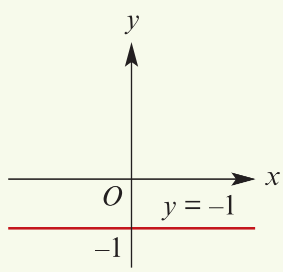
Solution

Note: Not all features of the graphs are known.

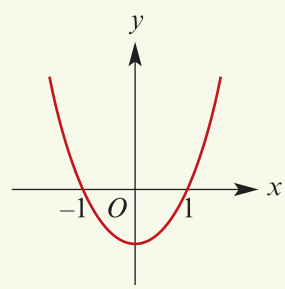
a



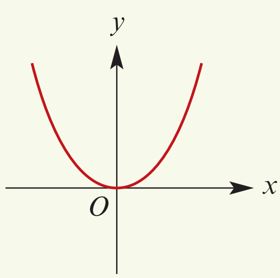
b



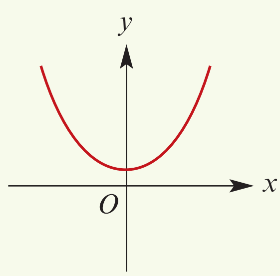
c



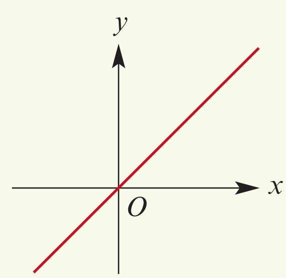
d



e



f



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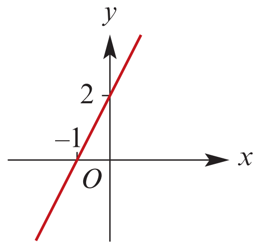
Exercise 4D

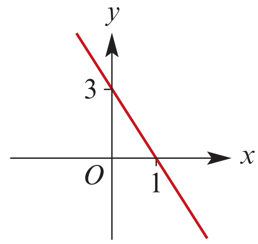
Example 19

1 Sketch the graph of the derivative function for each of the following functions:

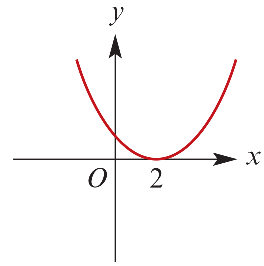
a



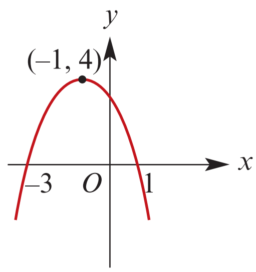
b



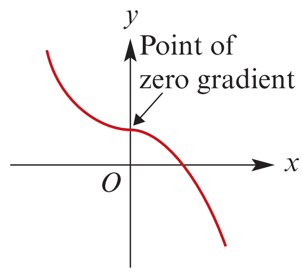
c



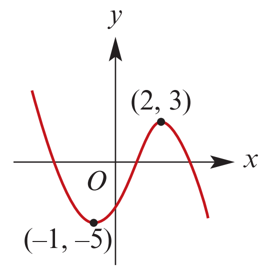
d



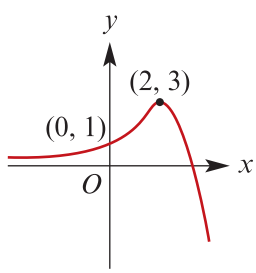
e



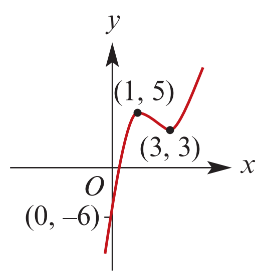
f



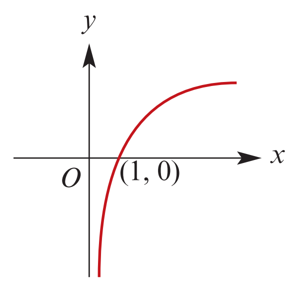
g



h

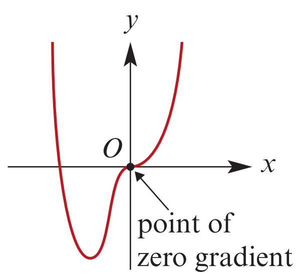


i

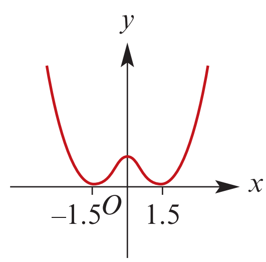


2 Sketch the graph of the derivative function for each of the following functions:

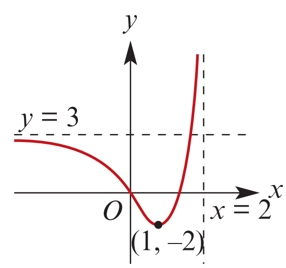
a



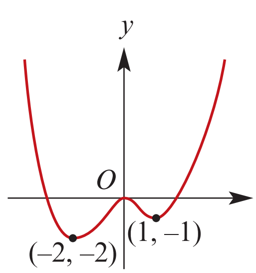
b



c



d



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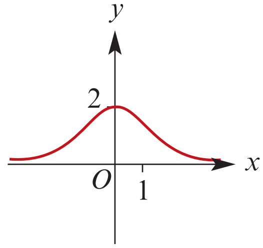
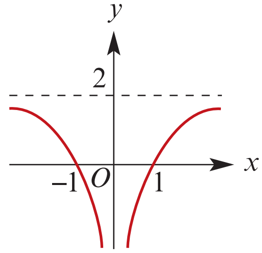
3 Match the graphs of the functions **a**–**d** with the graphs of their derivatives **A**–**D**:

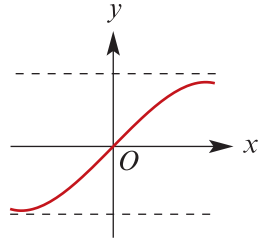
a



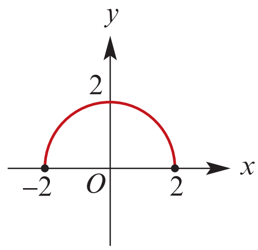
b



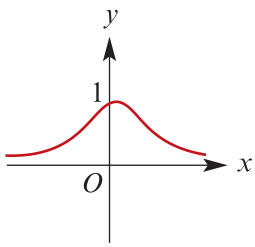
c



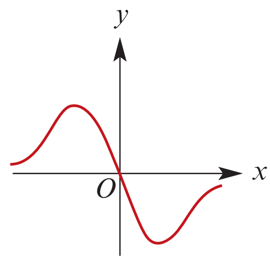
d



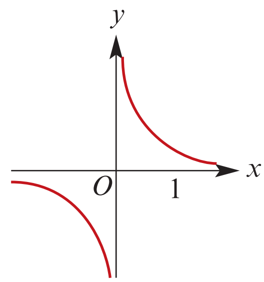
A



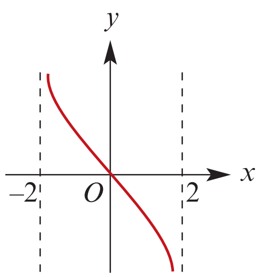
B



C



D



4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = (x^2 - 2x)^2$.

b Using the same screen, plot the graph of $y = f'(x)$. (Do not attempt to determine the rule for $f'(x)$ first.)

c Use a calculator to determine $f'(x)$ for:

i $x = 0$ **ii** $x = 2$ **iii** $x = 1$ **iv** $x = 4$

d For $0 \le x \le 1$, find the value of x for which:

i $f(x)$ is a maximum **ii** $f'(x)$ is a maximum.

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5 For $f(x) = \frac{x^3}{3} - x^2 + x + 1$, plot the graphs of $y = f(x)$ and $y = f'(x)$ on the same screen. Comment.

6 For $g(x) = x^3 + 2x + 1$, plot the graphs of $y = g(x)$ and $y = g'(x)$ on the same screen. Comment.

7 **a** For $h(x) = x^4 + 2x + 1$, plot the graphs of $y = h(x)$ and $y = h'(x)$ on the same screen.
b Find the value(s) of x such that:
i $h(x) = 3$ **ii** $h'(x) = 3$

4E The chain rule

An expression such as $q(x) = (x^3 + 1)^2$ may be differentiated by expanding and then differentiating each term separately. This method is a great deal more tiresome for an expression such as $q(x) = (x^3 + 1)^{30}$.

We can express $q(x) = (x^3 + 1)^2$ as the composition of two simpler functions defined by

$$u = g(x) = x^3 + 1 \quad \text{and} \quad y = f(u) = u^2$$

which are 'chained' together:

$$x \xrightarrow{g} u \xrightarrow{f} y$$

That is, $q(x) = (x^3 + 1)^2 = f(g(x))$, and so q is expressed as the composition $f \circ g$.

The chain rule gives a method of differentiating such functions.

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\, g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Proof To find the derivative of $q = f \circ g$ where $x = a$, consider the secant through the points $(a, f \circ g(a))$ and $(a + h, f \circ g(a + h))$. The gradient of this secant is

$$\frac{f \circ g(a + h) - f \circ g(a)}{h}$$

We carry out the trick of multiplying the numerator and the denominator by $g(a + h) - g(a)$. This gives

$$\frac{f(g(a + h)) - f(g(a))}{h} \times \frac{g(a + h) - g(a)}{g(a + h) - g(a)}$$

provided $g(a + h) - g(a) \neq 0$.

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Now write $b = g(a)$ and $b + k = g(a + h)$ so that $k = g(a + h) - g(a)$. The expression for the gradient becomes

$$\frac{f(b+k) - f(b)}{k} \times \frac{g(a+h) - g(a)}{h}$$

The function g is continuous, since its derivative exists, and therefore

$$\lim_{h \to 0} k = \lim_{h \to 0} [g(a+h) - g(a)] = 0$$

Thus, as h approaches 0, so does k. Hence $q'(a) = f'(g(a))\, g'(a)$.

Note that this proof does not hold for a function g such that $g(a + h) - g(a) = 0$ for arbitrarily chosen small h. However, a fully rigorous proof is beyond the scope of this course.

Example 20

Differentiate $y = (4x^3 - 5x)^{-2}$.

Solution

The differentiation is undertaken using both notations:

Let $u = 4x^3 - 5x$

Then $y = u^{-2}$

We have

$$\frac{dy}{du} = -2u^{-3}$$

$$\frac{du}{dx} = 12x^2 - 5$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -2u^{-3} \cdot (12x^2 - 5)$$

$$= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}$$

Let $h(x) = 4x^3 - 5x$

and $g(x) = x^{-2}$

Then $f(x) = g(h(x))$

We have

$$h'(x) = 12x^2 - 5$$

$$g'(x) = -2x^{-3}$$

Therefore

$$f'(x) = g'(h(x))\, h'(x)$$

$$= -2(h(x))^{-3}\, h'(x)$$

$$= -2(4x^3 - 5x)^{-3} \times (12x^2 - 5)$$

$$= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}$$

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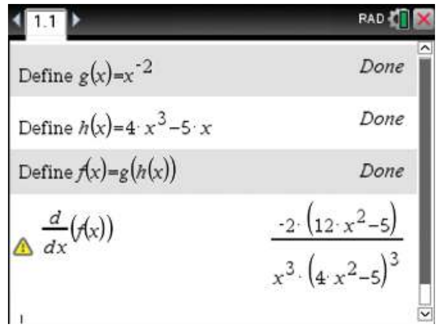
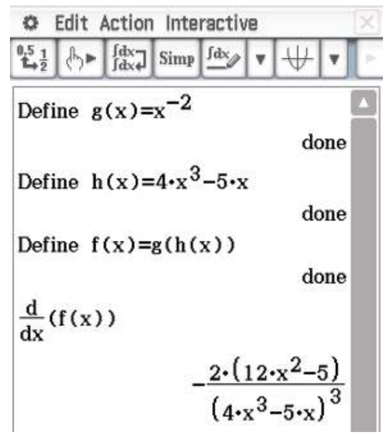
Using the TI-Nspire

- Define $g(x)$ and $h(x)$.
- Then define $f(x) = g(h(x))$.
- Use menu > **Calculus** > **Derivative** and complete as shown.



Using the Casio ClassPad

- Define $g(x)$ and $h(x)$.
- Then define $f(x) = g(h(x))$.
- Find the derivative of $f(x)$.



Example 21

Find the gradient of the tangent to the curve with equation $y = \dfrac{16}{3x^2 + 1}$ at the point $(1, 4)$.

Solution

Let $u = 3x^2 + 1$ then $y = 16u^{-1}$

So $\dfrac{du}{dx} = 6x$ and $\dfrac{dy}{du} = -16u^{-2}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -16u^{-2} \cdot 6x$$

$$= \frac{-96x}{(3x^2 + 1)^2}$$

$\therefore$ At $x = 1$, the gradient is $\dfrac{-96}{16} = -6$.

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Summary 4E

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\,g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Exercise 4E

Example 20

1 Differentiate each of the following with respect to x:

a $(x^2 + 1)^4$
b $(2x^2 - 3)^5$
c $(6x + 1)^4$
d $(ax + b)^n$
e $(ax^2 + b)^n$
f $(1 - x^2)^{-3}$
g $\left(x^2 - \frac{1}{x^2}\right)^{-3}$
h $(1 - x)^{-1}$

2 Differentiate each of the following with respect to x:

a $(x^2 + 2x + 1)^3$
b $(x^3 + 2x^2 + x)^4$
c $\left(6x^3 + \frac{2}{x}\right)^4$
d $(x^2 + 2x + 1)^{-2}$

Example 21

3 Find the gradient of the tangent to the curve with equation $y = \frac{16}{3x^3 + x}$ at the point $(1, 4)$.

4 Find the gradient of the tangent to the curve with equation $y = \frac{1}{x^2 + 1}$ at the points $\left(1, \frac{1}{2}\right)$ and $\left(-1, \frac{1}{2}\right)$.

5 Given that $f'(x) = \sqrt{3x + 4}$ and $g(x) = x^2 - 1$, find $F'(x)$ where $F(x) = f(g(x))$.

6 Differentiate each of the following with respect to x, giving the answer in terms of $f(x)$ and $f'(x)$:

a $[f(x)]^n$, where n is a positive integer
b $\frac{1}{f(x)}$, where $f(x) \neq 0$

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4F Differentiating rational powers

Before using the chain rule to differentiate rational powers, we will show how to differentiate $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ by first principles.

Example 22

Differentiate each of the following by first principles:

a $f(x) = x^{\frac{1}{2}}, \ x > 0$

b $g(x) = x^{\frac{1}{3}}, \ x \neq 0$

Solution

a

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

b We use the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

By observing that $\left(a^{\frac{1}{3}}\right)^3 = a$ and $\left(b^{\frac{1}{3}}\right)^3 = b$, we obtain

$$a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$$

and therefore

$$a^{\frac{1}{3}} - b^{\frac{1}{3}} = \frac{a-b}{a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$$

We now have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}\\ &= \frac{x+h-x}{h\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}\\ &= \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}\end{aligned}$$

Hence

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{1}{3x^{\frac{2}{3}}}$$

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Note: We can prove that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$ for $n \geq 2$. We could use this result to find the derivative of $x^{\frac{1}{n}}$ by first principles, but instead we will use the chain rule.

Using the chain rule

Suppose that two variables x and y are related in such a way that $y = f(x)$ and $x = g(y)$. Then $y = f(g(y))$ and using the chain rule gives

$$1 = \frac{dy}{dy} = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

Thus $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \text{for } \frac{dx}{dy} \neq 0$$

Now consider $y = x^{\frac{1}{n}}$, where $\{ n \in \mathbb{Z} : n \neq 0 \}$ and $x > 0$.

We have $x = y^n$ and so $\frac{dx}{dy} = ny^{n-1}$. Therefore

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n} x^{\frac{1}{n}-1}$$

For $y = x^{\frac{1}{n}}$, $\frac{dy}{dx} = \frac{1}{n} x^{\frac{1}{n}-1}$, where $\{ n \in \mathbb{Z} : n \neq 0 \}$ and $x > 0$.

This result may now be extended to rational powers.

Let $y = x^{\frac{p}{q}}$, where $\{ p, q \in \mathbb{Z} : p \neq 0, q \neq 0 \}$.

Write $y = \left(x^{\frac{1}{q}}\right)^p$. Let $u = x^{\frac{1}{q}}$. Then $y = u^p$. The chain rule yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= pu^{p-1} \cdot \frac{1}{q} x^{\frac{1}{q}-1} \\ &= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q} x^{\frac{1}{q}-1} \\ &= \frac{p}{q} x^{\frac{p}{q}-\frac{1}{q}} x^{\frac{1}{q}-1} \\ &= \frac{p}{q} x^{\frac{p}{q}-1} \end{aligned}$$

Thus the result for integer powers has been extended to rational powers. In fact, the analogous result holds for any non-zero real power:

For $f(x) = x^a$, $f'(x) = ax^{a-1}$, where $\{ a \in \mathbb{R} : a \neq 0 \}$ and $x > 0$.

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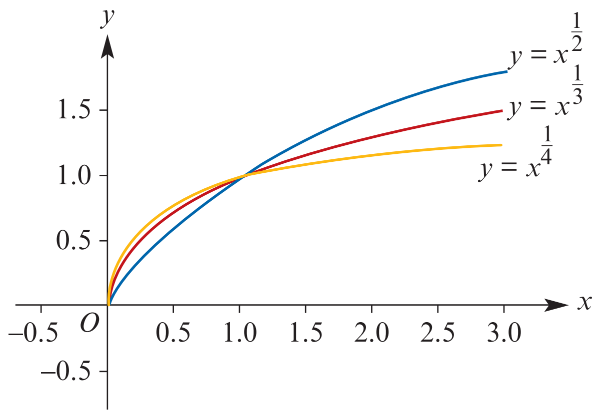
This result is stated for $x > 0$, as $(-3)^{\frac{1}{2}}$ is not defined, although $(-2)^{\frac{1}{3}}$ is defined.

The graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$ are shown.

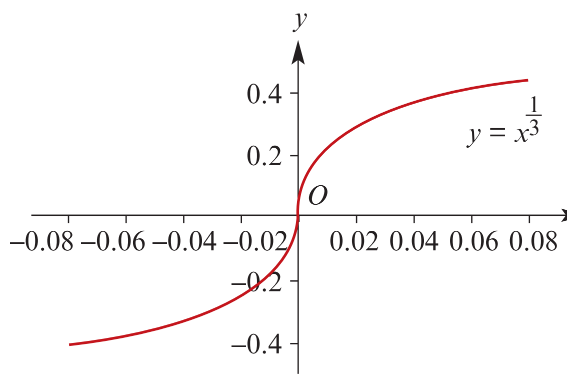
The domain of each has been taken to be $(0, \infty)$.



The figure to the right is the graph of the function $f(x) = x^{\frac{1}{3}}$.

Note that the values shown here are $-0.08 \le x \le 0.08$.

From this it can be seen that the tangent to $y = x^{\frac{1}{3}}$ at the origin is on the y-axis.



Example 23

Find the derivative of each of the following with respect to x:

a $2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}$

b $\sqrt[3]{x^2 + 2x}$

Solution

a $\dfrac{d}{dx}\left(2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}\right)$

$= 2\left(\dfrac{-1}{5}x^{-\frac{6}{5}}\right) + 3\left(\dfrac{2}{7}x^{-\frac{5}{7}}\right)$

$= -\dfrac{2}{5}x^{-\frac{6}{5}} + \dfrac{6}{7}x^{-\frac{5}{7}}$

b $\dfrac{d}{dx}\left(\sqrt[3]{x^2 + 2x}\right)$

$= \dfrac{d}{dx}\left((x^2 + 2x)^{\frac{1}{3}}\right)$

$= \dfrac{1}{3}(x^2 + 2x)^{-\frac{2}{3}}(2x + 2)$ (chain rule)

$= \dfrac{2x + 2}{3\sqrt[3]{(x^2 + 2x)^2}}$

Summary 4F

For any non-zero rational number $r = \dfrac{p}{q}$, if $f(x) = x^r$, then $f'(x) = rx^{r-1}$.

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Exercise 4F

Example 22

1 Differentiate $2x^{\frac{1}{2}}$ by first principles.

Example 23a

2 Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{5}}$ **b** $x^{\frac{5}{2}}$ **c** $x^{\frac{5}{2}} - x^{\frac{3}{2}},\ x > 0$

d $3x^{\frac{1}{2}} - 4x^{\frac{5}{3}}$ **e** $x^{-\frac{6}{7}}$ **f** $x^{-\frac{1}{4}} + 4x^{\frac{1}{2}}$

3 Find the gradient of the tangent to the curve for each of the following at the stated value for x:

a $f(x) = x^{\frac{1}{3}}$ where $x = 27$ **b** $f(x) = x^{\frac{1}{3}}$ where $x = -8$

c $f(x) = x^{\frac{2}{3}}$ where $x = 27$ **d** $f(x) = x^{\frac{5}{4}}$ where $x = 16$

Example 23b

4 Find the derivative of each of the following with respect to x:

a $\sqrt{2x+1}$ **b** $\sqrt{4-3x}$ **c** $\sqrt{x^2+2}$

d $\sqrt[3]{4-3x}$ **e** $\dfrac{x^2+2}{\sqrt{x}}$ **f** $3\sqrt{x}(x^2+2x)$

5 **a** Show that $\dfrac{d}{dx}\left(\sqrt{x^2 \pm a^2}\right) = \dfrac{x}{\sqrt{x^2 \pm a^2}}$.

b Show that $\dfrac{d}{dx}\left(\sqrt{a^2 - x^2}\right) = \dfrac{-x}{\sqrt{a^2 - x^2}}$.

6 If $y = (x + \sqrt{x^2+1})^2$, show that $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{x^2+1}}$.

7 Find the derivative with respect to x of each of the following:

a $\sqrt{x^2+2}$ **b** $\sqrt[3]{x^2-5x}$ **c** $\sqrt[5]{x^2+2x}$

4G Differentiation of e^x

In this section we investigate the derivative of functions of the form $f(x) = a^x$. We will see that Euler's number e has the special property that $f'(x) = f(x)$ where $f(x) = e^x$.

First consider $f(x) = 2^x$.

To find the derivative of f we recall that:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{2^{x+h}-2^x}{h} \\ &= 2^x\lim_{h\to 0}\frac{2^h-1}{h} \\ &= 2^x f'(0) \end{aligned}$$

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We can investigate this limit numerically to find that $f'(0) \approx 0.6931$ and therefore

$$f'(x) \approx 0.6931 \times 2^x$$

Now consider $g(x) = 3^x$. Then, as for f, it may be shown that $g'(x) = 3^x g'(0)$. We find $g'(0) \approx 1.0986$ and hence

$$g'(x) \approx 1.0986 \times 3^x$$

The question now arises:

> Can we find a number b between 2 and 3 such that, if $f(x) = b^x$, then $f'(0) = 1$ and therefore $f'(x) = b^x$?

Using a calculator or a spreadsheet, we can investigate the limit as $h \to 0$ of $\dfrac{b^h - 1}{h}$, for various values of b between 2 and 3.

This investigation is carried out in the spreadsheet shown on the right.

Start by taking values for b between 2.71 and 2.72 (first table) and finding $f'(0)$ for each of these values. From these results it may be seen that the required value of b lies between 2.718 and 2.719.

The investigation is continued with values of b between 2.718 and 2.719 (second table). From this the required value of b is seen to lie between 2.7182 and 2.7183.

b	$f'(0)$
2.710	0.996 949
2.711	0.997 318
2.712	0.997 686
2.713	0.998 055
2.714	0.998 424
2.715	0.998 792
2.716	0.999 160
2.717	0.999 528
2.718	0.999 896
2.719	1.000 264
2.720	1.000 632

b	$f'(0)$
2.7180	0.999 896
2.7181	0.999 933
2.7182	0.999 970
2.7183	1.000 007
2.7184	1.000 043
2.7185	1.000 080
2.7186	1.000 117
2.7187	1.000 154
2.7188	1.000 191
2.7189	1.000 227
2.7190	1.000 264

The required value of b is in fact Euler's number e, which was introduced in Chapter 2.

Our results can be recorded:

> For $f(x) = e^x$, $f'(x) = e^x$.

Next consider $y = e^{kx}$ where $k \in \mathbb{R}$. The chain rule can be used to find the derivative:

Let $u = kx$. Then $y = e^u$. The chain rule yields

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot k \\ &= ke^{kx}\end{aligned}$$

> For $f(x) = e^{kx}$, $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

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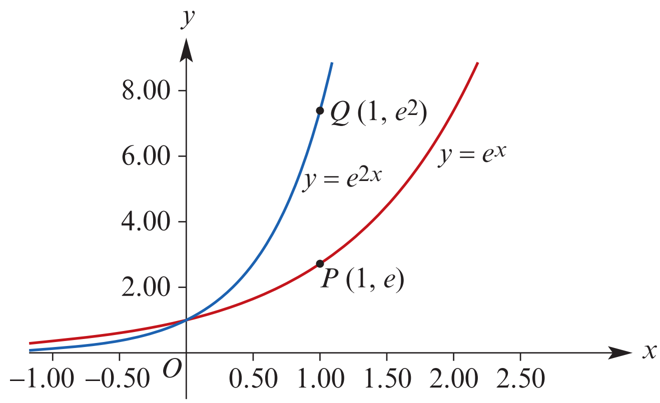
The graph illustrates the case where $k = 2$:

- the gradient of $y = e^x$ at the point $P(1, e)$ is e
- the gradient of $y = e^{2x}$ at the point $Q(1, e^2)$ is $2e^2$.



Example 24

Find the derivative of each of the following with respect to x:

a e^{3x} **b** e^{-2x} **c** e^{2x+1} **d** $\dfrac{1}{e^{2x}} + e^{3x}$

Solution

a Let $y = e^{3x}$. Then $\dfrac{dy}{dx} = 3e^{3x}$.

b Let $y = e^{-2x}$. Then $\dfrac{dy}{dx} = -2e^{-2x}$.

c Let $y = e^{2x+1}$. Then

$$\begin{aligned} y &= e^{2x} \cdot e \qquad \text{(index laws)} \\ &= e \cdot e^{2x} \\ \therefore \quad \frac{dy}{dx} &= 2e \cdot e^{2x} \\ &= 2e^{2x+1} \end{aligned}$$

d Let $y = \dfrac{1}{e^{2x}} + e^{3x}$. Then

$$\begin{aligned} y &= e^{-2x} + e^{3x} \\ \therefore \quad \frac{dy}{dx} &= -2e^{-2x} + 3e^{3x} \end{aligned}$$

Example 25

Find the derivative of each of the following with respect to x:

a e^{x^2} **b** e^{x^2+4x}

Solution

a Let $y = e^{x^2}$ and $u = x^2$.
Then $y = e^u$ and the chain rule yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot 2x \\ &= 2xe^{x^2} \end{aligned}$$

b Let $y = e^{x^2+4x}$ and $u = x^2 + 4x$.
Then $y = e^u$ and the chain rule yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u(2x + 4) \\ &= (2x + 4)\, e^{x^2+4x} \end{aligned}$$

In general, for $h(x) = e^{f(x)}$, the chain rule gives $h'(x) = f'(x)\, e^{f(x)}$.

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Example 26

Find the gradient of the tangent to the curve $y = e^{2x} + 4$ at the point:

a $(0, 5)$ **b** $(1, e^2 + 4)$

Solution

We have $\dfrac{dy}{dx} = 2e^{2x}$.

a When $x = 0$, $\dfrac{dy}{dx} = 2$.

The gradient at $(0, 5)$ is 2.

b When $x = 1$, $\dfrac{dy}{dx} = 2e^2$.

The gradient at $(1, e^2 + 4)$ is $2e^2$.

Example 27

For each of the following, first find the derivative with respect to x. Then evaluate the derivative at $x = 2$, given that $f(2) = 0$, $f'(2) = 4$ and $f'(e^2) = 5$.

a $e^{f(x)}$ **b** $f(e^x)$

Solution

a Let $y = e^{f(x)}$ and $u = f(x)$. Then $y = e^u$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= e^u f'(x)\\ &= e^{f(x)} f'(x)\end{aligned}$$

When $x = 2$, $\dfrac{dy}{dx} = e^0 \times 4 = 4$.

b Let $y = f(e^x)$ and $u = e^x$. Then $y = f(u)$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= f'(u)\cdot e^x\\ &= f'(e^x)\cdot e^x\end{aligned}$$

When $x = 2$, $\dfrac{dy}{dx} = f'(e^2)\cdot e^2 = 5e^2$.

Summary 4G

For $f(x) = e^{kx}$, $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

Exercise 4G

Example 24

1 Find the derivative of each of the following with respect to x:

a e^{5x} **b** $7e^{-3x}$ **c** $3e^{-4x} + e^x - x^2$

d $\dfrac{e^{2x} - e^x + 1}{e^x}$ **e** $\dfrac{4e^{2x} - 2e^x + 1}{2e^{2x}}$ **f** $e^{2x} + e^4 + e^{-2x}$

Example 25

2 Find the derivative of each of the following with respect to x:

a e^{-2x^3} **b** $e^{x^2} + 3x + 1$ **c** $e^{x^2-4x} + 3x + 1$

d $e^{x^2-2x+3} - x$ **e** $e^{\frac{1}{x}}$, $x \neq 0$ **f** $e^{x^{\frac{1}{2}}}$

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Example 26

3 Find the gradient of the tangent to the curve $y = e^{\frac{x}{2}} + 4x$ at the point:

a $(0, 1)$ **b** $(1, e^{\frac{1}{2}} + 4)$

4 Find the gradient of the tangent to the curve $y = e^{x^2+3x} + 2x$ at the point:

a $(0, 1)$ **b** $(1, e^4 + 2)$

Example 27

5 Find the derivative with respect to x of:

a $e^{2f(x)}$ **b** $f(e^{2x})$

6 Find the derivative with respect to x of:

a $(e^{2x} - 1)^4$ **b** $e^{\sqrt{x}}$

c $\sqrt{e^x - 1}$ **d** $e^{x^{\frac{2}{3}}}$

e $e^{(x-1)(x-2)}$ **f** e^{e^x}

4H Differentiation of the natural logarithm function

For the function with rule $f(x) = e^x$, we have seen that $f'(x) = e^x$.

This will be used to find the derivative of $g(x) = \ln(kx)$, $\{ x \in \mathbb{R} : x > 0 \}$ where $k > 0$.

Let $y = \ln(kx)$ and solve for x:

$$e^y = kx$$

$$\therefore \quad x = \frac{1}{k}e^y$$

From our observation above:

$$\frac{dx}{dy} = \frac{1}{k}e^y$$

Since $e^y = kx$, this gives

$$\frac{dx}{dy} = \frac{kx}{k} = x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

Let $f(x) = \ln(kx)$, $\{ x \in \mathbb{R} : x > 0 \}$ where $k > 0$.

Then $f'(x) = \dfrac{1}{x}$, $\{ x \in \mathbb{R} : x > 0 \}$.

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Example 28

Find the derivative of each of the following with respect to x:

a $\ln(5x)$, $x > 0$

b $\ln(5x+3)$, $x > \frac{-3}{5}$

Solution

a Let $y = \ln(5x)$ for $x > 0$.

Then $\frac{dy}{dx} = \frac{1}{x}$.

Alternatively, let $u = 5x$. Then $y = \ln u$ and the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 5$$
$$= \frac{5}{u}$$
$$= \frac{1}{x}$$

b Let $y = \ln(5x+3)$ for $x > \frac{-3}{5}$.

Let $u = 5x + 3$. Then $y = \ln u$ and the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times 5$$
$$= \frac{5}{u}$$
$$= \frac{5}{5x+3}$$

In general, if $y = \ln(ax+b)$ for $x > \frac{-b}{a}$, then $\frac{dy}{dx} = \frac{a}{ax+b}$.

Note: Let $y = \ln(-x)$, $x < 0$. Using the chain rule with $u = -x$ gives $\frac{dy}{dx} = \frac{1}{-x} \times (-1) = \frac{1}{x}$.

Example 29

Differentiate each of the following with respect to x:

a $\ln(x^2+2)$

b $(\ln x)^2$, $x > 0$

Solution

a We use the chain rule.

Let $y = \ln(x^2+2)$ and $u = x^2 + 2$.

Then $y = \ln u$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{u} \cdot 2x$$
$$= \frac{2x}{x^2+2}$$

b We use the chain rule.

Let $y = (\ln x)^2$ and $u = \ln x$.

Then $y = u^2$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 2u \cdot \frac{1}{x}$$
$$= \frac{2\ln x}{x}$$

Summary 4H

- If $y = \ln(ax+b)$ for $x > \frac{-b}{a}$, then $\frac{dy}{dx} = \frac{a}{ax+b}$.
- If $h(x) = \ln(f(x))$, then the chain rule gives $h'(x) = \frac{f'(x)}{f(x)}$.

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Exercise 4H

Example 28

1 Find the derivative of each of the following with respect to x:

a $y = 2\ln x$ **b** $y = 2\ln(2x)$ **c** $y = x^2 + 3\ln(2x)$

d $y = 3\ln x + \frac{1}{x}$ **e** $y = 3\ln(4x) + x$ **f** $y = \ln(x + 1)$

g $y = \ln(2x + 4)$ **h** $y = \ln(3x - 1)$ **i** $y = \ln(6x - 1)$

Example 29

2 Find the derivative of each of the following with respect to x:

a $y = \ln(x^3)$ **b** $y = (\ln x)^3$ **c** $y = \ln(x^2 + x - 1)$

d $y = \ln(x^3 + x^2)$ **e** $y = \ln((2x + 3)^2)$ **f** $y = \ln((3 - 2x)^2)$

3 For each of the following, find $f'(x)$:

a $f(x) = \ln(x^2 + 1)$

b $f(x) = \ln(e^x)$

4 Find the y-coordinate and the gradient of the tangent to the curve at the point corresponding to the given value of x:

a $y = \ln x,\ x > 0$, at $x = e$ **b** $y = \ln(x^2 + 1)$ at $x = e$

c $y = \ln(-x),\ x < 0$, at $x = -e$ **d** $y = x + \ln x$ at $x = 1$

e $y = \ln(x^2 - 2x + 2)$ at $x = 1$ **f** $y = \ln(2x - 1)$ at $x = \frac{3}{2}$

5 Find $f'(1)$ if $f(x) = \ln(\sqrt{x^2 + 1})$.

6 Differentiate $\ln(1 + x + x^2)$.

7 If $f(x) = \ln(x^2 + 1)$, find $f'(3)$.

8 Given that $f(0) = 2$ and $f'(0) = 4$, find $\frac{d}{dx}\big(\ln(f(x))\big)$ when $x = 0$.

4I Derivatives of trigonometric functions

In this section we find the derivatives of the sine, cosine and tangent functions.

In the proofs for sine and tangent, we will be using the addition formulas for sine, which were introduced in Mathematical Methods Units 1 & 2:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Note: The derivatives that we establish for $\sin\theta$, $\cos\theta$ and $\tan\theta$ only apply when the angle θ is measured in radians.

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The derivative of sin($k\theta$)

If $f(\theta) = \sin\theta$, then $f'(\theta) = \cos\theta$.

Proof Consider points $P(\theta, \sin\theta)$ and $Q(\theta + h, \sin(\theta + h))$ on the graph of $f(\theta) = \sin\theta$. The gradient of the secant PQ is

$$\frac{\sin(\theta + h) - \sin\theta}{h} = \frac{\sin\theta\cos h + \cos\theta\sin h - \sin\theta}{h}$$
$$= \frac{\sin\theta \cdot (\cos h - 1)}{h} + \frac{\cos\theta\sin h}{h}$$

We now consider what happens as $h \to 0$. We use two limit results (the second limit is proved below and the first limit then follows using a trigonometric identity):

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \quad \text{and} \quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

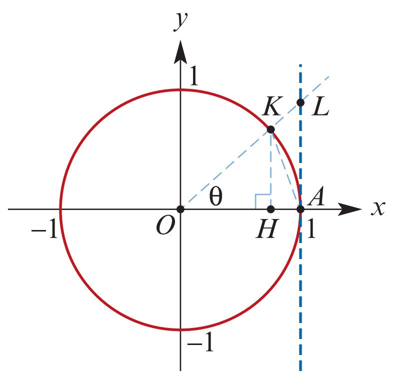
Therefore

$$f'(\theta) = \lim_{h\to 0}\left(\frac{\sin\theta \cdot (\cos h - 1)}{h} + \frac{\cos\theta\sin h}{h}\right)$$
$$= \sin\theta \times 0 + \cos\theta \times 1$$
$$= \cos\theta$$

We now prove the following result.

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proof Let K be a point on the unit circle as shown, and let $\angle KOH = \theta$. The coordinates of K are $(\cos\theta, \sin\theta)$. Point H is on the x-axis such that $\angle KHO$ is a right angle.



Draw a tangent to the circle at $A(1, 0)$. The line OK intersects this tangent at $L(1, \tan\theta)$.

The area of sector OAK is $\frac{1}{2}\theta$.

Thus $\quad$ area $\triangle OAK \leq \frac{1}{2}\theta \leq$ area $\triangle OAL$

i.e. $\quad \frac{1}{2}OA \cdot HK \leq \frac{1}{2}\theta \leq \frac{1}{2}OA \cdot AL$

This implies that $\sin\theta \leq \theta \leq \tan\theta$.

For $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta > 0$, and so we can divide both inequalities by $\sin\theta$ to obtain

$$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

As θ approaches 0, the value of $\cos\theta$ approaches 1, and so $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

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We now turn our attention to the function $f(\theta) = \sin(k\theta)$. The graph of $y = \sin(k\theta)$ is obtained from the graph of $y = \sin\theta$ by a dilation of factor $\frac{1}{k}$ from the y-axis (and so this immediately suggests that the gradient will change by a factor of k).

We use the chain rule to determine $f'(\theta)$.

Let $y = \sin(k\theta)$ and let $u = k\theta$. Then $y = \sin u$ and therefore

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} = \cos u \cdot k = k\cos(k\theta)$$

For $f(\theta) = \sin(k\theta)$

$f'(\theta) = k\cos(k\theta)$

The derivative of cos($k\theta$)

We next find the derivative of $\cos(k\theta)$. We first note the following:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

These results will be used in the following way.

Let $y = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

Now let $u = \frac{\pi}{2} - \theta$. Then $y = \sin u$. The chain rule gives

$$\begin{aligned}\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} &= \cos u \cdot (-1)\\ &= -\cos\left(\frac{\pi}{2} - \theta\right)\\ &= -\sin\theta\end{aligned}$$

We have the following results:

- For $f(\theta) = \cos\theta$
  $f'(\theta) = -\sin\theta$
- For $f(\theta) = \cos(k\theta)$
  $f'(\theta) = -k\sin(k\theta)$

The derivative of tan($k\theta$)

For convenience, we introduce a new function, called **secant**, given by

$$\sec\theta = \frac{1}{\cos\theta}$$

We can write $\sin^n\theta = (\sin\theta)^n$ and $\cos^n\theta = (\cos\theta)^n$.

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Here we find the derivative of $\tan\theta$ by first principles. In Section 4K we show another method.

If $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Proof Consider points $P(\theta, \tan\theta)$ and $Q(\theta + h, \tan(\theta + h))$ on the graph of $f(\theta) = \tan\theta$. The gradient of the secant PQ is

$$\begin{aligned}\frac{\tan(\theta+h)-\tan\theta}{h} &= \left(\frac{\sin(\theta+h)}{\cos(\theta+h)} - \frac{\sin\theta}{\cos\theta}\right)\times\frac{1}{h}\\ &= \left(\frac{\sin(\theta+h)\cos(\theta)-\cos(\theta+h)\sin(\theta)}{\cos(\theta+h)\cos(\theta)}\right)\times\frac{1}{h}\\ &= \frac{\sin h}{h\cos(\theta+h)\cos(\theta)}\end{aligned}$$

We now consider what happens as $h \to 0$:

$$\lim_{h\to 0}\cos(\theta+h) = \cos\theta \quad\text{and}\quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Therefore

$$f'(\theta) = \lim_{h\to 0}\left(\frac{\sin h}{h\cos(\theta+h)\cos(\theta)}\right) = \frac{1}{\cos^2\theta} = \sec^2\theta$$

We can use the chain rule to show that, if $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Example 30

Find the derivative with respect to θ of each of the following:

a $\sin(2\theta)$ **b** $\tan(3\theta)$ **c** $\sin^2(2\theta)$

d $\sin^2(2\theta + 1)$ **e** $\cos^3(4\theta + 1)$ **f** $\tan(3\theta^2 + 1)$

Solution

a Let $y = \sin(2\theta)$. Then $\dfrac{dy}{d\theta} = 2\cos(2\theta)$.

b Let $y = \tan(3\theta)$. Then $\dfrac{dy}{d\theta} = 3\sec^2(3\theta)$.

c Let $y = \sin^2(2\theta)$ and $u = \sin(2\theta)$.
Then $y = u^2$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= 2u\cdot 2\cos(2\theta)\\ &= 4u\cos(2\theta)\\ &= 4\sin(2\theta)\cos(2\theta)\end{aligned}$$

d Let $y = \sin^2(2\theta + 1)$ and $u = \sin(2\theta + 1)$.
Then $y = u^2$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= 2u\cdot 2\cos(2\theta+1)\\ &= 4\sin(2\theta+1)\cos(2\theta+1)\end{aligned}$$

e Let $y = \cos^3(4\theta + 1)$ and $u = \cos(4\theta + 1)$.
Then $y = u^3$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= 3u^2\cdot(-4)\sin(4\theta+1)\\ &= -12\cos^2(4\theta+1)\sin(4\theta+1)\end{aligned}$$

f Let $y = \tan(3\theta^2 + 1)$ and $u = 3\theta^2 + 1$.
Then $y = \tan u$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= \sec^2 u\cdot 6\theta\\ &= 6\theta\sec^2(3\theta^2+1)\end{aligned}$$

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Example 31

Find the y-coordinate and the gradient of the tangent at the points on the following curves corresponding to the given values of θ:

a $y = \cos\theta$ at $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$

b $y = \tan\theta$ at $\theta = 0$ and $\theta = \frac{\pi}{4}$

Solution

a Let $y = \cos\theta$. Then $\frac{dy}{d\theta} = -\sin\theta$.

When $\theta = \frac{\pi}{4}$, we have $y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\frac{dy}{d\theta} = -\sin\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$.

So the gradient at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is $\frac{-1}{\sqrt{2}}$.

When $\theta = \frac{\pi}{2}$, we have $y = 0$ and $\frac{dy}{d\theta} = -1$. The gradient at $\left(\frac{\pi}{2}, 0\right)$ is -1.

b Let $y = \tan\theta$. Then $\frac{dy}{d\theta} = \sec^2\theta$.

When $\theta = 0$, we have $y = 0$ and $\frac{dy}{d\theta} = 1$. The gradient at $(0, 0)$ is 1.

When $\theta = \frac{\pi}{4}$, we have $y = 1$ and $\frac{dy}{d\theta} = 2$. The gradient at $\left(\frac{\pi}{4}, 1\right)$ is 2.

Summary 4I

- If $f(\theta) = \sin(k\theta)$, then $f'(\theta) = k\cos(k\theta)$.
- If $f(\theta) = \cos(k\theta)$, then $f'(\theta) = -k\sin(k\theta)$.
- If $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Exercise 4I

Example 30

1 Find the derivative with respect to x of each of the following:

a $\sin(5x)$ **b** $\cos(5x)$ **c** $\tan(5x)$ **d** $\sin^2 x$ **e** $\tan(3x+1)$

f $\cos(x^2+1)$ **g** $\sin^2\left(x - \frac{\pi}{4}\right)$ **h** $\cos^2\left(x - \frac{\pi}{3}\right)$ **i** $\sin^3\left(2x + \frac{\pi}{6}\right)$ **j** $\cos^3\left(2x - \frac{\pi}{4}\right)$

Example 31

2 Find the y-coordinate and the gradient of the tangent at the points on the following curves corresponding to the given values of x:

a $y = \sin(2x)$ at $x = \frac{\pi}{8}$ **b** $y = \sin(3x)$ at $x = \frac{\pi}{6}$ **c** $y = 1 + \sin(3x)$ at $x = \frac{\pi}{6}$

d $y = \cos^2(2x)$ at $x = \frac{\pi}{4}$ **e** $y = \sin^2(2x)$ at $x = \frac{\pi}{4}$ **f** $y = \tan(2x)$ at $x = \frac{\pi}{8}$

3 For each of the following, find $f'(x)$:

a $f(x) = 5\cos x - 2\sin(3x)$ **b** $f(x) = \cos x + \sin x$

c $f(x) = \sin x + \tan x$ **d** $f(x) = \tan^2 x$

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4 Find the derivative of each of the following. (Change degrees to radians first.)

a $2\cos x°$

b $3\sin x°$

c $\tan(3x)°$

5 **a** If $y = -\ln(\cos x)$, find $\dfrac{dy}{dx}$.

b If $y = -\ln(\tan x)$, find $\dfrac{dy}{dx}$.

6 **a** If $y = e^{2\sin x}$, find $\dfrac{dy}{dx}$.

b If $y = e^{\cos(2x)}$, find $\dfrac{dy}{dx}$.

4J The product rule

In the next two sections, we introduce two more rules for differentiation. The first of these is the **product rule**.

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

For example, consider $F(x) = (x^2 + 3x)(4x + 5)$. Then F is the product of two functions f and g, where $f(x) = x^2 + 3x$ and $g(x) = 4x + 5$. The product rule gives:

$$\begin{aligned} F'(x) &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \\ &= (x^2 + 3x) \cdot 4 + (4x + 5) \cdot (2x + 3) \\ &= 4x^2 + 12x + 8x^2 + 22x + 15 \\ &= 12x^2 + 34x + 15 \end{aligned}$$

This could also have been found by multiplying $x^2 + 3x$ by $4x + 5$ and then differentiating.

The product rule (function notation)

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Proof By the definition of the derivative of F, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} \end{aligned}$$

Adding and subtracting $f(x+h)\,g(x)$:

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x) + [f(x+h)\,g(x) - f(x+h)\,g(x)]}{h} \\ &= \lim_{h\to 0} \left[f(x+h) \cdot \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \cdot \left(\frac{f(x+h) - f(x)}{h} \right) \right] \end{aligned}$$

Since f and g are differentiable, we obtain

$$\begin{aligned} F'(x) &= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \left(\frac{g(x+h) - g(x)}{h} \right) + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

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We can state the product rule in Leibniz notation and give a geometric interpretation.

The product rule (Leibniz notation)

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In the following figure, the white region represents $y = uv$ and the shaded region δy, as explained below.

δv	$u\delta v$	$\delta u \delta v$
v	uv	$v\delta u$
	u	δu

$$\begin{aligned}\delta y &= (u + \delta u)(v + \delta v) - uv \\ &= uv + v\delta u + u\delta v + \delta u\delta v - uv \\ &= v\delta u + u\delta v + \delta u\delta v\end{aligned}$$

$$\therefore \quad \frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x$$

In the limit, as $\delta x \to 0$, we have

$$\frac{\delta u}{\delta x} = \frac{du}{dx}, \qquad \frac{\delta v}{\delta x} = \frac{dv}{dx} \qquad \text{and} \qquad \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Therefore

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example 32

Differentiate each of the following with respect to x:

a $(2x^2 + 1)(5x^3 + 16)$

b $x^3(3x - 5)^4$

Solution

a Let $y = (2x^2 + 1)(5x^3 + 16)$. Let $u = 2x^2 + 1$ and $v = 5x^3 + 16$.

Then $\dfrac{du}{dx} = 4x$ and $\dfrac{dv}{dx} = 15x^2$.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (2x^2 + 1) \cdot 15x^2 + (5x^3 + 16) \cdot 4x \\ &= 30x^4 + 15x^2 + 20x^4 + 64x \\ &= 50x^4 + 15x^2 + 64x\end{aligned}$$

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b Let $y = x^3(3x-5)^4$. Let $u = x^3$ and $v = (3x-5)^4$.

Then $\dfrac{du}{dx} = 3x^2$ and $\dfrac{dv}{dx} = 12(3x-5)^3$, using the chain rule.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} &= 12x^3(3x-5)^3 + (3x-5)^4 \cdot 3x^2\\ &= (3x-5)^3[12x^3 + 3x^2(3x-5)]\\ &= (3x-5)^3[12x^3 + 9x^3 - 15x^2]\\ &= (3x-5)^3(21x^3 - 15x^2)\\ &= 3x^2(7x-5)(3x-5)^3\end{aligned}$$

Example 33

For $F(x) = x^{-3}(10x^2-5)^3$, $\{x \in \mathbb{R} : x \neq 0\}$ find $F'(x)$.

Solution

Let $f(x) = x^{-3}$ and $g(x) = (10x^2-5)^3$.

Then $f'(x) = -3x^{-4}$ and $g'(x) = 60x(10x^2-5)^2$ using the chain rule.

By the product rule:

$$\begin{aligned}F'(x) &= x^{-3} \cdot 60x(10x^2-5)^2 + (10x^2-5)^3 \cdot (-3x^{-4})\\ &= (10x^2-5)^2\,[60x^{-2} + (10x^2-5) \cdot (-3x^{-4})]\\ &= (10x^2-5)^2\left(\frac{60x^2 - 30x^2 + 15}{x^4}\right)\\ &= \frac{(10x^2-5)^2(30x^2+15)}{x^4}\end{aligned}$$

Example 34

Differentiate each of the following with respect to x:

a $e^x(2x^2+1)$

b $e^x\sqrt{x-1}$

Solution

a Use the product rule.

Let $y = e^x(2x^2+1)$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x(2x^2+1) + 4xe^x\\ &= e^x(2x^2+4x+1)\end{aligned}$$

b Use the product rule and the chain rule.

Let $y = e^x\sqrt{x-1}$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x\sqrt{x-1} + \frac{1}{2}e^x(x-1)^{-\frac{1}{2}}\\ &= e^x\sqrt{x-1} + \frac{e^x}{2\sqrt{x-1}}\\ &= \frac{2e^x(x-1) + e^x}{2\sqrt{x-1}}\\ &= \frac{2xe^x - e^x}{2\sqrt{x-1}}\end{aligned}$$

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Example 35

Find the derivative of each of the following with respect to x:

a $2x^2 \sin(2x)$ **b** $e^{2x} \sin(2x + 1)$ **c** $\cos(4x) \sin(2x)$

Solution

a Let $y = 2x^2 \sin(2x)$.

Applying the product rule:

$$\frac{dy}{dx} = 4x \sin(2x) + 4x^2 \cos(2x)$$

b Let $y = e^{2x} \sin(2x + 1)$.

Applying the product rule:

$$\frac{dy}{dx} = 2e^{2x} \sin(2x + 1) + 2e^{2x} \cos(2x + 1)$$

$$= 2e^{2x} [\sin(2x + 1) + \cos(2x + 1)]$$

c Let $y = \cos(4x) \sin(2x)$. Then the product rule gives

$$\frac{dy}{dx} = -4 \sin(4x) \sin(2x) + 2 \cos(2x) \cos(4x)$$

Summary 4J

The product rule

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Exercise 4J

Example 32
Example 33

1 Find the derivative of each of the following with respect to x, using the product rule:

a $(2x^2 + 6)(2x^3 + 1)$ **b** $3x^{\frac{1}{2}}(2x + 1)$ **c** $3x(2x - 1)^3$

d $4x^2(2x^2 + 1)^2$ **e** $(3x + 1)^{\frac{3}{2}}(2x + 4)$ **f** $(x^2 + 1)\sqrt{2x - 4}$

g $x^3(3x^2 + 2x + 1)^{-1}$ **h** $x^4\sqrt{2x^2 - 1}$ **i** $x^2\sqrt[3]{x^2 + 2x}$

j $x^{-2}(5x^2 - 4)^3$ **k** $x^{-3}(x^3 - 4)^2$ **l** $x^3\sqrt[5]{x^3 - x}$

Example 34

2 Find $f'(x)$ for each of the following:

a $f(x) = e^x(x^2 + 1)$ **b** $f(x) = e^{2x}(x^3 + 3x + 1)$

c $f(x) = e^{4x+1}(x + 1)^2$ **d** $f(x) = e^{-4x}\sqrt{x + 1},\ x \geq -1$

3 For each of the following, find $f'(x)$:

a $f(x) = x \ln x,\ x > 0$ **b** $f(x) = 2x^2 \ln x,\ x > 0$

c $f(x) = e^x \ln x,\ x > 0$ **d** $f(x) = x \ln(-x),\ x < 0$

4 Differentiate each of the following with respect to x:

a $x^4 e^{-2x}$ **b** e^{2x+3}

c $(e^{2x} + x)^{\frac{3}{2}}$ **d** $\frac{1}{x}e^x$

e $e^{\frac{1}{2}x^2}$ **f** $(x^2 + 2x + 2)e^{-x}$

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5 Find each of the following:

a $\frac{d}{dx}\left(e^x f(x)\right)$ **b** $\frac{d}{dx}\left(\frac{e^x}{f(x)}\right)$ **c** $\frac{d}{dx}\left(e^{f(x)}\right)$ **d** $\frac{d}{dx}\left(e^x(f(x))^2\right)$

Example 35

6 Differentiate each of the following with respect to x:

a $x^3 \cos x$ **b** $(1 + x^2)\cos x$ **c** $e^{-x}\sin x$ **d** $6x\cos x$ **e** $\sin(3x)\cos(4x)$

f $\tan(2x)\sin(2x)$ **g** $12x\sin x$ **h** $x^2 e^{\sin x}$ **i** $x^2\cos^2 x$ **j** $e^x \tan x$

7 For each of the following, find $f'(\pi)$:

a $f(x) = e^x \sin x$

b $f(x) = \cos^2(2x)$

8 Given that $f(1) = 2$ and $f'(1) = 4$, find the derivative of $f(x)\ln(x)$ when $x = 1$.

4K The quotient rule

Let $F(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

For example, if

$$F(x) = \frac{x^3 + 2x}{x^5 + 2}$$

then F can be considered as a quotient of two functions f and g, where $f(x) = x^3 + 2x$ and $g(x) = x^5 + 2$. The quotient rule gives

$$\begin{aligned} F'(x) &= \frac{(x^5 + 2)(3x^2 + 2) - (x^3 + 2x)5x^4}{(x^5 + 2)^2} \\ &= \frac{3x^7 + 6x^2 + 2x^5 + 4 - 5x^7 - 10x^5}{(x^5 + 2)^2} \\ &= \frac{-2x^7 - 8x^5 + 6x^2 + 4}{(x^5 + 2)^2} \end{aligned}$$

The quotient rule (function notation)

Let $F(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Proof The quotient rule can be proved from first principles, but instead we will use the product rule and the chain rule.

We can write $F(x) = f(x) \cdot h(x)$, where $h(x) = [g(x)]^{-1}$. Using the chain rule, we have

$$h'(x) = -[g(x)]^{-2} \cdot g'(x)$$

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Therefore, using the product rule, we obtain

$$\begin{aligned} F'(x) &= f(x) \cdot h'(x) + h(x) \cdot f'(x) \\ &= -f(x) \cdot [g(x)]^{-2} \cdot g'(x) + [g(x)]^{-1} \cdot f'(x) \\ &= [g(x)]^{-2}\big(-f(x) \cdot g'(x) + g(x) \cdot f'(x)\big) \\ &= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2} \end{aligned}$$

The quotient rule (Leibniz notation)

If $y = \dfrac{u}{v}$, where u and v are functions of x and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example 36

Find the derivative of $\dfrac{x-2}{x^2+4x+1}$ with respect to x.

Solution

Let $y = \dfrac{x-2}{x^2+4x+1}$. The quotient rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2+4x+1-(x-2)(2x+4)}{(x^2+4x+1)^2} \\ &= \frac{x^2+4x+1-(2x^2-8)}{(x^2+4x+1)^2} \\ &= \frac{-x^2+4x+9}{(x^2+4x+1)^2} \end{aligned}$$

Example 37

Differentiate each of the following with respect to x:

a $\dfrac{e^x}{e^{2x}+1}$

b $\dfrac{\sin x}{x+1}, \quad x \neq -1$

Solution

a Let $y = \dfrac{e^x}{e^{2x}+1}$.

Applying the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^{2x}+1)e^x - e^x \cdot 2e^{2x}}{(e^{2x}+1)^2} \\ &= \frac{e^{3x}+e^x-2e^{3x}}{(e^{2x}+1)^2} \\ &= \frac{e^x-e^{3x}}{(e^{2x}+1)^2} \end{aligned}$$

b Let $y = \dfrac{\sin x}{x+1}$ for $x \neq -1$.

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{(x+1)\cos x - \sin x}{(x+1)^2}$$

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Using the quotient rule to find the derivative of tan θ

Let $y = \tan\theta$. We write $y = \dfrac{\sin\theta}{\cos\theta}$ and apply the quotient rule to find the derivative:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{\cos\theta\cos\theta - \sin\theta\cdot(-\sin\theta)}{(\cos\theta)^2}\\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\\ &= \frac{1}{\cos^2\theta} \qquad \text{(by the Pythagorean identity)}\\ &= \sec^2\theta\end{aligned}$$

Summary 4K

The quotient rule

Let $F(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Exercise 4K

Example 36

1 Find the derivative of each of the following with respect to x:

a $\dfrac{x}{x+4}$ **b** $\dfrac{x^2-1}{x^2+1}$ **c** $\dfrac{x^{\frac{1}{2}}}{1+x}$ **d** $\dfrac{(x+2)^3}{x^2+1}$

e $\dfrac{x-1}{x^2+2}$ **f** $\dfrac{x^2+1}{x^2-1}$ **g** $\dfrac{3x^2+2x+1}{x^2+x+1}$ **h** $\dfrac{2x+1}{2x^3+2x}$

2 Find the y-coordinate and the gradient at the point on the curve corresponding to the given value of x:

a $y = (2x+1)^4x^2$ at $x = 1$ **b** $y = x^2\sqrt{x+1}$ at $x = 0$ **c** $y = x^2(2x+1)^{\frac{1}{2}}$ at $x = 0$

d $y = \dfrac{x}{x^2+1}$ at $x = 1$ **e** $y = \dfrac{2x+1}{x^2+1}$ at $x = 1$

3 For each of the following, find $f'(x)$:

a $f(x) = (x+1)\sqrt{x^2+1}$ **b** $f(x) = (x^2+1)\sqrt{x^3+1},\ x > -1$

c $f(x) = \dfrac{2x+1}{x+3},\ x \neq -3$

Example 37

4 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{e^x}{e^{3x}+3}$ **b** $f(x) = \dfrac{\cos x}{x+1},\ x \neq -1$

c $f(x) = \dfrac{\ln x}{x+1},\ x > 0$

5 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{\ln x}{x},\ x > 0$ **b** $f(x) = \dfrac{\ln x}{x^2+1},\ x > 0$

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6 Find $f'(x)$ for each of the following:

a $f(x) = \dfrac{e^{3x}}{e^{3x} + 3}$ **b** $f(x) = \dfrac{e^x + 1}{e^x - 1}$ **c** $f(x) = \dfrac{e^{2x} + 2}{e^{2x} - 2}$

7 For each of the following, find $f'(\pi)$:

a $f(x) = \dfrac{2x}{\cos x}$ **b** $f(x) = \dfrac{3x^2 + 1}{\cos x}$ **c** $f(x) = \dfrac{e^x}{\cos x}$ **d** $f(x) = \dfrac{\sin x}{x}$

4L Second derivatives

For the function f with rule $f(x)$, the derivative is denoted by f' and has rule $f'(x)$. This notation is extended to taking the derivative of the derivative: the new function is denoted by f'' and has rule $f''(x)$. This new function is known as the **second derivative**.

Consider the function g with rule $g(x) = 2x^3 - 4x^2$. The derivative has rule $g'(x) = 6x^2 - 8x$, and the second derivative has rule $g''(x) = 12x - 8$.

Note: The second derivative might not exist at a point even if the first derivative does.
For example, let $f(x) = x^{\frac{4}{3}}$. Then $f'(x) = \dfrac{4}{3}x^{\frac{1}{3}}$ and $f''(x) = \dfrac{4}{9}x^{-\frac{2}{3}}$.
We see that $f'(0) = 0$, but the second derivative $f''(x)$ is not defined at $x = 0$.

In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Example 38

Find the second derivative of each of the following with respect to x:

a $f(x) = 6x^4 - 4x^3 + 4x$ **b** $y = e^x \sin x$

Solution

a
$$f(x) = 6x^4 - 4x^3 + 4x$$
$$f'(x) = 24x^3 - 12x^2 + 4$$
$$f''(x) = 72x^2 - 24x$$

b
$$y = e^x \sin x$$
$$\frac{dy}{dx} = e^x \sin x + e^x \cos x \quad \text{(by the product rule)}$$
$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$
$$= 2e^x \cos x$$

A CAS calculator has the capacity to find the second derivative directly.

Using the TI-Nspire

- Press to obtain the second-derivative template $\frac{d^2}{d\square}\square$.
- Complete as shown.

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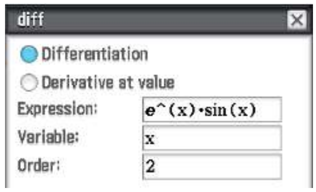
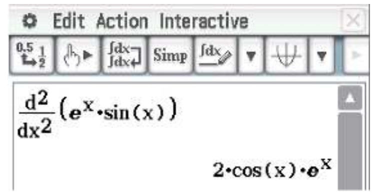
Using the Casio ClassPad

- Enter and highlight the expression $e^x \cdot \sin(x)$.
- Go to **Interactive** > **Calculation** > **diff** and change to order 2. Tap OK.





Example 39

If $f(x) = e^{2x}$, find $f''(0)$.

Solution

$$f(x) = e^{2x}$$
$$f'(x) = 2e^{2x}$$
$$f''(x) = 4e^{2x}$$

Therefore $f''(0) = 4e^0 = 4$.

Example 40

Consider $f(x) = x^3 - 2x^2 + 4x - 6$.

a Find $f''(x)$.

b Solve the equation $f''(x) = 0$ for x.

Solution

a
$$f(x) = x^3 - 2x^2 + 4x - 6$$
$$f'(x) = 3x^2 - 4x + 4$$
$$f''(x) = 6x - 4$$

b
$$f''(x) = 0$$
$$6x - 4 = 0$$
$$\therefore \quad x = \frac{2}{3}$$

Example 41

Consider $y = x^2 e^x$.

a Find $\dfrac{d^2y}{dx^2}$.

b Solve the equation $\dfrac{d^2y}{dx^2} = 0$ for x.

Solution

a
$$y = x^2 e^x$$
$$\frac{dy}{dx} = 2xe^x + x^2 e^x$$
$$\frac{d^2y}{dx^2} = 2e^x + 2xe^x + 2xe^x + x^2 e^x$$
$$= 2e^x + 4xe^x + x^2 e^x$$

b $\dfrac{d^2y}{dx^2} = 0$ implies

$$2e^x + 4xe^x + x^2 e^x = 0$$
$$e^x(2 + 4x + x^2) = 0$$
$$x^2 + 4x + 2 = 0$$

Therefore $x = -2 + \sqrt{2}$ or $x = -2 - \sqrt{2}$.

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Summary 4L

- The **second derivative** of a function f is the derivative of the derivative of f.
- For a function f with rule $f(x)$, the second derivative of f is denoted by f'' and has rule $f''(x)$.
- In Leibniz notation, the second derivative of y with respect to x is denoted by $\frac{d^2y}{dx^2}$.

Exercise 4L

Example 38

1 Find the second derivative of each of the following:

a $2x + 5$ **b** x^8 **c** $\sqrt{x}$ **d** $(2x + 1)^4$ **e** $\sin x$

f $\cos x$ **g** e^x **h** $\ln x$ **i** $\frac{1}{x + 1}$ **j** $\sin\left(2x + \frac{\pi}{4}\right)$

2 Find the second derivative of each of the following:

a $\sqrt{x^5}$ **b** $(x^2 + 3)^4$ **c** $\sin\left(\frac{x}{2}\right)$

d $3\cos(4x + 1)$ **e** $\frac{1}{2}e^{2x+1}$ **f** $\ln(2x + 1)$

g $x^4 + 3x^2 - 7x + 2$ **h** x^3e^x **i** $x\ln x$

3 For each of the following, find $f''(x)$:

a $f(x) = 6e^{3-2x}$ **b** $f(x) = -8e^{-0.5x^2}$ **c** $f(x) = e^{\ln x}$

d $f(x) = \ln(x^2 + 2x)$ **e** $f(x) = 2(1 - 3x)^5$ **f** $f(x) = e^{x^2}$

g $f(x) = \frac{x - 1}{x + 1}$ **h** $f(x) = \frac{1}{\sqrt{1 - x}}$ **i** $f(x) = 5\sin(3 - x)$

j $f(x) = \cos(1 - 3x)$ **k** $f(x) = \sin\left(\frac{x}{3}\right)$ **l** $f(x) = \cos\left(\frac{x}{4}\right)$

Example 39

4 For each of the following, find $f''(0)$:

a $f(x) = e^{\sin x}$ **b** $f(x) = e^{-\frac{1}{2}x^2}$ **c** $f(x) = \sqrt{1 - x^2}$ **d** $f(x) = \cos(x^2)$

Example 40

5 For each of the following, solve the equation $f''(x) = 0$ for x:

a $f(x) = 2x^3 + 4x^2$

b $f(x) = 5 - x - x^2 + 5x^3$

c $f(x) = x^4 - 3x^2 - 4x$

Example 41

6 For each of the following, solve the equation $\frac{d^2y}{dx^2} = 0$ for x:

a $y = 2xe^x$

b $y = x^2e^x - xe^x$

c $y = x^3e^x$

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Review

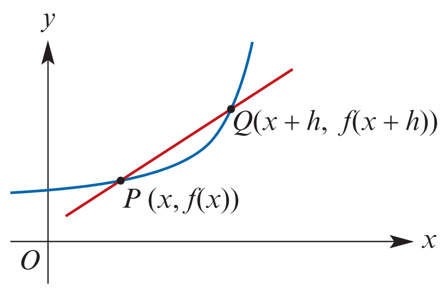
Chapter summary

Assignment

Nrich

The derivative

- The notation for the limit as h approaches 0 is $\lim_{h \to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
    $$\frac{f(x+h) - f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
    $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

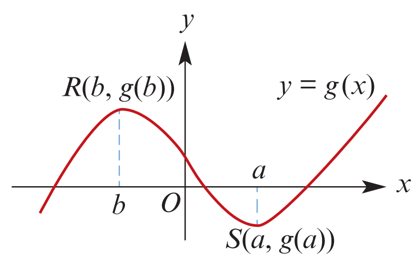


- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- At a point $(a, g(a))$ on the curve $y = g(x)$, the gradient is $g'(a)$.

  For the graph shown:
  - $g'(x) > 0$ for $x < b$ and for $x > a$
  - $g'(x) < 0$ for $b < x < a$
  - $g'(x) = 0$ for $x = b$ and for $x = a$.



Basic derivatives

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^n	nx^{n-1}	where n is a non-zero integer
x^a	ax^{a-1}	where $\{a \in \mathbb{R} : a \neq 0\}$
e^{kx}	ke^{kx}	
$\ln(kx)$	$\frac{1}{x}$	
$\sin(kx)$	$k\cos(kx)$	
$\cos(kx)$	$-k\sin(kx)$	
$\tan(kx)$	$k\sec^2(kx)$	Note: $\sec\theta = \frac{1}{\cos\theta}$

Rules for differentiation

- For $f(x) = k\,g(x)$, where k is a constant, $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- For $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of a sum is the sum of the derivatives.

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- The chain rule
  - If $q(x) = f(g(x))$, then $q'(x) = f'(g(x))\, g'(x)$
  - $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- The product rule
  - If $F(x) = f(x) \cdot g(x)$, then $F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
  - If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
- The quotient rule
  - If $F(x) = \dfrac{f(x)}{g(x)}$, then $F'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
  - If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

The second derivative

- The **second derivative** of a function f is the derivative of the derivative of f, and it is denoted by f''.
- In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Short-answer questions

1 For $y = x^2 + 1$:

a Find the average rate of change of y with respect to x over the interval $[3, 5]$.

b Find the instantaneous rate of change of y with respect to x at the point where $x = -4$.

2 Differentiate each of the following with respect to x:

a $x + \sqrt{1 - x^2}$ **b** $\dfrac{4x + 1}{x^2 + 3}$ **c** $\sqrt{1 + 3x}$ **d** $\dfrac{2 + \sqrt{x}}{x}$

e $(x - 9)\sqrt{x - 3}$ **f** $x\sqrt{1 + x^2}$ **g** $\dfrac{x^2 - 1}{x^2 + 1}$ **h** $\dfrac{x}{x^2 + 1}$

i $(2 + 5x^2)^{\frac{1}{3}}$ **j** $\dfrac{2x + 1}{x^2 + 2}$ **k** $(3x^2 + 2)^{\frac{2}{3}}$

3 For each of the following functions, find the gradient of the tangent to the curve at the point corresponding to the given x-value:

a $y = 3x^2 - 4$ at $x = -1$

b $y = \dfrac{x - 1}{x^2 + 1}$ at $x = 0$

c $y = (x - 2)^5$ at $x = 1$

d $y = (2x + 2)^{\frac{1}{3}}$ at $x = 3$

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4 Differentiate each of the following with respect to x:

a $\ln(x+2)$ **b** $\sin(3x+2)$

c $\cos\left(\dfrac{x}{2}\right)$ **d** e^{x^2-2x}

e $\ln(3-x)$ **f** $\sin(2\pi x)$

g $\sin^2(3x+1)$ **h** $\sqrt{\ln x},\ x>1$

i $\dfrac{2\ln(2x)}{x}$ **j** $x^2\sin(2\pi x)$

5 Differentiate each of the following with respect to x:

a $e^x\sin(2x)$ **b** $2x^2\ln x$ **c** $\dfrac{\ln x}{x^3}$ **d** $\sin(2x)\cos(3x)$

e $\dfrac{\sin(2x)}{\cos(2x)}$ **f** $\cos^3(3x+2)$ **g** $x^2\sin^2(3x)$

6 Find the gradient of each of the following curves at the stated value of x:

a $y=e^{2x}+1,\ x=1$ **b** $y=e^{x^2+1},\ x=0$

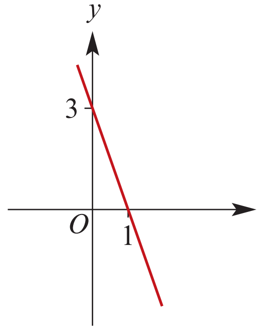
c $y=5e^{3x}+x^2,\ x=1$ **d** $y=5-e^{-x},\ x=0$

7 Differentiate each of the following with respect to x:

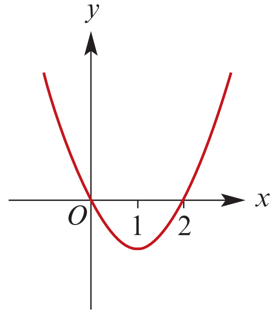
a e^{ax} **b** e^{ax+b} **c** e^{a-bx} **d** $be^{ax}-ae^{bx}$ **e** $\dfrac{e^{ax}}{e^{bx}}$

8 Sketch the graph of the derivative function for each of the following functions:

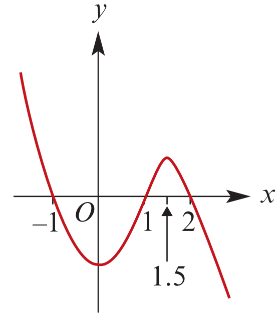
a



b



c



9 Find the derivative of $\left(4x+\dfrac{9}{x}\right)^2$ and find the values of x at which the derivative is zero.

10 **a** For $y=\dfrac{2x-3}{x^2+4}$, show that $\dfrac{dy}{dx}=\dfrac{8+6x-2x^2}{(x^2+4)^2}$.

b Find the values of x for which both y and $\dfrac{dy}{dx}$ are positive.

11 For each of the following, find $\dfrac{d^2y}{dx^2}$:

a $y=(2x-1)^3$ **b** $y=e^{x^2+1}$ **c** $y=x\sin x$

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Extended-response questions

1 **a** For functions f and g, which are defined and differentiable for all real numbers, it is known that:

- $f(1) = 6$, $g(1) = -1$, $g(6) = 7$ and $f(-1) = 8$
- $f'(1) = 6$, $g'(1) = -2$, $f'(-1) = 2$ and $g'(6) = -1$

Find:

i $(f \circ g)'(1)$ **ii** $(g \circ f)'(1)$

iii $(fg)'(1)$ **iv** $(gf)'(1)$

v $\left(\frac{f}{g}\right)'(1)$ **vi** $\left(\frac{g}{f}\right)'(1)$

b It is known that f is a cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

2 For a function f, which is differentiable for $\mathbb{R}$, it is known that:

- $f'(x) = 0$ for $x = 1$ and $x = 5$
- $f'(x) > 0$ for $x > 5$ and $x < 1$
- $f'(x) < 0$ for $1 < x < 5$
- $f(1) = 6$ and $f(5) = 1$

a For $y = f(x + 2)$, find the values of x for which:

i $\frac{dy}{dx} = 0$ **ii** $\frac{dy}{dx} > 0$

b Find the coordinates of the points on the graph of $y = f(x - 2)$ where $\frac{dy}{dx} = 0$.

c Find the coordinates of the points on the graph of $y = f(2x)$ where $\frac{dy}{dx} = 0$.

d Find the coordinates of the points on the graph of $y = f\left(\frac{x}{2}\right)$ where $\frac{dy}{dx} = 0$.

e Find the coordinates of the points on the graph of $y = 3f\left(\frac{x}{2}\right)$ where $\frac{dy}{dx} = 0$.

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5 Applications of differentiation

In this chapter

Syllabus references

Topic: The second derivative and applications of differentiation

Subtopics: 3.1.10 – 3.1.16

In this chapter we continue our study of differential calculus. There are two main aspects of this chapter. One is to apply our knowledge of the derivative to sketching graphs and solving maximum and minimum problems. The other is to see that the derivative can be used to define instantaneous rate of change.

The use of the derivative to determine instantaneous rates of change is a very important application of calculus. One of the first areas of applied mathematics to be studied in the seventeenth century was motion in a straight line. The problems of kinematics were the motivation for Newton's work on calculus.

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5A Tangents and normals

The derivative of a function is a new function that gives the measure of the gradient of the tangent at each point on the curve. Having the gradient, we can find the equation of the tangent line at a given point on the curve.

Suppose that (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

Example 1

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$, and so $\left(1, \frac{3}{2}\right)$ is a point on the tangent.

Since $\frac{dy}{dx} = 3x^2 + x$, the gradient of the tangent at $x = 1$ is 4.

Hence the equation of the tangent is

$$y - \frac{3}{2} = 4(x - 1)$$

i.e. $$y = 4x - \frac{5}{2}$$

The **normal** to a curve at a point on the curve is the line that passes through the point and is perpendicular to the tangent at that point.

Recall that two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

Thus, if a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Example 2

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ is on the normal.

Since $\frac{dy}{dx} = 3x^2 - 4x$, the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = 1$.

Hence the equation of the normal is

$$y - (-1) = 1(x - 1)$$

i.e. $$y = x - 2$$

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Example 3

Find the equation of the tangent to the curve with equation $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$ at the point on the graph where $x = 4$.

Solution

Let $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$. Then $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$.

When $x = 4$,

$$y = 4^{\frac{3}{2}} - 4 \times 4^{\frac{1}{2}} = 0$$

and $\frac{dy}{dx} = \frac{3}{2} \times 4^{\frac{1}{2}} - 2 \times 4^{-\frac{1}{2}} = 2$

Hence the equation of the tangent is

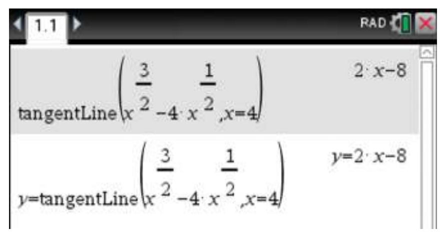
$$y - 0 = 2(x - 4)$$

i.e. $\quad y = 2x - 8$

Using the TI-Nspire

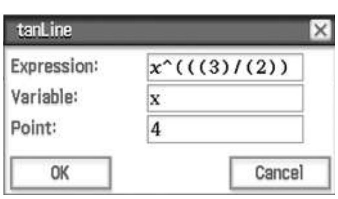
Use (menu) > **Calculus** > **Tangent Line** and complete as shown.

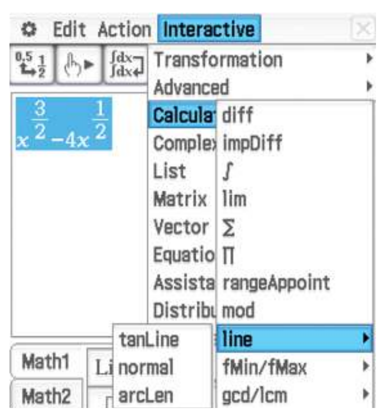
Note: The equation of the tangent can also be found in a **Graphs** application.



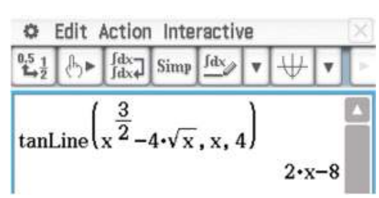
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$.
- Go to **Interactive** > **Calculation** > **line** > **tanLine**.
- Enter the x-value 4 in the tanLine window and tap `OK`.





- Write your answer as an equation: $y = 2x - 8$.



Note: You can also obtain the tangent line by sketching the graph and using **Analysis** > **Sketch** > **Tangent**.

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Example 4

Find the equation of the tangent to the graph of $y = \sin x$ at the point where $x = \frac{\pi}{3}$.

Solution

Let $y = \sin x$. Then $\frac{dy}{dx} = \cos x$. When $x = \frac{\pi}{3}$, $y = \frac{\sqrt{3}}{2}$ and $\frac{dy}{dx} = \frac{1}{2}$.

Therefore the equation of the tangent is

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

i.e. $$y = \frac{x}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

Example 5

Find the equations of the tangent and normal to the graph of $y = -\cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Solution

First find the gradient of the curve at this point:

$\frac{dy}{dx} = \sin x$ and so, when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 1$.

The equation of the tangent is

$$y - 0 = 1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = x - \frac{\pi}{2}$$

The gradient of the normal is -1 and therefore the equation of the normal is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = -x + \frac{\pi}{2}$$

The following example shows two situations in which we can view a graph as having a 'vertical tangent line' at a point where the derivative is not defined.

Example 6

Find the equation of the tangent to:

a $f(x) = x^{\frac{1}{3}}$ where $x = 0$ **b** $f(x) = x^{\frac{2}{3}}$ where $x = 0$.

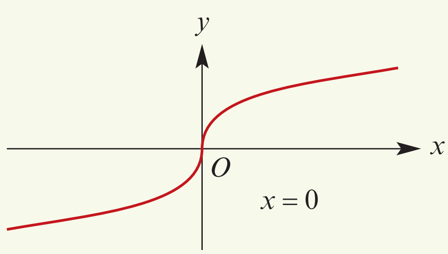
Solution

a The derivative of f is not defined at $x = 0$.

For $\{ x \in \mathbb{R} : x \neq 0 \}$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0$.

The graph has a **vertical tangent** at $x = 0$.



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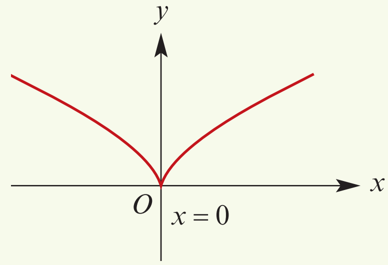
b $f(x) = x^{\frac{2}{3}}$

The derivative of f is not defined at $x = 0$.

For $\{ x \in \mathbb{R} : x \neq 0 \}$, $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0^+$ and $f'(x) \to -\infty$ as $x \to 0^-$.

There is a **cusp** at $x = 0$, and the graph of $y = f(x)$ has a **vertical tangent** at $x = 0$.



Summary 5A

- **Equation of a tangent** Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.
- **Gradient of normal** If a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Exercise 5A

Example 1

1 Find the equation of the tangent to the curve $y = x^2 - 1$ at the point $(2, 3)$.

Example 2

2 Find the equation of the normal to the curve $y = x^2 + 3x - 1$ at the point where the curve cuts the y-axis.

3 Find the equations of the normals to the curve $y = x^2 - 5x + 6$ at the points where it cuts the x-axis.

4 Find the equations of the tangent and the normal to the curve $y = (2x + 1)^9$ at the point $(0, 1)$.

5 Find the coordinates of the point on $y = x^2 - 5$ at which the curve has gradient 3. Hence find the value of c for which the line $y = 3x + c$ is tangent to $y = x^2 - 5$.

6 Find the equations of **i** the tangent and **ii** the normal at the point corresponding to the given x-value on each of the following curves:

a $y = x^2 - 2;\ x = 1$

b $y = x^2 - 3x - 1;\ x = 0$

c $y = \frac{1}{x};\ x = -1$

d $y = (x - 2)(x^2 + 1);\ x = -1$

e $y = \sqrt{3x + 1};\ x = 0$

f $y = \sqrt{x};\ x = 1$

g $y = x^{\frac{2}{3}} + 1;\ x = 1$

h $y = x^3 - 8x;\ x = 2$

i $y = x^3 - 3x^2 + 2;\ x = 2$

j $y = 2x^3 + x^2 - 4x + 1;\ x = 1$

Example 3

7 Use a CAS calculator to find the equation of the tangent to the curve with equation $y = 4x^{\frac{5}{2}} - 8x^{\frac{3}{2}}$ at the point on the graph where $x = 4$.

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8 Find the equation of the tangent at the point corresponding to the given x-value on each of the following curves:

a $y = \dfrac{x^2 - 1}{x^2 + 1}$; $x = 0$

b $y = \sqrt{3x^2 + 1}$; $x = 1$

c $y = \dfrac{1}{2x - 1}$; $x = 0$

d $y = \dfrac{1}{(2x - 1)^2}$; $x = 1$

Example 4
Example 5

9 Find the equation of the tangent to each of the following curves at the given x-value:

a $y = \sin(2x)$; $x = 0$

b $y = \cos(2x)$; $x = \dfrac{\pi}{2}$

c $y = \tan x$; $x = \dfrac{\pi}{4}$

d $y = \tan(2x)$; $x = 0$

e $y = \sin x + x\sin(2x)$; $x = 0$

f $y = x - \tan x$; $x = \dfrac{\pi}{4}$

10 For each function, find the equation of the tangent to the graph at the given value of x:

a $f(x) = e^x + e^{-x}$; $x = 0$

b $f(x) = \dfrac{e^x - e^{-x}}{2}$; $x = 0$

c $f(x) = x^2e^{2x}$; $x = 1$

d $f(x) = e^{\sqrt{x}}$; $x = 1$

e $f(x) = xe^{x^2}$; $x = 1$

f $f(x) = x^2e^{-x}$; $x = 2$

11 **a** Find the equation of the tangent and the normal to the graph of $f(x) = \ln x$ at the point $(1, 0)$.

b Find the equation of the tangent to the graph of $f(x) = \ln(2x)$ at the point $\left(\dfrac{1}{2}, 0\right)$.

c Find the equation of the tangent to the graph of $f(x) = \ln(kx)$ at the point $\left(\dfrac{1}{k}, 0\right)$, where $k \in (0, \infty)$.

Example 6

12 Find the equation of the tangent at the point where $y = 0$ for each of the following curves:

a $y = x^{\frac{1}{5}}$

b $y = x^{\frac{3}{5}}$

c $y = (x - 4)^{\frac{1}{3}}$

d $y = (x + 5)^{\frac{2}{3}}$

e $y = (2x + 1)^{\frac{1}{3}}$

f $y = (x + 5)^{\frac{4}{5}}$

13 The tangent to the curve with equation $y = \tan(2x)$ at the point where $x = \dfrac{\pi}{8}$ meets the y-axis at the point A. Find the distance OA, where O is the origin.

14 The tangent to the curve with equation $y = 2e^x$ at the point $(a, 2e^a)$ passes through the origin. Find the value of a.

15 The tangent to the curve with equation $y = \ln x$ at the point $(a, \ln a)$ passes through the origin. Find the value of a.

16 The tangent to the curve with equation $y = x^2 + 2x$ at the point $(a, a^2 + 2a)$ passes through the origin. Find the value of a.

17 The tangent to the curve with equation $y = x^3 + x$ at the point $(a, a^3 + a)$ passes through the point $(1, 1)$. Find the value of a.

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5B Rates of change

The derivative was defined geometrically in the previous chapter. However, the process of differentiation may also be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$:

- The **average rate of change** for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The **instantaneous rate of change** of f with respect to x when $x = a$ is given by $f'(a)$.

The derivative $\dfrac{dy}{dx}$ gives the instantaneous rate of change of y with respect to x.

- If $\dfrac{dy}{dx} > 0$, then y is increasing as x increases.
- If $\dfrac{dy}{dx} < 0$, then y is decreasing as x increases.

Example 7

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b
$$\begin{aligned}\text{Average rate of change} &= \frac{f(2+h) - f(2)}{2 + h - 2} \\ &= \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \frac{6h + h^2}{h} = 6 + h\end{aligned}$$

c The derivative is $f'(x) = 2x + 2$. When $x = 2$, the instantaneous rate of change is $f'(2) = 6$. This can also be seen from the result of part **b**.

Example 8

A balloon develops a microscopic leak and gradually decreases in volume. Its volume, V cm^3, at time t seconds is $V = 600 - 10t - \dfrac{1}{100}t^2$, $t \geq 0$.

a Find the rate of change of volume after:

  i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

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Solution

a $\frac{dV}{dt} = -10 - \frac{t}{50}$

i When $t = 10$, $\frac{dV}{dt} = -10\frac{1}{5}$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$, $\frac{dV}{dt} = -10\frac{2}{5}$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$. Consider $V = 0$.

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = 100(\sqrt{31} - 5) \quad \text{or} \quad t = -100(\sqrt{31} + 5)$$

The model may be suitable for $0 \le t \le 100(\sqrt{31} - 5)$.

Example 9

A pot of liquid is put on the stove. When the temperature of the liquid reaches 80°C, the pot is taken off the stove and placed on the kitchen bench. The temperature in the kitchen is 20°C. The temperature of the liquid, T°C, at time t minutes is given by

$$T = 20 + 60e^{-0.3t}$$

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

i $T = 80$ **ii** $T = 30$

Solution

a By rearranging $T = 20 + 60e^{-0.3t}$, we see that $e^{-0.3t} = \frac{T - 20}{60}$.

Now $T = 20 + 60e^{-0.3t}$

$$\therefore \quad \frac{dT}{dt} = -18e^{-0.3t}$$

Hence
$$\frac{dT}{dt} = -18\left(\frac{T-20}{60}\right) = -3\left(\frac{T-20}{10}\right) = 0.3(20 - T)$$

b **i** When $T = 80$, $\frac{dT}{dt} = 0.3(20 - 80) = -18$

The liquid is cooling at a rate of 18°C per minute.

ii When $T = 30$, $\frac{dT}{dt} = 0.3(20 - 30) = -3$

The liquid is cooling at a rate of 3°C per minute.

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Motion in a straight line

Position, velocity and acceleration were introduced for an object moving in a straight line in Mathematical Methods Units 1 & 2.

Position (x m) is specified with respect to a reference point O on the line. Velocity (v m/s) and acceleration (a m/s^2) are given by:

velocity $v = \dfrac{dx}{dt}$ acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

Example 10

A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by the formula $x = t^3 - 6t^2 + 9t$. Find:

a at what times and in what positions the particle will have zero velocity

b its acceleration at those instants

c its velocity when its acceleration is zero.

Solution

Velocity $v = \dfrac{dx}{dt} = 3t^2 - 12t + 9$

a When $v = 0$,

$$3(t^2 - 4t + 3) = 0$$
$$(t - 1)(t - 3) = 0$$
$$t = 1 \text{ or } t = 3$$

i.e. the velocity is zero when $t = 1$ and $t = 3$ and where $x = 4$ and $x = 0$.

b Acceleration $a = \dfrac{dv}{dt} = 6t - 12$

When $t = 1$, $a = -6$ m/s^2
When $t = 3$, $a = 6$ m/s^2

c The acceleration is zero when $6t - 12 = 0$, i.e. when $t = 2$.
When $t = 2$, the velocity $v = 3 \times 4 - 24 + 9$
$= -3$ m/s

Summary 5B

For the function with rule $f(x)$:

- The average rate of change for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The instantaneous rate of change of f with respect to x when $x = a$ is given by $f'(a)$.

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Skill-sheet

Exercise 5B

Example 7

1 For the function with rule $f(x) = 3x^2 + 6x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

2 Express each of the following in symbols:

a the rate of change of volume (V) with respect to time (t)

b the rate of change of surface area (S) of a sphere with respect to radius (r)

c the rate of change of volume (V) of a cube with respect to edge length (x)

d the rate of change of area (A) with respect to time (t)

e the rate of change of volume (V) of water in a glass with respect to depth of water (h)

Example 8

3 If your interest (I) in Mathematical Methods can be expressed as

$$I = \frac{4}{(t+1)^2}$$

where t is the time in days measured from the first day of Term 1, how fast is your interest waning when $t = 10$?

4 A reservoir is being emptied and the quantity of water, $V\text{ m}^3$, remaining in the reservoir t days after it starts to empty is given by

$$V(t) = 10^3(90 - t)^3$$

a At what rate is the reservoir being emptied at time t?

b How long does it take to empty the reservoir?

c What is the volume of water in the reservoir when $t = 0$?

d After what time is the reservoir being emptied at $3 \times 10^5\text{ m}^3\text{/day}$?

e Sketch the graph of $V(t)$ against t.

f Sketch the graph of $V'(t)$ against t.

5 A coffee percolator allows 1000 mL of water to flow into a filter in 20 minutes. The volume which has flowed into the filter at time t minutes is given by

$$V(t) = \frac{1}{160}\left(5t^4 - \frac{t^5}{5}\right), \quad 0 \le t \le 20$$

a At what rate is the water flowing into the filter at time t minutes?

b Sketch the graph of $\dfrac{dV}{dt}$ against t for $0 \le t \le 20$.

c When is the rate of flow greatest?

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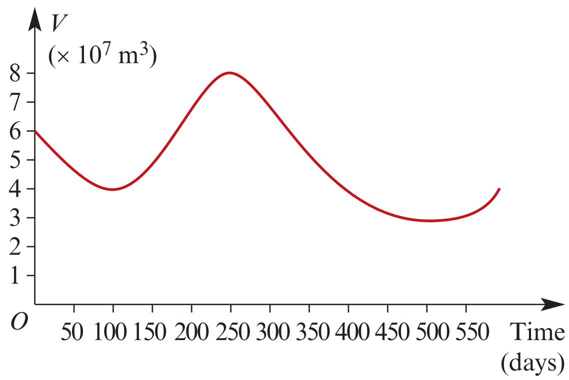
6 The graph shows the volume, V m^3, of water in a reservoir at time t days.



a At what times is the rate of flow from the reservoir 0 m^3/day?

b Find an estimate for the rate of flow at $t = 200$.

c Find the average rate of flow for the interval $[100, 250]$.

d State the times for which there is net flow into the reservoir.

7 A car tyre is inflated to a pressure of 30 units. Eight hours later it is found to have deflated to a pressure of 10 units. The pressure, P, at time t hours is given by

$$P = P_0 e^{-\lambda t}$$

a Find the values of P_0 and λ.

b At what time would the pressure be 8 units?

c Find the rate of loss of pressure at:

  i time $t = 0$

  ii time $t = 8$

Example 9

8 A liquid is heated to a temperature of 90°C and then allowed to cool in a room in which the temperature is 15°C. While the liquid is cooling, its temperature, T°C, at time t minutes is given by $T = 15 + 75e^{-0.3t}$.

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

  i $T = 90$

  ii $T = 60$

  iii $T = 30$

9 If $y = 3x + 2\cos x$, find $\dfrac{dy}{dx}$ and hence show that y increases as x increases.

10 The volume of water in a reservoir at time t is given by $V(t) = 3 + 2\sin\left(\dfrac{t}{4}\right)$.

a Find the volume in the reservoir at time $t = 10$.

b Find the rate of change of the volume of water in the reservoir at time $t = 10$.

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Example 10 **11** A particle moves along a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = 2t^3 - 9t^2 + 12t$.

a Find the velocity, v, as a function of t.

b At what times and in what positions will the particle have zero velocity?

c Find its acceleration at those instants.

d Find its velocity when its acceleration is zero.

12 A particle moves in a straight line such that its position, x cm, relative to a point O at time t seconds is given by the equation $x = 8 + 2t - t^2$. Find:

a its initial position

b its initial velocity

c when and where the velocity is zero

d its acceleration at time t.

13 A particle is moving in a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = \sqrt{2t^2 + 2}$. Find:

a the velocity as a function of t

b the acceleration as a function of t

c the velocity and acceleration when $t = 1$.

14 A vehicle is travelling in a straight line away from point O. Its distance from O after t seconds is $0.4e^t$ metres. Find the velocity of the vehicle when $t = 0$, $t = 1$, $t = 2$.

15 A manufacturing company has a daily output on day t of a production run given by $y = 600(1 - e^{-0.5t})$, where the first day of the production run is $t = 0$.

a Sketch the graph of y against t. (Assume a continuous model.)

b Find the instantaneous rate of change of output y with respect to t on the 10th day.

16 For each of the following, find $\dfrac{dy}{dx}$ in terms of y:

a $y = e^{-2x}$

b $y = Ae^{kx}$

17 The mass, m kg, of radioactive lead remaining in a sample t hours after observations began is given by $m = 2e^{-0.2t}$.

a Find the mass left after 12 hours.

b Find how long it takes for the mass to fall to half of its value at $t = 0$.

c Find how long it takes for the mass to fall to **i** one-quarter and **ii** one-eighth of its value at $t = 0$.

d Express the rate of decay as a function of m.

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5C Linear approximation using the increments formula: $\delta y \approx \frac{dy}{dx} \times \delta x$

In the graph, the line ℓ is the tangent to the curve at the point $P(x, f(x))$.

The gradient of the tangent ℓ is $f'(x)$.

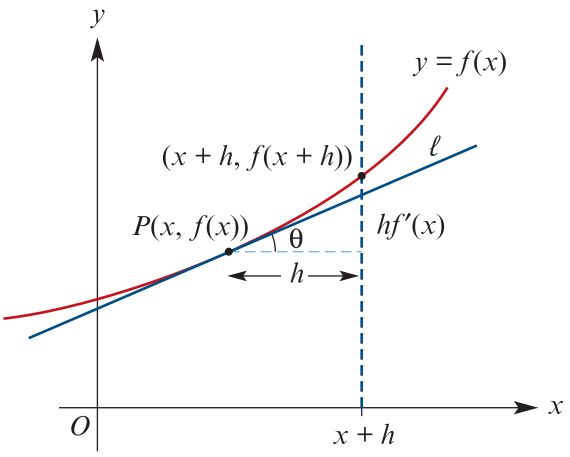
From the diagram, it can be seen that if h is small, then $f(x+h) \approx f(x) + hf'(x)$.

This can also be seen by considering the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{for small } h$$

$$\therefore \quad f(x+h) \approx f(x) + hf'(x) \quad \text{for small } h$$



For a small number h, the change in the value of $y = f(x)$ as x changes from a to $a + h$ can be approximated by

$$f(a+h) - f(a) \approx hf'(a)$$

Note: We can express this using Leibniz notation by setting $\delta x = h$ and $\delta y = f(a+h) - f(a)$, so that δx represents a small change in x and δy the corresponding change in y. Then we have $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ and so $\delta y \approx \frac{dy}{dx} \times \delta x$.

The linear approximation of a function may be regarded as using the tangent to the curve at a point to approximate the curve in the immediate neighbourhood of this point.

Example 11

Let $f(x) = x^4 - x^3$. Find the approximate change in $f(x)$ as x changes from 2 to $2 + h$, where h is small.

Solution

$$f(x) = x^4 - x^3$$

$$f'(x) = 4x^3 - 3x^2$$

$$\therefore \quad f'(2) = 4 \times 2^3 - 3 \times 2^2 = 20$$

The approximate change in $f(x)$ is given by

$$f(2+h) - f(2) \approx hf'(2) = 20h$$

Explanation

Alternatively, using Leibniz notation we have

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\therefore \quad \delta y \approx \frac{dy}{dx} \times \delta x$$

For $x = 2$ and $\delta x = h$, this gives

$$\delta y \approx 20 \times h = 20h$$

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The following table of values indicates the accuracy of the approximation in Example 11.

Linear approximation of $f(x) = x^4 - x^3$ near $x = 2$

h	$2 + h$	$f(2 + h)$	$f(2) + 20h$
−0.01	1.99	7.8018	7.8
0	2	8	8
0.01	2.01	8.2018	8.2
0.02	2.02	8.4073	8.4
0.05	2.05	9.0459	9
0.1	2.1	10.1871	10

In practical problems, you may be required to find the percentage change in a quantity resulting from a given change in another quantity.

The **percentage change** in $f(x)$ between $x = a$ and $x = a + h$ is defined to be

$$\left(\frac{f(a+h) - f(a)}{f(a)} \times 100\right)\%$$

provided $f(a) \neq 0$. Using the approximation $f(a + h) - f(a) \approx hf'(a)$, we obtain

$$\text{percentage change} \approx \left(\frac{100hf'(a)}{f(a)}\right)\%$$

Example 12

The time for a pendulum of length ℓ cm to complete one swing is given by the function with rule $f(\ell) = c\sqrt{\ell}$, where c is a constant. If an error is made in the measurement of the length so that the measured length is 2.5% greater than the actual length, find the approximate percentage error if the function f is used to calculate the time of a swing.

Solution

Let a cm be the actual length of the pendulum. Then

$$\text{percentage error} \approx \frac{100hf'(a)}{f(a)}$$

where $f(a) = ca^{\frac{1}{2}}$, $f'(a) = \frac{1}{2}ca^{-\frac{1}{2}}$ and $h = \frac{2.5}{100} \times a = \frac{a}{40}$. Therefore

$$\begin{aligned} \text{percentage error} &\approx \left(100 \times \frac{a}{40} \times \frac{1}{2}ca^{-\frac{1}{2}}\right) \div \left(ca^{\frac{1}{2}}\right) \\ &= \left(\frac{5}{4}ca^{\frac{1}{2}}\right) \div \left(ca^{\frac{1}{2}}\right) \\ &= \frac{5}{4} \end{aligned}$$

The estimated error is 1.25%.

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Example 13

Given that $y = 10 - \frac{5}{x}$ and that the value of y increases from 5 by a small amount k, find:

a the approximate change in x

b the corresponding percentage change in x.

Solution

a $y = 10 - \frac{5}{x}$ and $\frac{dy}{dx} = \frac{5}{x^2}$

When $y = 5$, $x = 1$ and so $\frac{dy}{dx} = 5$.

Thus, for $y = 5$ and $\delta y = k$, we have

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\therefore \quad \frac{k}{\delta x} \approx 5$$

$$\therefore \quad \delta x \approx \frac{k}{5}$$

b For $x = 1$ and $\delta x \approx \frac{k}{5}$, the percentage change in x is

$$\frac{\delta x}{x} \times 100\% \approx \left(\frac{k}{5} \times 100\right)\%$$

$$= 20k\%$$

Example 14

By differentiating $\frac{1}{\sqrt{x}}$ with respect to x, find an approximate value for $\frac{1}{\sqrt{100.5}}$.

Solution

Let $y = x^{-\frac{1}{2}}$. Then $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$.

When $x = 100$,

$$y = \frac{1}{10} \quad \text{and} \quad \frac{dy}{dx} = -\frac{1}{2}(100)^{-\frac{3}{2}} = -\frac{1}{2000}$$

Using $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

gives $\delta y \approx \frac{dy}{dx} \times \delta x$

$$= -\frac{1}{2000} \times 0.5 \qquad (\text{for } x = 100 \text{ and } \delta x = 0.5)$$

$$\therefore \quad \delta y = -\frac{1}{4000}$$

Hence $\frac{1}{\sqrt{100.5}} \approx \frac{1}{10} - \frac{1}{4000}$

$$= \frac{399}{4000}$$

$$= 0.09\,975$$

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Example 15

Given that $f(x) = x - e^{2x}$, find the approximate change in $f(x)$ as x changes from 0 to h, where h is small.

Solution

$$f(x) = x - e^{2x}$$
$$f'(x) = 1 - 2e^{2x}$$
$$f'(0) = 1 - 2 = -1$$

Therefore

$$\begin{aligned} f(h) - f(0) &= f(0 + h) - f(0) \\ &\approx hf'(0) \\ &= -h \end{aligned}$$

Summary 5C

For a small number h, the change in the value of $y = f(x)$ as x changes from a to $a + h$ can be approximated by

$$f(a + h) - f(a) \approx hf'(a)$$

Using Leibniz notation, we can write this as

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

Exercise 5C

Example 11

1 Let $f(x) = x^4 - 5x^3$. Find the approximate change in $f(x)$ as x changes from 2 to $2 + h$, where h is small.

2 The equation of a curve is $y = 4x^3 - 8x^2 + 10$.

a Find $\frac{dy}{dx}$. **b** Find the value of $\frac{dy}{dx}$ when $x = 2$.

c Find the approximate change in y as x changes from 2 to 2.02.

3 A curve has equation $y = \frac{5x^2 + 8}{x}$. Find an expression for $\frac{dy}{dx}$ and hence find the approximate increase in y as x increases from 2 to $2 + h$, where h is small.

4 For each of the following, write down an expression for the approximate change in y when x changes from a to $a + h$, where h is small:

a $y = 3x + 7$ **b** $y = \sqrt{x}$ **c** $y = \frac{2}{3x + 1}$ **d** $y = (2x + 1)^4$

e $y = (6x^2 - 1)^3$ **f** $y = \frac{x^2 + 2x + 6}{x^2 + 1}$ **g** $y = \sqrt[3]{x^2 + 10}$ **h** $y = \frac{6x + 1}{x + 1}$

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5 Given that $\frac{dy}{dx} = 6x - 4$, calculate the approximate change in the value of y when the value of x changes from -2 to -1.97.

6 The time, T seconds, for one complete swing of a pendulum of length ℓ m is given by the rule $T = 2\pi\sqrt{\frac{\ell}{g}}$, where g is a constant.

a Find $\frac{dT}{d\ell}$.

b Find the approximate increase in T when ℓ is increased from 1.6 to 1.7. Give the answer in terms of g.

7 Find the approximate increase in the radius of a circular disc if its area increases from $100\pi\text{ cm}^2$ to $101\pi\text{ cm}^2$.

Example 12 **8** A 2% error is made in measuring the radius of a sphere. Find the approximate percentage error in the surface area. (The surface area of a sphere is $A = 4\pi r^2$.)

Example 13 **9** Let $y = 6 - 4x + 5x^2$. If the value of x increases from 5 by a small amount h, determine in terms of h:

a the approximate change in y **b** the corresponding percentage change in y.

10 Let $f(x) = x^{-\frac{1}{2}}$.

a **i** Find $f'(x)$. **ii** Find $f'(100)$.

b Find the approximate change in $f(x)$ as x changes from 100 to 103.

c Find an approximate value for $\frac{1}{\sqrt{103}}$.

d Find an approximation for $\frac{1}{\sqrt{a+h}}$, where h is small.

Example 14 **11** By differentiating $y = x^{-\frac{1}{3}}$, determine an approximate value for $\frac{1}{\sqrt[3]{0.9}}$.

12 The radius r of a circle is 5 cm. Find the approximate increase in the area, $A\text{ cm}^2$, of the circle when the radius expands by h cm, where h is small.

13 The length of a rectangle is three times its width. If the perimeter increases by 2%, what is the approximate percentage increase in the area?

14 Let $f(x) = \frac{1}{1-x}$.

a **i** Find $f'(x)$. **ii** Find $f'(0)$.

b Show that $f(h) \approx 1 + h$ when h is small. Hint: Use $f(0+h) \approx f(0) + hf'(0)$.

15 Let $f(x) = \sqrt{1+x}$.

a **i** Find $f'(x)$. **ii** Find $f'(0)$.

b Show that $f(h) \approx 1 + \frac{h}{2}$ when h is small.

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16 Show that $f(x) = \frac{1}{2-x}$ can be approximated by $g(x) = \frac{1}{2} + \frac{x}{4}$ for x close to 0.

17 The radius of a sphere is measured as 4 cm with a possible error of 0.05 cm.

a What is the approximate error for the surface area? (Use $S = 4\pi r^2$.)

b What is the approximate error for the volume? (Use $V = \frac{4}{3}\pi r^3$.)

Example 15 **18** Given that h is small, find the approximate increase in $y = e^{2x}$ if x increases from 0 to h.

19 Consider the function $f(x) = e^{ax}$.

a Find $f'(x)$.

b Find an approximation for $f(h)$, where h is small, in terms of h and a.

c Find an approximation for $f(b+h)$, where h is small, in terms of $f(b)$, h and a.

20 For each of the following, write down an expression for the approximate change in y when x changes from a to $a+h$, where h is small:

a $y = 2e^{\frac{x}{2}}$ **b** $y = 3 - 2e^x$ **c** $y = xe^x$ **d** $y = \frac{x}{e^x}$

21 For $y = \ln(1+x^2)$, find the approximate increase in y as x increases from 0 to h, where h is a small positive number.

22 For $y = \ln(\sqrt{1+x+x^2})$, find the approximate increase in y as x increases from 0 to h, where h is a small positive number.

23 Find an approximation for $\ln(1.01)$.

24 For each of the following, write down an expression for the approximate change in y when x changes from a to $a+h$, where h is small:

a $y = \cos(2x)$ **b** $y = \sin\left(\frac{x}{2}\right)$ **c** $y = \tan(2x)$

d $y = 1 - \tan\left(\frac{x}{2}\right)$ **e** $y = \cos\left(\frac{\pi}{4} - x\right)$ **f** $y = \sin\left(\frac{-x}{2}\right)$

25 Let $y = \tan\theta$.

a **i** If θ is increased by a small amount h, find an estimate for the corresponding increase in y in terms of h and θ.

ii If $\theta = \frac{\pi}{4}$, find an estimate for the increase in y in terms of h.

b Estimate the value of $\tan(46^\circ)$ using the results of part **a**.

26 **a** Given $f(x) = \cos x$, find $f\left(\frac{\pi}{4}\right)$ and $f'\left(\frac{\pi}{4}\right)$.

b Use the results of part **a** to find an approximate value for:

i $\cos\left(\frac{\pi}{4} + h\right)$ where h is small

ii $\cos 0.8$

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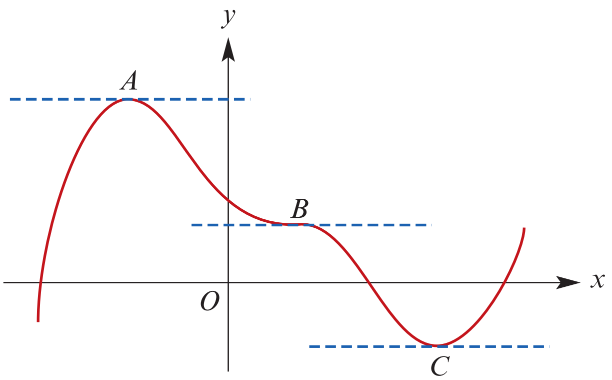
5D Stationary points

In the previous chapter, we have seen that the gradient of the tangent at a point $(a, f(a))$ on the curve with rule $y = f(x)$ is given by $f'(a)$.

> A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
>
> Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

In the graph shown, there are stationary points at A, B and C.

At such points, the tangents are parallel to the x-axis (illustrated as dashed lines).



The reason for the name *stationary point* becomes clear if we look at an application to the motion of a particle.

Example 16

A particle is moving in a straight line. Its position, x metres, relative to a point O on the line at time t seconds is given by

$$x = 9t - \frac{1}{3}t^3, \quad 0 \leq t \leq 4$$

Find the particle's maximum distance from O. (Here the particle is always on the right of O and so its distance from O is its position.)

Solution

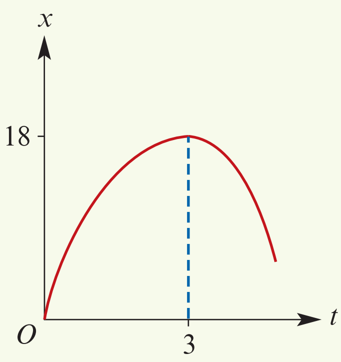
$$\frac{dx}{dt} = 9 - t^2$$

The maximum distance from O occurs when $\frac{dx}{dt} = 0$.

So $t = 3$ or $t = -3$. But $t = -3$ lies outside the domain.

At $t = 3$, $x = 18$.

Thus the stationary point is $(3, 18)$ and the maximum distance from O is 18 metres.



Note: The stationary point occurs when the rate of change of position with respect to time (the velocity) is zero. At this moment, the particle is stationary.

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Example 17

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $y = 4 + 3x - x^3$ **c** $p = 2t^3 - 5t^2 - 4t + 13,\ t > 0$

Solution

a $y = 9 + 12x - 2x^2$

$$\frac{dy}{dx} = 12 - 4x$$

A stationary point occurs when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$.

Hence $x = 3$ and $y = 9 + 12 \times 3 - 2 \times 3^2$

$= 27$

The stationary point is $(3, 27)$.

b $y = 4 + 3x - x^3$

$$\frac{dy}{dx} = 3 - 3x^2$$

$$\frac{dy}{dx} = 0 \text{ implies } 3(1 - x^2) = 0$$

$$\therefore \quad x = \pm 1$$

The stationary points are $(1, 6)$ and $(-1, 2)$.

c $p = 2t^3 - 5t^2 - 4t + 13$

$$\frac{dp}{dt} = 6t^2 - 10t - 4, \quad t > 0$$

$$\frac{dp}{dt} = 0 \text{ implies } 2(3t^2 - 5t - 2) = 0$$

$$(3t + 1)(t - 2) = 0$$

$$\therefore \quad t = -\frac{1}{3} \text{ or } t = 2$$

But $t > 0$, and so the only acceptable solution is $t = 2$. The corresponding stationary point is $(2, 1)$.

Example 18

Find the stationary points of the following functions:

a $y = \sin(2x),\ x \in [0, 2\pi]$ **b** $y = e^{2x} - x$ **c** $y = x\ln(2x),\ x \in (0, \infty)$

Solution

a $y = \sin(2x)$

$$\frac{dy}{dx} = 2\cos(2x)$$

So $\frac{dy}{dx} = 0$ implies $2\cos(2x) = 0$

$$\cos(2x) = 0$$

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$

$$\therefore \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The stationary points are $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{3\pi}{4}, -1\right)$, $\left(\frac{5\pi}{4}, 1\right)$ and $\left(\frac{7\pi}{4}, -1\right)$.

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b $y = e^{2x} - x$

$\dfrac{dy}{dx} = 2e^{2x} - 1$

So $\dfrac{dy}{dx} = 0$ implies

$$2e^{2x} - 1 = 0$$
$$e^{2x} = \frac{1}{2}$$
$$\therefore \quad x = \frac{1}{2}\ln\left(\frac{1}{2}\right)$$
$$= -\frac{1}{2}\ln 2$$

When $x = -\dfrac{1}{2}\ln 2$,

$$y = e^{2\times\frac{1}{2}\ln\left(\frac{1}{2}\right)} + \frac{1}{2}\ln 2$$
$$= \frac{1}{2} + \frac{1}{2}\ln 2$$

The coordinates of the stationary point are $\left(-\dfrac{1}{2}\ln 2, \dfrac{1}{2} + \dfrac{1}{2}\ln 2\right)$.

c $y = x\ln(2x)$

$\dfrac{dy}{dx} = \ln(2x) + 1$

So $\dfrac{dy}{dx} = 0$ implies

$$\ln(2x) + 1 = 0$$
$$\ln(2x) = -1$$
$$2x = e^{-1}$$
$$\therefore \quad x = \frac{1}{2e}$$

When $x = \dfrac{1}{2e}$, $y = \dfrac{1}{2e}\ln\left(\dfrac{2}{2e}\right)$

$$= \frac{-1}{2e}$$

The coordinates of the stationary point are $\left(\dfrac{1}{2e}, \dfrac{-1}{2e}\right)$.

Example 19

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through the point $(0, 5)$ and has a stationary point at $(2, 7)$. Find a, b and c.

Solution

When $x = 0$, $y = 5$. Thus $5 = c$.

$\dfrac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\dfrac{dy}{dx} = 0$. Therefore

$$12 + 4a + b = 0 \qquad (1)$$

The point $(2, 7)$ is on the curve and so

$$8 + 4a + 2b + 5 = 7$$
$$\therefore \quad 6 + 4a + 2b = 0 \qquad (2)$$

Subtracting (1) from (2) gives $-6 + b = 0$. Thus $b = 6$. Substitute in (1):

$$12 + 4a + 6 = 0$$
$$4a = -18$$

Hence $a = -\dfrac{9}{2}$, $b = 6$ and $c = 5$.

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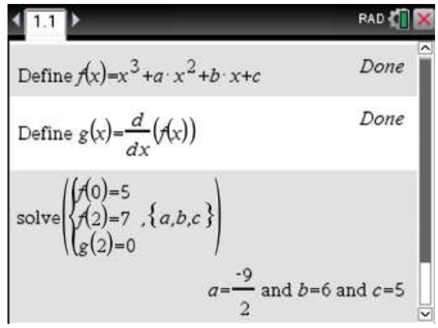
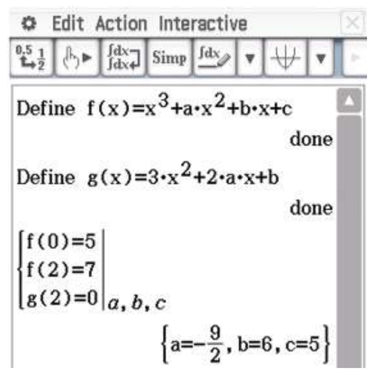
Using the TI-Nspire

- Use (menu) > **Actions** > **Define** to define $f(x) = x^3 + ax^2 + bx + c$.
- Define $g(x)$ to be the derivative ((menu) > **Calculus** > **Derivative**) of $f(x)$ as shown.
- Use the simultaneous equations solver ((menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) to find a, b and c given that $f(0) = 5$, $f(2) = 7$ and $g(2) = 0$.



Using the Casio ClassPad

- Use **Interactive** > **Define** to define the functions $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = 3x^2 + 2ax + b$.
- In (Math1), tap (simultaneous equations key) twice.
- Enter the equations and variables as shown and tap (EXE).



Summary 5D

- A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
- Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

Skill-sheet

Exercise 5D

Example 17

1 Find the stationary points for each of the following:

a $f(x) = x^3 - 12x$
b $g(x) = 2x^2 - 4x$
c $h(x) = 5x^4 - 4x^5$
d $f(t) = 8t + 5t^2 - t^3$ for $t > 0$
e $g(z) = 8z^2 - 3z^4$
f $f(x) = 5 - 2x + 3x^2$
g $h(x) = x^3 - 4x^2 - 3x + 20,\ x > 0$
h $f(x) = 3x^4 - 16x^3 + 24x^2 - 10$

Example 18

2 Find the stationary points of the following functions:

a $y = e^{2x} - 2x$
b $y = x\ln(3x),\ x \in (0, \infty)$
c $y = \cos(2x),\ x \in [-\pi, \pi]$
d $y = xe^x$
e $y = x^2e^{-x}$
f $y = 2x\ln x,\ x \in (0, \infty)$

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3 **a** The curve with rule $f(x) = x^2 - ax + 9$ has a stationary point when $x = 3$. Find a.
b The curve with rule $h(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$. Find b.

Example 19

4 The curve with equation $y = x^3 + bx^2 + cx + d$ passes through the point $(0, 3)$ and has a stationary point at $(1, 3)$. Find b, c and d.

5 The tangent to the curve of $y = ax^2 + bx + c$ at the point where $x = 2$ is parallel to the line $y = 4x$. There is a stationary point at $(1, -3)$. Find the values of a, b and c.

6 The graph of $y = ax^3 + bx^2 + cx + d$ touches the line $2y + 6x = 15$ at the point $A(0, 7\frac{1}{2})$ and has a stationary point at $B(3, -6)$. Find the values of a, b, c and d.

7 The curve with equation $y = ax + \dfrac{b}{2x - 1}$ has a stationary point at $(2, 7)$. Find:
a the values of a and b **b** the coordinates of the other stationary point.

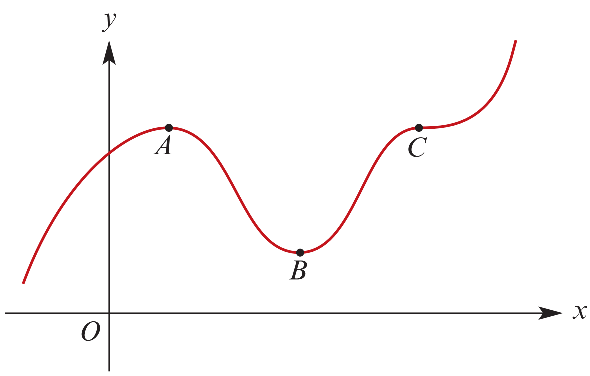
8 Find the x-coordinates, in terms of n, of the stationary points of the curve with equation $y = (2x - 1)^n(x + 2)$, where n is a natural number.

9 Find the x-coordinates of the stationary points of the curve with equation $y = (x^2 - 1)^n$ where n is an integer greater than 1.

10 Find the coordinates of the stationary points of the curve with equation $y = \dfrac{x}{x^2 + 1}$.

5E Types of stationary points

The graph of $y = f(x)$ shown has three stationary points A, B, C.



A Point A is called a **local maximum** point.
Notice that immediately to the left of A the gradient is positive, and immediately to right the gradient is negative.

gradient	+	0	−
shape of f	╱	—	╲

B Point B is called a **local minimum** point.
Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

gradient	−	0	+
shape of f	╲	—	╱

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c Point C is called a **stationary point of inflection**. The gradient is positive immediately to the left and right of C.

gradient	+	0	+
shape of f	/	—	/

Clearly it is also possible to have stationary points of inflection such that the gradient is negative immediately to the left and right.

gradient	–	0	–
shape of f	\	—	\

Stationary points of types A and B are referred to as **turning points**.

Example 20

For the function $f(x) = 3x^3 - 4x + 1$:

a Find the stationary points and state their nature. **b** Sketch the graph.

Solution

a The derivative is $f'(x) = 9x^2 - 4$.

The stationary points occur where $f'(x) = 0$:

$$9x^2 - 4 = 0$$
$$\therefore \quad x = \pm\frac{2}{3}$$

There are stationary points at $\left(-\frac{2}{3}, f\left(-\frac{2}{3}\right)\right)$ and $\left(\frac{2}{3}, f\left(\frac{2}{3}\right)\right)$, that is, at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$ and $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

So $f'(x)$ is of constant sign for each of

$$\left\{x : x < -\frac{2}{3}\right\}, \quad \left\{x : -\frac{2}{3} < x < \frac{2}{3}\right\} \quad \text{and} \quad \left\{x : x > \frac{2}{3}\right\}$$

To calculate the sign of $f'(x)$ for each of these sets, simply choose a representative number in the set.

Thus $f'(-1) = 9 - 4 = 5 > 0$

$f'(0) = 0 - 4 = -4 < 0$

$f'(1) = 9 - 4 = 5 > 0$

We can now put together the table shown on the right.

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	–	0	+
shape of f	/	—	\	—	/

There is a local maximum at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$ and a local minimum at $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

b To sketch the graph of this function we need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

The y-axis intercept is $f(0) = 1$.

To find the x-axis intercepts, consider $f(x) = 0$, which implies $3x^3 - 4x + 1 = 0$.
Using the factor theorem, we find that $x - 1$ is a factor of $3x^3 - 4x + 1$.
By division:

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

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Now $f(x) = (x - 1)(3x^2 + 3x - 1) = 0$ implies that $x = 1$ or $3x^2 + 3x - 1 = 0$.

We have

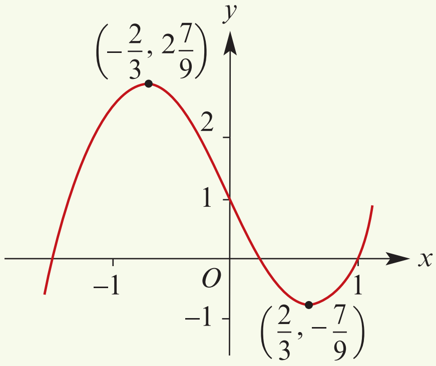
$$\begin{aligned} 3x^2 + 3x - 1 &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{3}\right] \\ &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{21}{36}\right] \\ &= 3\left(x + \frac{1}{2} - \frac{\sqrt{21}}{6}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{6}\right) \end{aligned}$$

Thus the x-axis intercepts are at

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{6}, \quad x = -\frac{1}{2} - \frac{\sqrt{21}}{6}, \quad x = 1$$

For $x > \frac{2}{3}$, $f(x)$ becomes larger.

For $x < \frac{2}{3}$, $f(x)$ becomes smaller.



A CAS calculator can be used to plot the graph of a function and determine its key features, including:

- the value of the function at any point
- the value of its derivative at any point
- the axis intercepts
- the local maximum and local minimum points.

Example 21

Plot the graph of $y = x^3 - 19x + 20$ and determine:

a the value of y when $x = -4$

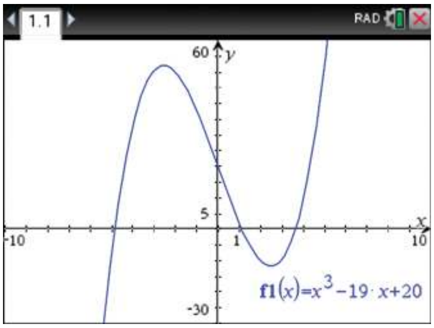
b the values of x when $y = 0$

c the value of $\frac{dy}{dx}$ when $x = -1$

d the coordinates of the local maximum.

Using the TI-Nspire

Graph $y = x^3 - 19x + 20$ in an appropriate window (menu > **Window/Zoom** > **Window Settings**).



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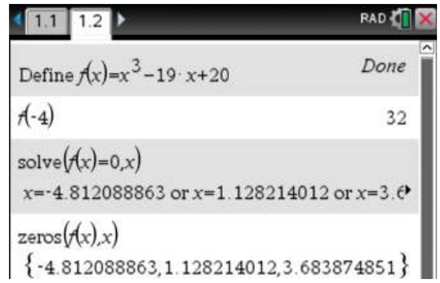
Define $f(x) = x^3 - 19x + 20$.

a $f(-4) = 32$

b Use solve$(f(x) = 0, \ x)$.

Note: Alternatively, (menu) > **Algebra** > **Zeros** can be used to solve equations equal to zero as shown.

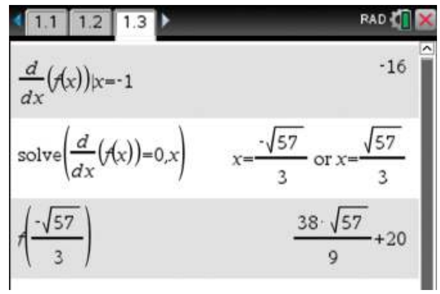


c Find the derivative of $f(x)$ at $x = -1$ as shown.

d To find the stationary points, use

$$\text{solve}\left(\frac{d}{dx}(f(x)) = 0, \ x\right)$$

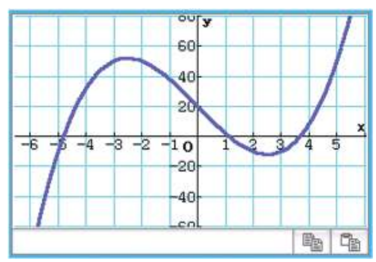
and then substitute to find the y-coordinate.



Note: Since the function was also defined in the **Graphs** application as $f1$, the name $f1$ could have been used in place of f in these calculations.

Using the Casio ClassPad

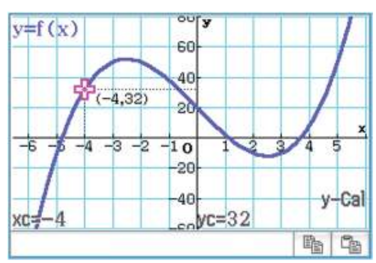
- Define $f(x) = x^3 - 19x + 20$.
- Tap [graph icon] to open the graph window.
- Drag $f(x)$ into the graph window.
- Adjust the window using [window icon].



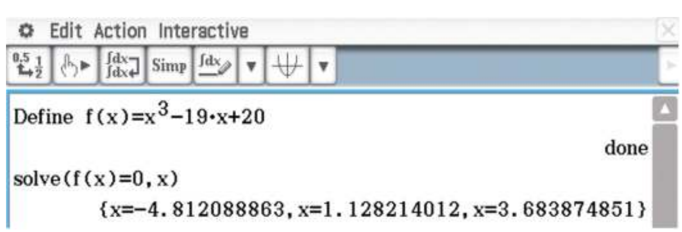
a To find $f(-4) = 32$, there are two methods:

1 In Main $\sqrt{\alpha}$, type $f(-4)$ and tap (EXE).

2 In [graph icon], go to **Analysis** > **G-Solve** > **x-Cal/y-Cal** > **y-Cal** and type -4.



b
- In Main $\sqrt{\alpha}$, enter and highlight $f(x) = 0$.
- Go to **Interactive** > **Equation/Inequality** > **solve**.
- Rotate the screen and press ► to view all solutions.



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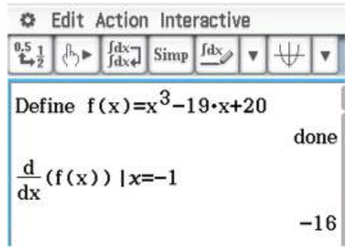
c To find $\frac{dy}{dx}$ when $x = -1$:

- In Main $\sqrt{\alpha}$, enter and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and then tap OK.
- Select | from Math3 and type $x = -1$ as shown.
- Tap EXE.



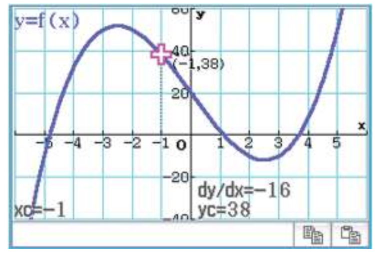
Using the graph window

To view the derivative at any point on a graph, first ensure that the Derivative/Slope setting is activated:

- Go to settings ⚙, select **Graph Format**, tick **Derivative/Slope** and tap Set.

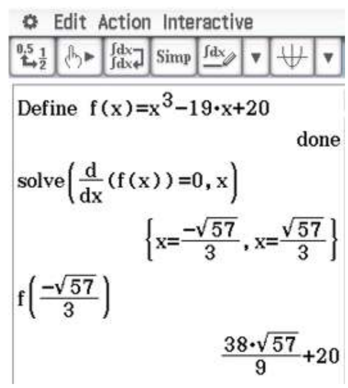
Now in ⊞:

- Go to **Analysis** > **Trace**, type −1 and tap OK.



d To find the local maximum:

- In Main $\sqrt{\alpha}$, solve $\frac{d}{dx}(f(x)) = 0$ as shown.
- Substitute to find the y-coordinate.



Alternatively, find **fMax** using an appropriate domain:

fMax(f(x), x, −6, 0)

$\left\{\text{MaxValue}=\frac{38\cdot\sqrt{57}}{9}+20,\ x=\frac{-\sqrt{57}}{3}\right\}$

Example 22

Sketch the graph of $f(x) = e^{x^3}$.

Solution

As $x \to -\infty$, $f(x) \to 0$.

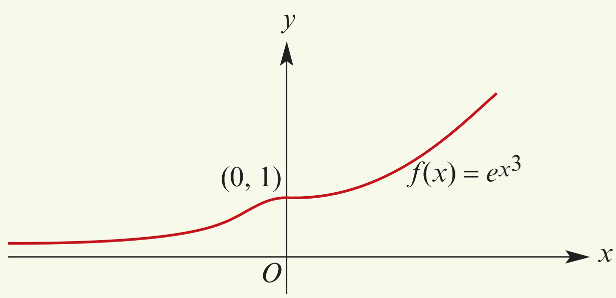
Axis intercepts

When $x = 0$, $f(x) = 1$.

Stationary points

$f'(x) = 3x^2 e^{x^3}$

So $f'(x) = 0$ implies $x = 0$.



The gradient of f is always greater than or equal to 0, which means that (0, 1) is a stationary point of inflection.

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Example 23

For $f(x) = x \ln x$, $\{ x \in \mathbb{R} : x > 0 \}$:

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

Solution

a $f'(x) = x \times \dfrac{1}{x} + \ln x$ (product rule)

$= 1 + \ln x$

b $f(x) = x \ln x$

Thus $f(x) = 0$ implies $x = 0$ or $\ln x = 0$.

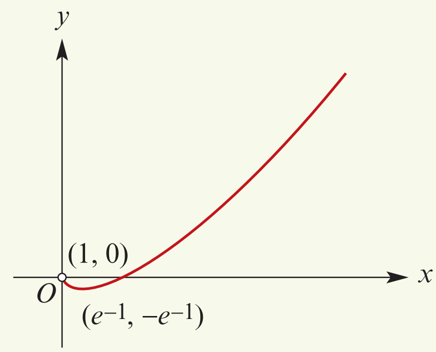
Since $x \in (0, \infty)$, the only solution is $x = 1$.

c $f'(x) = 0$ implies $1 + \ln x = 0$.

Therefore $\ln x = -1$ and so $x = e^{-1}$.

d When $x = e^{-1}$, $y = e^{-1} \ln(e^{-1})$

$= e^{-1} \times (-1) = -e^{-1}$



Example 24

Find the local maximum and local minimum points of $f(x) = 2 \sin x + 1 - 2 \sin^2 x$, where $0 < x < 2\pi$.

Solution

Find $f'(x)$ and solve $f'(x) = 0$:

$$f(x) = 2 \sin x + 1 - 2 \sin^2 x$$

$$\therefore \quad f'(x) = 2 \cos x - 4 \sin x \cos x$$

$$= 2 \cos x \cdot (1 - 2 \sin x)$$

Thus $f'(x) = 0$ implies

$$\cos x = 0 \quad \text{or} \quad 1 - 2 \sin x = 0$$

$$\text{i.e.} \quad \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$$

$$\text{i.e.} \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$$

We have $f\left(\dfrac{\pi}{2}\right) = 1$, $f\left(\dfrac{3\pi}{2}\right) = -3$, $f\left(\dfrac{\pi}{6}\right) = \dfrac{3}{2}$ and $f\left(\dfrac{5\pi}{6}\right) = \dfrac{3}{2}$

x		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$		$\frac{3\pi}{2}$	
$f'(x)$	+	0	−	0	+	0	−	0	+
shape of f	/	—	\	—	/	—	\	—	/

Local maxima at $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right)$ and $\left(\dfrac{5\pi}{6}, \dfrac{3}{2}\right)$. Local minima at $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(\dfrac{3\pi}{2}, -3\right)$.

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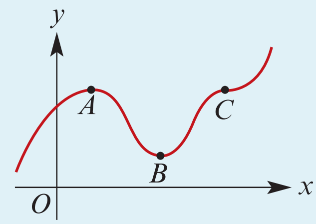
Summary 5E

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Types of stationary points



A Point A is a **local maximum**:
- $f'(x) > 0$ immediately to the left of A
- $f'(x) < 0$ immediately to the right of A.

B Point B is a **local minimum**:
- $f'(x) < 0$ immediately to the left of B
- $f'(x) > 0$ immediately to the right of B.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are called **turning points**.

Skill-sheet

Exercise 5E

Example 20

1 For each of the following derivative functions, write down the values of x at which the derivative is zero and prepare a gradient table (as in Example 20) showing whether the corresponding points on the graph of $y = f(x)$ are local maxima, local minima or stationary points of inflection:

a $f'(x) = 4x^2$
b $f'(x) = (x - 2)(x + 5)$
c $f'(x) = (x + 1)(2x - 1)$
d $f'(x) = -x^2 + x + 12$
e $f'(x) = x^2 - x - 12$
f $f'(x) = 5x^4 - 27x^3$
g $f'(x) = (x - 1)(x - 3)$
h $f'(x) = -(x - 1)(x - 3)$

2 Find the stationary points on each of the following curves and state their nature:

a $y = x(x^2 - 12)$
b $y = x^2(3 - x)$
c $y = x^3 - 5x^2 + 3x + 2$
d $y = 3 - x^3$
e $y = 3x^4 + 16x^3 + 24x^2 + 3$
f $y = x(x^2 - 1)$

3 Sketch the graph of each of the following, finding **i** axis intercepts and **ii** stationary points:

a $y = 4x^3 - 3x^4$
b $y = x^3 - 6x^2$
c $y = 3x^2 - x^3$
d $y = x^3 + 6x^2 + 9x + 4$
e $y = (x^2 - 1)^5$
f $y = (x^2 - 1)^4$

4 **a** Find the stationary points of the graph of $y = 2x^3 + 3x^2 - 12x + 7$, stating the nature of each.
b Show that the graph passes through $(1, 0)$.
c Find the other axis intercepts.
d Sketch the graph.

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5 **a** Show that the polynomial $P(x) = x^3 + ax^2 + b$ has a stationary point at $x = 0$ for all a and b.

b Given that $P(x)$ has a second stationary point at $(-2, 6)$, find the values of a and b and the nature of both stationary points.

6 Sketch the graph of $f(x) = (2x-1)^5(2x-4)^4$.

a State the coordinates of the axis intercepts.

b State the coordinates and nature of each stationary point.

7 **a** Sketch the graphs of $f(x) = (4x^2-1)^6$ and $g(x) = (4x^2-1)^5$ on the one set of axes.

b **i** Find $\{ x : (4x^2-1)^6 > (4x^2-1)^5 \}$.

ii Find $\{ x : f'(x) > g'(x) \}$.

8 Sketch the graph of each of the following. State the axis intercepts and the coordinates of stationary points.

a $y = x^3 + x^2 - 8x - 12$

b $y = 4x^3 - 18x^2 + 48x - 290$

9 For each of the following, find the coordinates of the stationary points and determine their nature:

a $f(x) = 3x^4 + 4x^3$

b $f(x) = x^4 + 2x^3 - 1$

c $f(x) = 3x^3 - 3x^2 + 12x + 9$

10 Consider the function f defined by $f(x) = \frac{1}{8}(x-1)^3(8-3x) + 1$.

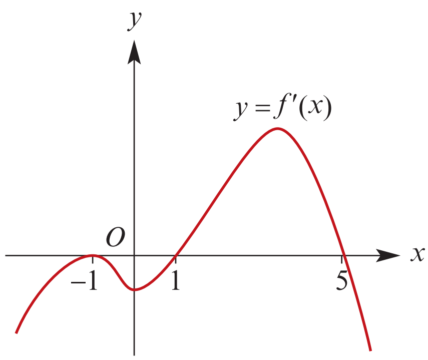
a Show that $f(0) = 0$ and $f(3) = 0$.

b Show that $f'(x) = \frac{3}{8}(x-1)^2(9-4x)$ and specify the values of x for which $f'(x) \geq 0$.

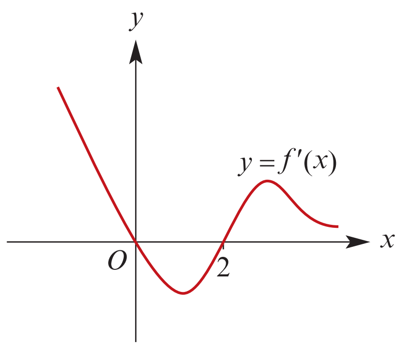
c Sketch the graph of $y = f(x)$.

11 Each graph below shows the graph of f' for a function f. Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point.

a



b



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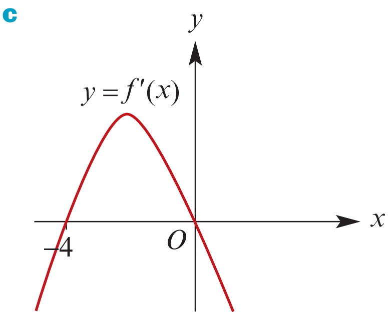
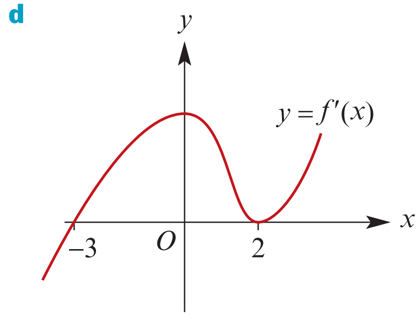
c



d



12 Find the coordinates of the stationary points, and state the nature of each, for the curve with equation:

a $y = x^4 - 16x^2$

b $y = x^{2m} - 16x^{2m-2}$, where m is a natural number greater than or equal to 2.

Example 22 **13** Sketch the graph of $f(x) = e^{-\frac{x^2}{2}}$.

14 Let $f(x) = x^2 e^x$. Find $\{ x : f'(x) < 0 \}$.

15 Find the values of x for which $100e^{-x^2+2x-5}$ increases as x increases and hence find the maximum value of $100e^{-x^2+2x-5}$.

16 Let $f(x) = e^x - 1 - x$.

a Find the minimum value of $f(x)$.

b Hence show $e^x \geq 1 + x$ for all real x.

17 For $f(x) = x + e^{-x}$:

a Find the position and nature of any stationary points.

b Find, if they exist, the equations of any asymptotes.

c Sketch the graph of $y = f(x)$.

18 The curve $y = e^x(px^2 + qx + r)$ is such that the tangents at $x = 1$ and $x = 3$ are parallel to the x-axis. The point with coordinates $(0, 9)$ is on the curve. Find p, q and r.

19 **a** Let $y = e^{4x^2-8x}$. Find $\frac{dy}{dx}$.

b Find the coordinates of the stationary point on the curve of $y = e^{4x^2-8x}$ and state its nature.

c Sketch the graph of $y = e^{4x^2-8x}$.

d Find the equation of the normal to the curve of $y = e^{4x^2-8x}$ at the point where $x = 2$.

20 On the same set of axes, sketch the graphs of $y = \ln x$ and $y = \ln(5x)$, and use them to explain why $\frac{d}{dx}\big(\ln x\big) = \frac{d}{dx}\big(\ln(5x)\big)$.

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Example 23

21 For the function $f(x) = x^2 \ln x$, $\{x \in \mathbb{R} : x > 0\}$:

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

22 Let $f(x) = x^3 - 3x^2 - 9x + 11$. Sketch the graph of:

a $y = f(x)$

b $y = 2f(x)$

c $y = f(x + 2)$

d $y = f(x - 2)$

e $y = -f(x)$

23 Let $f(x) = 2 + 3x - x^3$. Sketch the graph of:

a $y = f(x)$

b $y = -2f(x)$

c $y = 2f(x - 1)$

d $y = f(x) - 3$

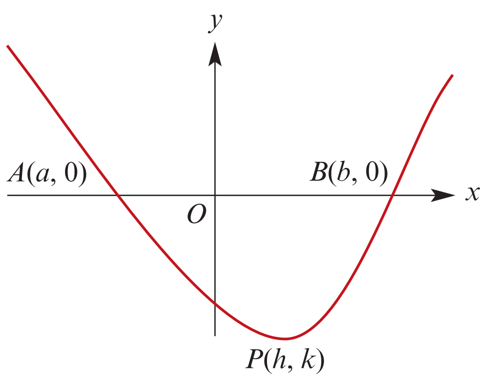
e $y = 3f(x + 1)$

24 The graph shown opposite has equation $y = f(x)$. Suppose a dilation of factor p parallel to the y-axis followed by a translation of ℓ units in the positive direction of the x-axis is applied to the graph.

For the graph of the image, state:

a the axis intercepts

b the coordinates of the turning point.



Example 24

25 Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point. Consider $0 \le x \le 2\pi$ only.

a $f(x) = 2\cos x - (2\cos^2 x - 1)$

b $f(x) = 2\cos x + 2\sin x \cos x$

c $f(x) = 2\sin x - (2\cos^2 x - 1)$

d $f(x) = 2\sin x + 2\sin x \cos x$

26 The graph of a quartic function passes through the points with coordinates $(1, 21)$, $(2, 96)$, $(5, 645)$, $(6, 816)$ and $(7, 861)$.

a Find the rule of the quartic and plot the graph. Determine the turning points and axis intercepts.

b Plot the graph of the derivative on the same screen.

c Find the value of the function when $x = 10$.

d For what value(s) of x is the value of the function 500?

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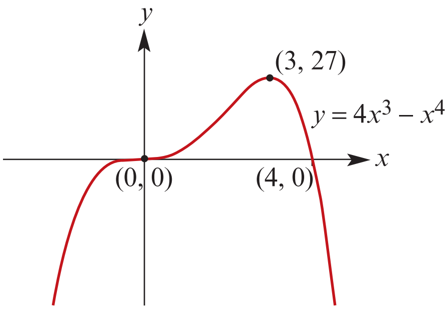
5F Using the second derivative in graph sketching

In the previous section, you have undertaken sketching the graphs of polynomial functions. The second derivative enables us to find out more information about these and other graphs. We start this section by considering the graph of $y = 4x^3 - x^4$.

The graph of $y = 4x^3 - x^4$

The graph of this function is shown in the diagram on the right.

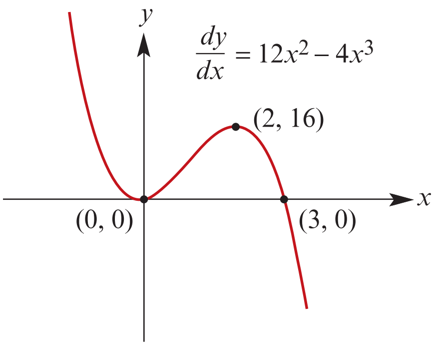
There is a local maximum at $(3, 27)$ and a stationary point of inflection at $(0, 0)$. These points have been determined by considering the derivative function $\dfrac{dy}{dx} = 12x^2 - 4x^3$.



The graph of the derivative function

Note that the local maximum and the stationary point of inflection of the original graph correspond to the x-axis intercepts of the graph of the derivative.

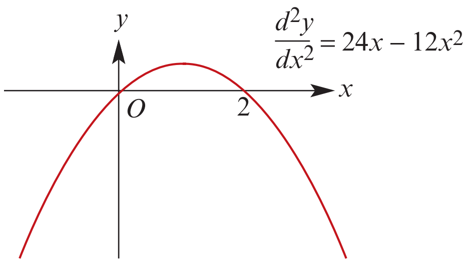
Also, it can be seen that the gradient of the original graph is positive for $x < 0$ and $0 < x < 3$ and negative for $x > 3$.



The graph of the second derivative function

Further information can be obtained by considering the graph of the second derivative.

The graph of the second derivative reveals that, at the points on the original graph where $x = 0$ and $x = 2$, there are important changes in the gradient.



- At the point where $x = 0$, the gradient of $y = 4x^3 - x^4$ changes from decreasing (positive) to increasing (positive). This point is also a stationary point, but it is neither a local maximum nor a local minimum. It is known as a **stationary point of inflection**.
- At the point where $x = 2$, the gradient of $y = 4x^3 - x^4$ changes from increasing (positive) to decreasing (positive). This point is called a **point of inflection**. In this case, the point corresponds to a local maximum of the derivative graph.

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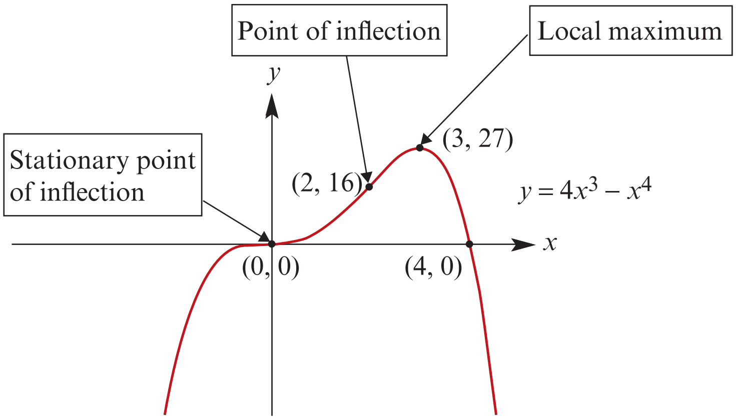
The gradient of $y = 4x^3 - x^4$ increases on the interval $(0, 2)$ and then decreases on the interval $(2, 3)$. The point $(2, 16)$ is the point of maximum gradient of $y = 4x^3 - x^4$ for the interval $(0, 3)$.



Behaviour of tangents

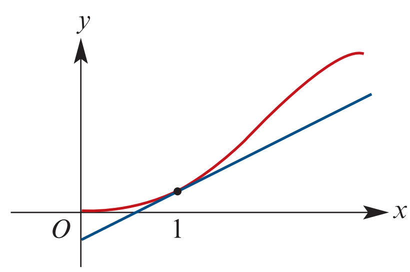
A closer look at the graph of $y = 4x^3 - x^4$ for the interval $(0, 3)$ and, in particular, the behaviour of the tangents to the graph in this interval will reveal more.

The tangents at $x = 1$, 2 and 2.5 have equations $y = 8x - 5$, $y = 16x - 16$ and $y = \frac{25}{2}x - \frac{125}{16}$ respectively. The following graphs illustrate the behaviour.

- The first diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 1$.

  The tangent lies *below* the graph in the immediate neighbourhood of where $x = 1$.

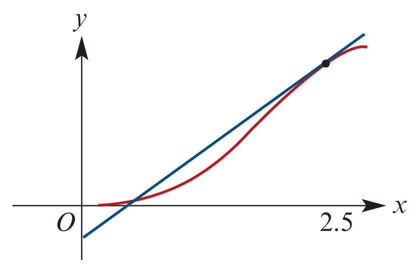
  For the interval $(0, 2)$, the gradient of the graph is increasing; the graph is said to be *concave up*.



- The second diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2.5$.

  The tangent lies *above* the graph in the immediate neighbourhood of where $x = 2.5$.

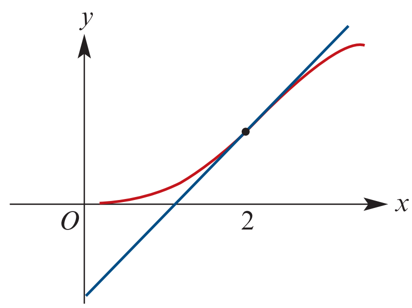
  For the interval $(2, 3)$, the gradient of the graph is decreasing; the graph is said to be *concave down*.



- The third diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2$.

  The tangent *crosses* the graph at the point $(2, 16)$.

  At $x = 2$, the gradient of the graph changes from increasing to decreasing; the point $(2, 16)$ is said to be a *point of inflection*.



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Concavity and points of inflection

For a curve $y = f(x)$:

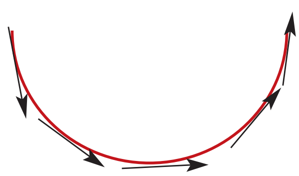
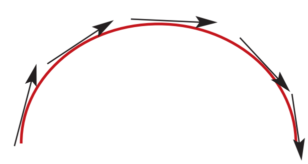
- the derivative $f'(a)$ gives the gradient of the curve at $x = a$
- the second derivative $f''(a)$ gives the rate of change of the gradient of the curve at $x = a$.

We have met the ideas of concave up and concave down in the example at the beginning of this section. We now give the definitions of these ideas.

Concave up and concave down

For a curve $y = f(x)$:

- If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval (a, b). The curve is said to be **concave up**.
- If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval (a, b). The curve is said to be **concave down**.

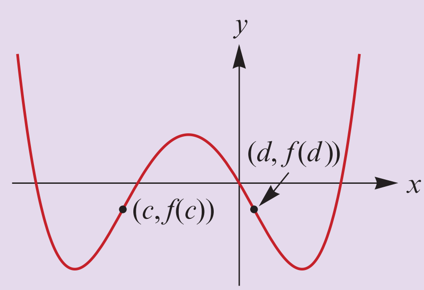
Concave up for an interval	Concave down for an interval
	
The tangent is below the curve at each point and the gradient is increasing i.e. $f''(x) > 0$	The tangent is above the curve at each point and the gradient is decreasing i.e. $f''(x) < 0$

Point of inflection

A point where a curve changes from concave up to concave down or from concave down to concave up is called a **point of inflection**.

That is, a point of inflection occurs where the sign of the second derivative changes.

In the graph on the right, there are points of inflection at $x = c$ and $x = d$.



Note: At a point of inflection, the tangent will pass through the curve.

At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.

For example, consider $f(x) = x^4$. Then $f''(x) = 12x^2$ and so $f''(0) = 0$. But the graph of $y = x^4$ has a local minimum at $x = 0$.

From now on, we can use these new ideas in our graphing.

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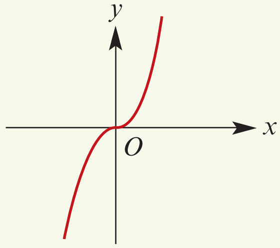
Example 25

For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

a $f(x) = x^3$ **b** $f(x) = -x^3$ **c** $f(x) = x^3 - 3x^2 + 1$

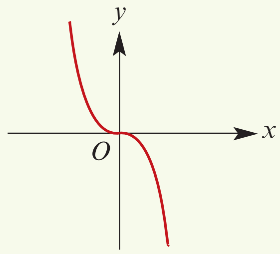
Solution

a
- There is a stationary point of inflection at $(0, 0)$. At $x = 0$, the gradient is zero and the curve changes from concave down to concave up.
- The curve is concave up on the interval $(0, \infty)$. The second derivative is positive on this interval.



Note: The tangent at $x = 0$ is the line $y = 0$.

b
- There is a stationary point of inflection at $(0, 0)$. At $x = 0$, the gradient is zero and the curve changes from concave up to concave down.
- The curve is concave up on the interval $(-\infty, 0)$. The second derivative is positive on this interval.



Note: The tangent at $x = 0$ is the line $y = 0$.

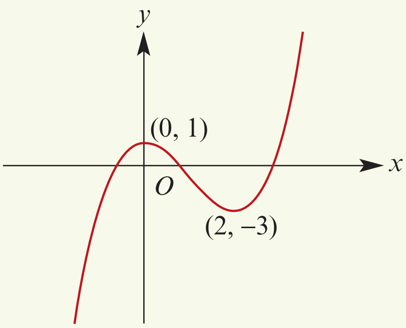
c

$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$

There is a local maximum at $(0, 1)$ and a local minimum at $(2, -3)$.

The second derivative is zero at $x = 1$, it is positive for $x > 1$, and it is negative for $x < 1$.

- There is a point of inflection at $(1, -1)$.
- The curve is concave up on the interval $(1, \infty)$.



Test for local maximum or minimum

The following test provides a useful method for identifying local maxima and minima.

Second derivative test

For the graph of $y = f(x)$:
- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum, as the curve is concave up.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum, as the curve is concave down.
- If $f''(a) = 0$, then further investigation is necessary.

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Example 26

Consider the graph of $y = f(x)$, where $f(x) = x^2(10 - x)$.

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 10]$.

Solution

We have $f(x) = x^2(10 - x) = 10x^2 - x^3$, $f'(x) = 20x - 3x^2$ and $f''(x) = 20 - 6x$.

a $f'(x) = 0$ implies $x(20 - 3x) = 0$, and therefore $x = 0$ or $x = \frac{20}{3}$.

Since $f''(0) = 20 > 0$, there is a local minimum at $(0, 0)$.

Since $f''\left(\frac{20}{3}\right) = -20 < 0$, there is a local maximum at $\left(\frac{20}{3}, \frac{4000}{27}\right)$.

b $f''(x) = 0$ implies $x = \frac{10}{3}$.

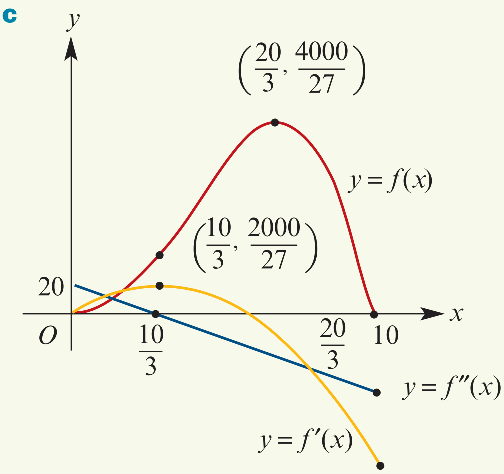
We have $f''(x) > 0$ for $x < \frac{10}{3}$ and $f''(x) < 0$ for $x > \frac{10}{3}$.

Hence there is a point of inflection at $\left(\frac{10}{3}, \frac{2000}{27}\right)$.

The gradient at this point is $\frac{100}{3}$.

Note: The maximum gradient of $y = f(x)$ is at the point of inflection.

c

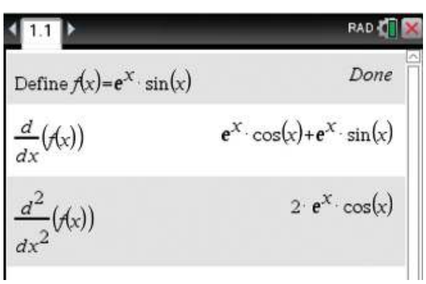


Example 27

Use a CAS calculator to find the stationary points and the points of inflection of the graph of $f(x) = e^x \sin x$ for $x \in [0, 2\pi]$.

Using the TI-Nspire

- Define $f(x) = e^x \sin(x)$.
- To find the derivative, press [template key] to obtain the template $\frac{d}{d\square}\square$ and then complete as shown.
- To find the second derivative, press [template key] to obtain the template $\frac{d^2}{d\square^2}\square$ and then complete as shown.



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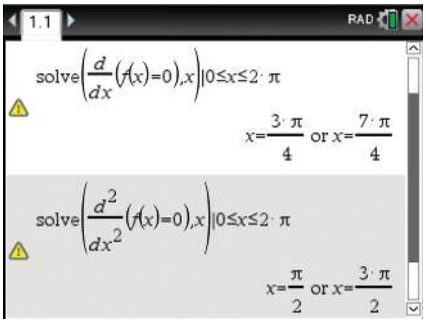
Stationary points



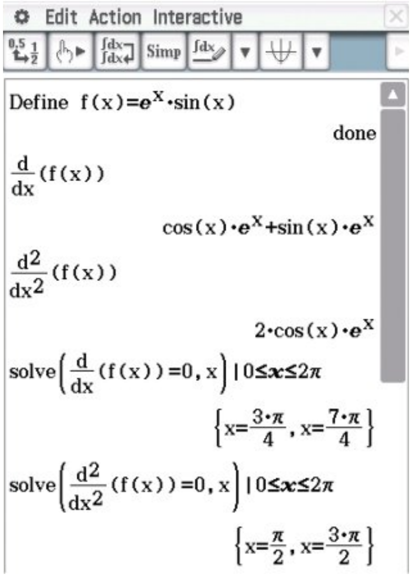
- Solve the equation $\frac{d}{dx}(f(x)) = 0$ for x such that $0 \leq x \leq 2\pi$.
- Substitute to find the y-coordinates.
- The stationary points are $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}} e^{\frac{3\pi}{4}}\right)$ and $\left(\frac{7\pi}{4}, \frac{-1}{\sqrt{2}} e^{\frac{7\pi}{4}}\right)$.

Points of inflection

- Solve the equation $\frac{d^2}{dx^2}(f(x)) = 0$ for x such that $0 \leq x \leq 2\pi$.
- Note that the second derivative changes sign at each of these x-values.
- Substitute to find the y-coordinates.
- The points of inflection are $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right)$.

Using the Casio ClassPad

- Define $f(x) = e^x \sin(x)$.
- Find $\frac{d}{dx}(f(x))$ and $\frac{d^2}{dx^2}(f(x))$.



Stationary points

- Solve the equation $\frac{d}{dx}(f(x)) = 0$ for x such that $0 \leq x \leq 2\pi$.
- Substitute to find the y-coordinates.
- The stationary points are $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}} e^{\frac{3\pi}{4}}\right)$ and $\left(\frac{7\pi}{4}, \frac{-1}{\sqrt{2}} e^{\frac{7\pi}{4}}\right)$.

Points of inflection

- Solve the equation $\frac{d^2}{dx^2}(f(x)) = 0$ for x such that $0 \leq x \leq 2\pi$.
- Note that the second derivative changes sign at each of these x-values.
- Substitute to find the y-coordinates.
- The points of inflection are $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right)$.

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Example 28

Sketch the graph of $f(x) = x^4 - 8x^3 + 18x^2 + 4$, locating the stationary points and the points of inflection.

Solution

$$f(x) = x^4 - 8x^3 + 18x^2 + 4$$
$$f'(x) = 4x^3 - 24x^2 + 36x = 4x(x-3)^2$$
$$f''(x) = 12x^2 - 48x + 36 = 12(x-1)(x-3)$$

Stationary points

$f'(x) = 0$ implies $x = 0$ or $x = 3$

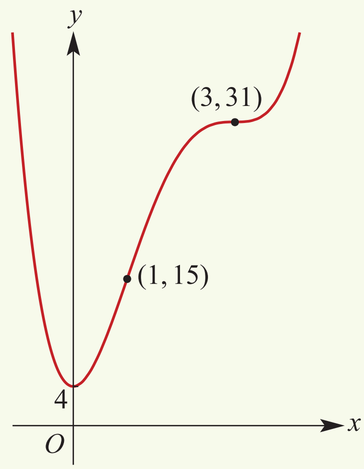
- Since $f''(0) = 36 > 0$, there is a local minimum at $(0, 4)$.
- Since $f''(3) = 0$, the test is inconclusive; further investigation is required.

Points of inflection

$f''(x) = 0$ implies $x = 1$ or $x = 3$

When $x < 1$, $f''(x) > 0$.
When $1 < x < 3$, $f''(x) < 0$.
When $x > 3$, $f''(x) > 0$.

- There is a point of inflection at $(1, 15)$ and a stationary point of inflection at $(3, 31)$.



Summary 5F

- For a curve $y = f(x)$:
  - If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval (a, b). The curve is said to be **concave up**.
  - If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval (a, b). The curve is said to be **concave down**.
- A **point of inflection** is where the curve changes from concave up to concave down or from concave down to concave up.
- At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.
- Second derivative test

  For the graph of $y = f(x)$:
  - If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
  - If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
  - If $f''(a) = 0$, then further investigation is necessary.

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Skill-sheet

Exercise 5F

1 Sketch a small portion of a continuous curve around a point $x = a$ having the property:

a $\frac{dy}{dx} > 0$ when $x = a$ and $\frac{d^2y}{dx^2} > 0$ when $x = a$

b $\frac{dy}{dx} < 0$ when $x = a$ and $\frac{d^2y}{dx^2} < 0$ when $x = a$

c $\frac{dy}{dx} > 0$ when $x = a$ and $\frac{d^2y}{dx^2} < 0$ when $x = a$

d $\frac{dy}{dx} < 0$ when $x = a$ and $\frac{d^2y}{dx^2} > 0$ when $x = a$

Example 25

2 For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

a $f(x) = x^3 - x$ **b** $f(x) = x^3 - x^2$ **c** $f(x) = x^2 - x^3$ **d** $f(x) = x^4 - x^3$

Example 26

3 Let $f(x) = \frac{x^2}{10}(20 - x)$, $\{ x \in \mathbb{R} : 0 \le x \le 20 \}$.

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 20]$.

4 Let $f(x) = 2x^3 + 6x^2 - 12$.

a **i** Find $f'(x)$. **ii** Find $f''(x)$.

b Find the coordinates of the stationary points and use the second derivative test to establish their nature.

c Use $f''(x)$ to find the point of inflection.

5 Repeat Question 4 for each of the following functions:

a $f(x) = \sin x$, $\{ x \in \mathbb{R} : 0 \le x \le 2\pi \}$

b $f(x) = xe^x$

Example 28

6 Sketch the graph of $y = x^3(4 - x)$, locating the stationary points and the points of inflection.

7 Sketch the graph of $y = 3x^4 - 44x^3 + 144x^2$, locating the stationary points and the points of inflection.

8 Let $f(x) = x(10 - x)e^x$, $\{ x \in \mathbb{R} : 0 \le x \le 10 \}$.

a Find $f'(x)$ and $f''(x)$.

b Sketch the graphs of $y = f(x)$ and $y = f''(x)$ on the one set of axes for $x \in [0, 10]$.

c Find the value of x for which the gradient of the graph of $y = f(x)$ is a maximum and indicate this point on the graph of $y = f(x)$.

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9 Find the coordinates of the points of inflection of $y = x - \sin x$ for $x \in [0, 4\pi]$.

10 For each of the following functions, find the values of x in the interval $[0, 2\pi]$ for which the graph of the function has a point of inflection:

a $y = \sin x$ **b** $y = \tan x$ **c** $y = \sin(2x)$

11 Show that the parabola with equation $y = ax^2 + bx + c$ has no points of inflection.

12 For the curve with equation $y = 2x^3 - 9x^2 + 12x + 8$, find the values of x for which:

a $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$ **b** $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} < 0$

13 For each of the following functions, determine the coordinates of any points of inflection and the gradient of the graph at these points:

a $y = x^3 - 6x$ **b** $y = x^4 - 6x^2 + 4$ **c** $y = 3 - 10x^3 + 10x^4 - 3x^5$

d $y = (x^2 - 1)(x^2 + 1)$ **e** $y = x\sqrt{x + 1}$ **f** $y = \dfrac{2x}{x^2 + 1}$

14 Determine the values of x in $[-2\pi, 2\pi]$ for which the graph of $y = e^{-x} \sin x$ has:

a stationary points **b** points of inflection.

15 Given that $f(x) = x^3 + bx^2 + cx$ and $b^2 > 3c$, prove that:

a the graph of f has two stationary points

b the graph of f has one point of inflection

c the point of inflection is the midpoint of the interval joining the stationary points.

16 Consider the function with rule $f(x) = 2x^2 \ln(x)$.

a Find $f'(x)$.

b Find $f''(x)$.

c Find the stationary points and the points of inflection of the graph of $y = f(x)$.

17 The graph of $y = x^3 - ax^2 + bx + c$ passes through the point $(2, 7)$, has a local maximum when $x = 1$ and a point of inflection when $x = 3$.

a Find the values of a, b and c.

b Sketch the graph.

18 The graph of $y = ax^3 + bx + c$ has a point of inflection where $y = 3$, a local maximum where $x = 1$ and passes through the point $(2, 1)$.

a Find the values of a, b and c.

b Give the value of x for which there is a local minimum.

c Sketch the graph.

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5G Absolute maximum and minimum values

Local maximum and minimum values were discussed in the previous two sections. These are often not the actual maximum and minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

More precisely, for a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the absolute maximum value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the absolute minimum value of the function.

Example 29

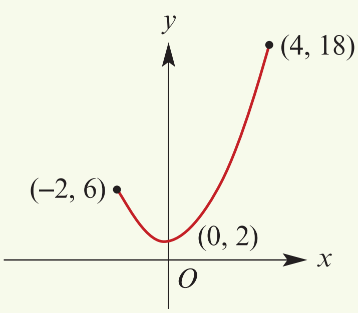
Let $f(x) = x^2 + 2$, $\{ x \in \mathbb{R} : -2 \leq x \leq 4 \}$. Find the absolute maximum value and the absolute minimum value of the function.

Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

(Note that the absolute minimum occurs at a stationary point of the graph. The absolute maximum occurs at an endpoint, not at a stationary point.)



Example 30

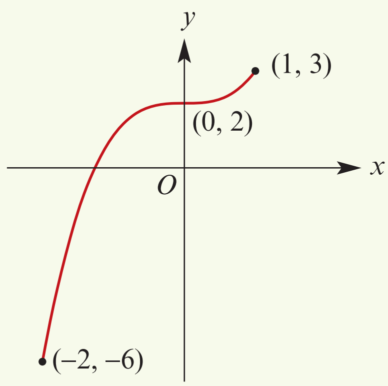
Let $f(x) = x^3 + 2$, $\{ x \in \mathbb{R} : -2 \leq x \leq 1 \}$. Find the maximum and minimum values of the function.

Solution

The maximum value is 3 and occurs when $x = 1$.

The minimum value is -6 and occurs when $x = -2$.

(Note that the absolute maximum and minimum values do not occur at stationary points.)



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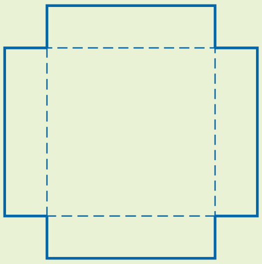
Example 31

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d If the height of the box must be less than 0.3 m, i.e. $x \leq 0.3$, what will be the maximum volume of the box?

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$\begin{aligned} V &= (2-2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \end{aligned}$$

b Let $V(x) = 4x^3 - 8x^2 + 4x$. A local maximum will occur when $V'(x) = 0$.
We have $V'(x) = 12x^2 - 16x + 4$, and so $V'(x) = 0$ implies that

$$\begin{aligned} 12x^2 - 16x + 4 &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x-1)(x-1) &= 0 \\ \therefore \quad x &= \frac{1}{3} \text{ or } x = 1 \end{aligned}$$

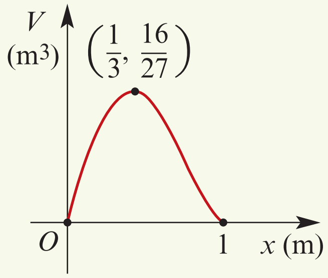
But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

x		$\frac{1}{3}$		1
$V'(x)$	+	0	−	0
shape of V	⁄	—	╲	—

The maximum occurs when $x = \frac{1}{3}$.

$$\therefore \text{Maximum volume} = \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3} = \frac{16}{27} \text{ m}^3$$

c



d The local maximum of $V(x)$ defined on $[0, 1]$ is at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

But $\frac{1}{3}$ is not in the interval $[0, 0.3]$.

Since $V'(x) > 0$ for all $x \in [0, 0.3]$, the maximum volume for this situation occurs when $x = 0.3$ and is 0.588 m^3.

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Summary 5G

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Skill-sheet

Exercise 5G

Example 29

1 Let $f(x) = 2 - 8x^2$, $\{ x \in \mathbb{R} : -3 \leq x \leq 3 \}$. Find the absolute maximum value and the absolute minimum value of the function.

Example 30

2 Let $f(x) = x^3 + 2x + 3$, $\{ x \in \mathbb{R} : -3 \leq x \leq 2 \}$. Find the absolute maximum value and the absolute minimum value of the function for its domain.

3 Let $f(x) = 2x^3 - 6x^2$, $\{ x \in \mathbb{R} : -1.5 \leq x \leq 2.5 \}$. Find the absolute maximum and absolute minimum values of the function.

4 Let $f(x) = 2x^4 - 8x^2$, $\{ x \in \mathbb{R} : -2 \leq x \leq 6 \}$. Find the absolute maximum and absolute minimum values of the function.

Example 31

5 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, of the block is given by $V = 15x^2 - 12x^3$.

b Find $\dfrac{dV}{dx}$.

c Find the coordinates of the local maximum of the graph of V against x for $x \in [0, 1.25]$.

d If $x \in [0, 0.8]$, find the absolute maximum value of V and the value of x for which this occurs.

e If $x \in [0, 1]$, find the absolute maximum value of V and the value of x for which this occurs.

6 Variables x, y and z are such that $x + y = 30$ and $z = xy$.

a If $x \in [2, 5]$, find the possible values of y.

b Find the absolute maximum and absolute minimum values of z.

7 Consider the function $f(x) = \dfrac{1}{x-1} + \dfrac{1}{4-x}$, $\{ x \in \mathbb{R} : 2 \leq x \leq 3 \}$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

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8 A piece of string 10 metres long is cut into two pieces to form two squares.

a If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.

b Find $\frac{dA}{dx}$.

c Find the value of x that makes A a minimum.

d If two squares are formed but $x \in [0, 1]$, find the maximum possible combined area of the two squares.

9 Find the absolute maximum and minimum values of the function $g(x) = x + \frac{1}{x-2}$, $\{x \in \mathbb{R} : 2.1 \le x \le 8\}$.

10 Consider the function $f(x) = \frac{1}{x+1} + \frac{1}{4-x}$, $\{x \in \mathbb{R} : 0 \le x \le 3\}$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

11 For the function $f(x) = \sin(2x)$, $\left\{x \in \mathbb{R} : -\frac{\pi}{2} \le x \le \frac{\pi}{8}\right\}$, state the absolute maximum and minimum values of the function.

12 For the function $f(x) = \cos(2x)$, $\left\{x \in \mathbb{R} : 0 \le x \le \frac{\pi}{8}\right\}$, state the absolute maximum and minimum values of the function.

13 For the function $f(x) = 2 - x^{\frac{2}{3}}$, $\{x \in \mathbb{R} : -1 \le x \le 8\}$, sketch the graph and state the absolute maximum and minimum values of the function.

14 For the function $f(x) = 2e^x + e^{-x}$, $\{x \in \mathbb{R} : -1 \le x \le 2\}$, sketch the graph and state the absolute maximum and minimum values of the function.

15 For the function $f(x) = (x-5)\ln\left(\frac{x-5}{10}\right)$, $\{x \in \mathbb{R} : 6 \le x \le 10\}$, sketch the graph and state the absolute maximum and minimum values of the function.

5H Maximum and minimum problems

Many practical problems require that some quantity (for example, cost of manufacture or fuel consumption) be **minimised**, that is, be made as small as possible. Other problems require that some quantity (for example, profit on sales or attendance at a concert) be **maximised**, that is, be made as large as possible. We can use differential calculus to solve many of these problems.

We now have two methods for identifying whether a stationary point corresponds to a local maximum or local minimum value:

Method 1 create a gradient chart

Method 2 use the second derivative test.

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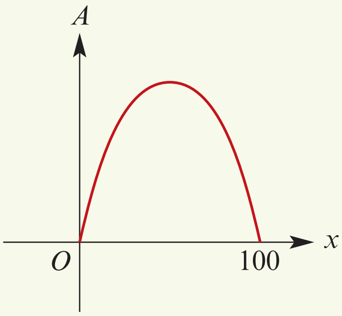
Example 32

A farmer has sufficient fencing to make a rectangular pen of perimeter 200 metres. What dimensions will give an enclosure of maximum area?

Solution

Let the length of the rectangle be x metres. Then the width is $100 - x$ metres and the area is A m^2, where

$$A = x(100 - x)$$
$$= 100x - x^2$$



The maximum value of A occurs when $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 100 - 2x$$

$$\therefore \quad \frac{dA}{dx} = 0 \text{ implies } x = 50$$

From the gradient chart, the maximum area occurs when $x = 50$.

The pen with maximum area has dimensions 50 m by 50 m, and so has area 2500 m^2.

x		50	
$\frac{dA}{dx}$	+	0	−
shape of A	/	—	\

Example 33

Two variables x and y are such that $x^4y = 8$. A third variable z is defined by $z = x + y$. Find the values of x and y that give z a stationary value. Use the second derivative test to show that this value of z is a minimum.

Solution

Obtain y in terms of x from the equation $x^4y = 8$:

$$y = 8x^{-4}$$

Substitute in the equation $z = x + y$:

$$z = x + 8x^{-4} \qquad (1)$$

Now z is expressed in terms of one variable, x. Differentiate with respect to x:

$$\frac{dz}{dx} = 1 - 32x^{-5}$$

A stationary point occurs where $\frac{dz}{dx} = 0$:

$$1 - 32x^{-5} = 0$$
$$32x^{-5} = 1$$
$$x^5 = 32$$
$$\therefore \quad x = 2$$

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There is a stationary point at $x = 2$. The corresponding value of y is $8 \times 2^{-4} = \frac{1}{2}$.

Now substitute in equation (1) to find z:

$$z = 2 + \frac{8}{16} = 2\frac{1}{2}$$

Second derivative test:

When $x = 2$, we have

$$\frac{d^2z}{dx^2} = 160x^{-6} = \frac{160}{2^6} > 0$$

which corresponds to a local minimum.

The minimum value of z is $2\frac{1}{2}$ and occurs when $x = 2$ and $y = \frac{1}{2}$.

Example 34

A cylindrical tin canister closed at both ends has a surface area of 100 cm^2. Find, correct to two decimal places, the greatest volume it can have. If the radius of the canister can be at most 2 cm, find the greatest volume it can have.

Solution

Let the radius of the circular end of the tin be r cm, let the height of the tin be h cm and let the volume of the tin be V cm^3.

Obtain equations for the surface area and the volume.

Surface area: $100 = 2\pi r^2 + 2\pi rh$ (1)

Volume: $V = \pi r^2 h$ (2)

The process we follow now is very similar to Example 33. Obtain h in terms of r from equation (1):

$$h = \frac{1}{2\pi r}(100 - 2\pi r^2)$$

Substitute in equation (2):

$$V = \pi r^2 \times \frac{1}{2\pi r}(100 - 2\pi r^2)$$

$$\therefore \quad V = 50r - \pi r^3 \qquad (3)$$

A stationary point of the graph of $V = 50r - \pi r^3$ occurs when $\frac{dV}{dr} = 0$.

$$\frac{dV}{dr} = 0 \text{ implies } 50 - 3\pi r^2 = 0$$

$$\therefore \quad r = \pm\sqrt{\frac{50}{3\pi}} \approx \pm 2.3$$

But $r = -2.3$ does not fit the practical situation.

Substitute $r = 2.3$ in equation (3) to find $V \approx 76.78$.

So there is a stationary point at (2.3, 76.8).

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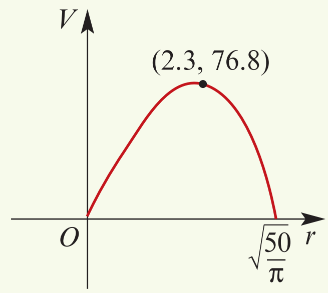
Use a gradient chart to determine the nature of this stationary point.

The maximum volume is 76.78 cm^3 correct to two decimal places.

r		2.3	
$\frac{dV}{dr}$	+	0	−
shape of V	╱	—	╲

It can be observed that the volume is given by a function f with rule $f(r) = 50r - \pi r^3$ and domain $\left[0, \sqrt{\frac{50}{\pi}}\right]$, giving the graph on the right.



If the greatest radius the canister can have is 2 cm, then the function f has domain $[0, 2]$. It has been seen that $f'(r) > 0$ for all $r \in [0, 2]$. The maximum value occurs when $r = 2$. The maximum volume in this case is $f(2) = 100 - 8\pi \approx 74.87$ cm^3.

In some situations the variables may not be continuous. For instance, one of them may only take integer values. In such cases, we may be able to model the non-continuous case with a continuous function so that the techniques of differential calculus can be used.

Example 35

A TV cable company has 1000 subscribers who are paying \$5 per month. It can get 100 more subscribers for each \$0.10 decrease in the monthly fee. What monthly fee will yield the maximum revenue and what will this revenue be?

Solution

Let x denote the monthly fee. Then the number of subscribers is $1000 + 100\left(\frac{5-x}{0.1}\right)$.

(Note that we are treating a discrete situation with a continuous function.)

Let R denote the revenue. Then

$$R = x(1000 + 1000(5 - x))$$
$$= 1000(6x - x^2)$$
$$\therefore \quad \frac{dR}{dx} = 1000(6 - 2x)$$

Thus $\frac{dR}{dx} = 0$ implies $6 - 2x = 0$ and hence $x = 3$.

Second derivative test:

When $x = 3$, we have

$$\frac{d^2R}{dx^2} = -2000 < 0$$

which corresponds to a local maximum.

For maximum revenue, the monthly fee should be \$3, and this gives a total revenue of \$9000.

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Example 36

A manufacturer annually produces and sells 10 000 shirts. Sales are uniformly distributed throughout the year. The production cost of each shirt is \$23 and the carrying costs (storage, insurance, interest) depend on the total number of shirts in a production run. (A production run is the number, x, of shirts which are under production at a given time.)

The set-up costs for a production run are \$40. The annual carrying costs are $\$x^{\frac{3}{2}}$. Find the size of a production run that minimises the total set-up and carrying costs for a year.

Solution

Number of production runs per year $= \dfrac{10\,000}{x}$

Set-up costs for these production runs $= 40\left(\dfrac{10\,000}{x}\right)$

Let C be the total set-up and carrying costs. Then

$$C = x^{\frac{3}{2}} + \frac{400\,000}{x}$$

$$= x^{\frac{3}{2}} + 400\,000x^{-1}, \quad x > 0$$

$$\therefore \quad \frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{400\,000}{x^2}$$

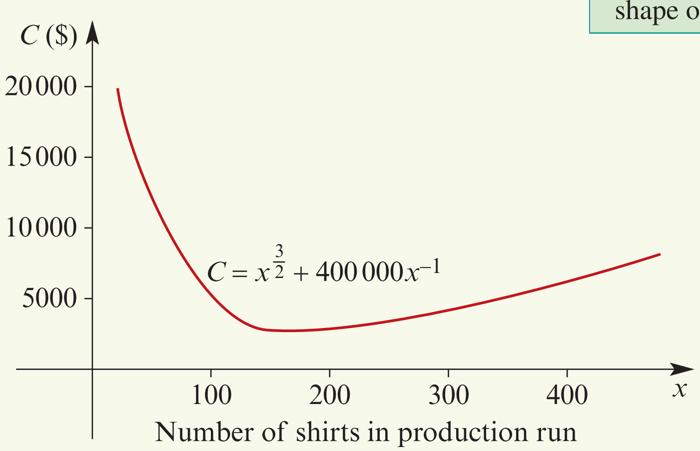
Thus $\dfrac{dC}{dx} = 0$ implies $\dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{400\,000}{x^2}$

$$x^{\frac{5}{2}} = \frac{400\,000 \times 2}{3}$$

$$\therefore \quad x \approx 148.04$$

Each production run should be 148 shirts.

x		148.04	
$\dfrac{dC}{dx}$	−	0	+
shape of C	\	—	/



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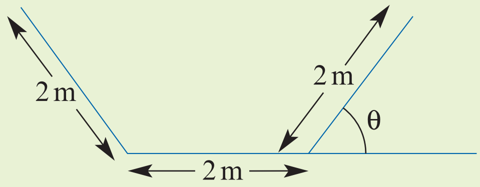
Example 37

The cross-section of a drain is to be an isosceles trapezium, with three sides of length 2 metres, as shown. Find the angle θ that maximises the cross-sectional area, and find this maximum area.



Solution

Let A m^2 be the area of the trapezium. Then

$$A = \frac{1}{2} \times 2\sin\theta \times (2 + 2 + 4 \times \cos\theta)$$
$$= \sin\theta \cdot (4 + 4\cos\theta)$$

and
$$A'(\theta) = \cos\theta \cdot (4 + 4\cos\theta) - 4\sin^2\theta$$
$$= 4\cos\theta + 4\cos^2\theta - 4(1 - \cos^2\theta)$$
$$= 4\cos\theta + 8\cos^2\theta - 4$$

The maximum will occur when $A'(\theta) = 0$:

$$8\cos^2\theta + 4\cos\theta - 4 = 0$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\therefore \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

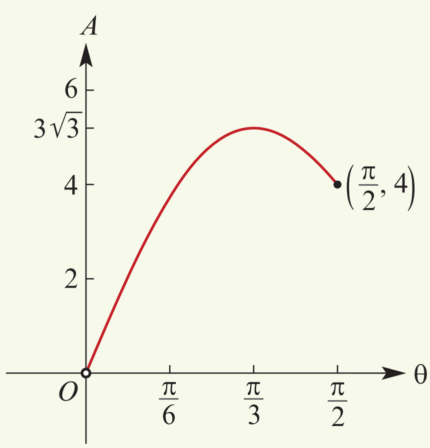
The practical restriction on θ is that $0 < \theta \le \frac{\pi}{2}$.

Therefore the only possible solution is $\theta = \frac{\pi}{3}$, and a gradient chart confirms that $\frac{\pi}{3}$ gives a maximum.

θ		$\frac{\pi}{3}$	
$A'(\theta)$	+	0	−
shape of A	╱	—	╲

When $\theta = \frac{\pi}{3}$, $A = \frac{\sqrt{3}}{2}(4 + 2) = 3\sqrt{3}$,

i.e. the maximum cross-sectional area is $3\sqrt{3}$ m^2.



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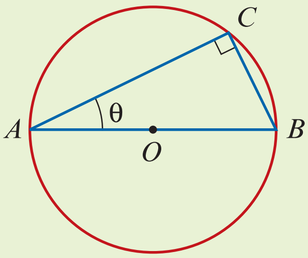
Example 38

The figure shows a circular lake, centre O, of radius 2 km. A man swims across the lake from A to C at 3 km/h and then walks around the edge of the lake from C to B at 4 km/h.



a If $\angle BAC = \theta$ radians and the total time taken is T hours, show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta)$$

b Find the value of θ for which $\frac{dT}{d\theta} = 0$ and determine whether this gives a maximum or minimum value of T $(0° < \theta° < 90°)$.

Solution

a Time taken $= \dfrac{\text{distance travelled}}{\text{speed}}$

Therefore the swim takes $\dfrac{4\cos\theta}{3}$ hours and the walk takes $\dfrac{4\theta}{4}$ hours.

Thus the total time taken is given by $T = \frac{1}{3}(4\cos\theta + 3\theta)$.

b $\dfrac{dT}{d\theta} = \frac{1}{3}(-4\sin\theta + 3)$

The stationary point occurs where $\frac{dT}{d\theta} = 0$, and $\frac{1}{3}(-4\sin\theta + 3) = 0$ implies $\sin\theta = \frac{3}{4}$.

Therefore $\theta = 48.59°$ to two decimal places.

From the gradient chart, the value of T is a maximum when $\theta = 48.59°$.

θ		$48.59°$	
$\frac{dT}{d\theta}$	+	0	−
shape of T	/	—	\

Notes:

- The maximum time taken is 1.73 hours.
- If the man swims straight across the lake, it takes $1\frac{1}{3}$ hours.
- If he walks around all the way around the edge, it takes approximately 1.57 hours.

Example 39

Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Solution

$$N(t) = 36te^{-0.1t}$$

$$\therefore \quad N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$$

$$= e^{-0.1t}(36 - 3.6t)$$

Thus $N'(t) = 0$ implies $t = 10$.

The maximum population is $N(10) = 360e^{-1} \approx 132$.

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Maximum rates of increase and decrease

We know that we can use the derivative of a function to help find the maximum and minimum values of the function. Similarly, we can use the second derivative to help find the maximum rate of increase or decrease of the function.

The second derivative gives the instantaneous rate of change of the derivative.

- If $\frac{d^2y}{dx^2} > 0$, then $\frac{dy}{dx}$ is increasing as x increases.
- If $\frac{d^2y}{dx^2} < 0$, then $\frac{dy}{dx}$ is decreasing as x increases.

Example 40

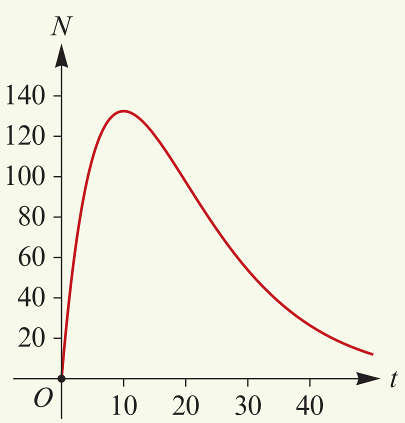
Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

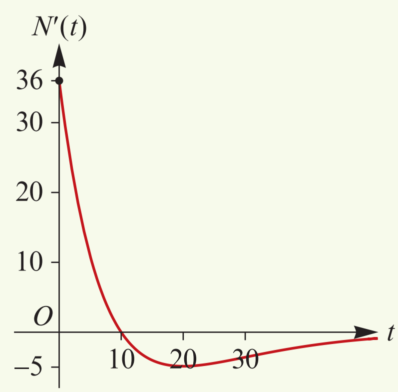
b Find the maximum rates of increase and decrease of the population and the times at which these occur.

Solution

a $N(t) = 36te^{-0.1t}$



$N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$



b The rate of change of the population is $N'(t) = e^{-0.1t}(36 - 3.6t)$.

From the graph, the maximum value of $N'(t)$ occurs at $t = 0$. Thus the maximum rate of increase of the population is $N'(0) = 36$ bacteria per unit of time.

We now calculate

$$\begin{aligned} N''(t) &= -7.2e^{-0.1t} + 0.36te^{-0.1t} \\ &= e^{-0.1t}(-7.2 + 0.36t) \end{aligned}$$

Thus $N''(t) = 0$ implies $t = 20$.

The minimum value of $N'(t)$ occurs at $t = 20$. Since $N'(20) = -36e^{-2} \approx -4.9$, the maximum rate of decrease of the population is 4.9 bacteria per unit of time.

Note: The graph of the original function N has a point of inflection at $t = 20$.

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Summary 5H

Here are some steps for solving maximum and minimum problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Exercise 5H

Example 32

1 Find the maximum area of a rectangular field that can be enclosed by 100 m of fencing.

Example 33

2 Find two positive numbers that sum to 4 and such that the sum of the cube of the first and the square of the second is as small as possible.

3 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$ and find the maximum value of P. (Use the second derivative test.)

4 A farmer has 4 km of fencing wire and wishes to fence a rectangular piece of land through which flows a straight river, which is to be utilised as one side of the enclosure. How can this be done to enclose as much land as possible?

5 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum. (Use the second derivative test.)

Example 34

6 A cuboid has a total surface area of 150 cm^2 with a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \dfrac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

Example 35
Example 36

7 A manufacturer finds that the daily profit, \$$P$, from selling n articles is given by $P = 100n - 0.4n^2 - 160$.

a **i** Find the value of n which maximises the daily profit.

ii Find the maximum daily profit.

b Sketch the graph of P against n. (Use a continuous graph.)

c State the allowable values of n for a profit to be made.

d Find the value of n which maximises the profit per article.

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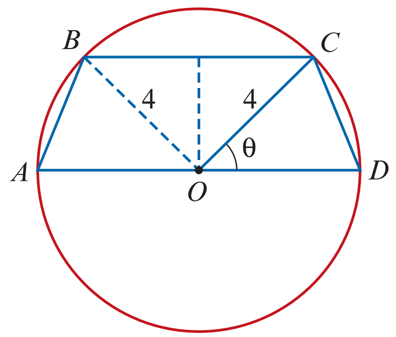
8 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$ with x representing the temperature of the water in degrees (°C). (This function is valid only if $6 \le x \le 20$.) Find the water temperature that produces the maximum number of salmon swimming upstream.

9 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Find the rainfall that will produce the maximum and the minimum number of mosquitos.

Example 37 **10** $ABCD$ is a trapezium with $AB = CD$. The vertices are on a circle with centre O and radius 4 units. The line segment AD is a diameter of the circle.

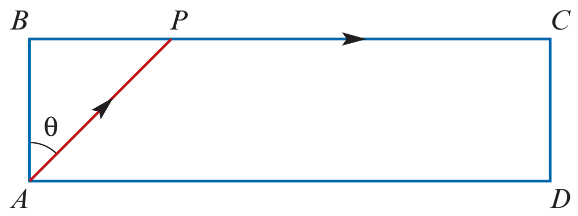


a Find BC in terms of θ.

b Find the area of the trapezium in terms of θ and hence find the maximum area.

11 Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$.

Example 38 **12** The figure shows a rectangular field in which $AB = 300$ m and $BC = 1100$ m.



$AB = 300$ m

$BC = 1100$ m

a An athlete runs across the field from A to P at 4 m/s. Find the time taken to run from A to P in terms of θ.

b The athlete, on reaching P, immediately runs to C at 5 m/s. Find the time taken to run from P to C in terms of θ.

c Use the results from **a** and **b** to show that the total time taken, T seconds, is given by $T = 220 + \dfrac{75 - 60\sin\theta}{\cos\theta}$.

d Find $\dfrac{dT}{d\theta}$.

e Find the value of θ for which $\dfrac{dT}{d\theta} = 0$ and show that this is the value of θ for which T is a minimum.

f Find the minimum value of T and find the distance of point P from B that will minimise the athlete's running time.

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Example 39 **13** The number $N(t)$ of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Example 40 **14** The number $N(t)$ of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

b Find the maximum rates of increase and decrease of the population and the times at which these occur.

15 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by

$$V(t) = \frac{3}{4}\left(10t^2 - \frac{t^3}{3}\right), \quad 0 \le t \le 20$$

a Find the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Find $V'(t)$, the rate of flow of water into the flask.

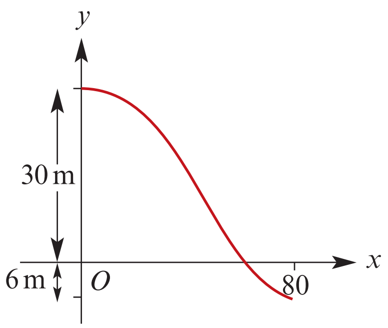
c Sketch the graph of $V(t)$ against t for $0 \le t \le 20$.

d Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

e At what time is the flow greatest and what is the flow at this time?

16 A section of a roller coaster can be described by the rule

$$y = 18\cos\left(\frac{\pi x}{80}\right) + 12, \quad 0 \le x \le 80$$



a Find the gradient function, $\frac{dy}{dx}$.

b Sketch the graph of $\frac{dy}{dx}$ against x.

c State the coordinates of the point on the track for which the magnitude of the gradient is maximum.

17 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Find the rate at which the depth is changing when:

i $t = 3$ **ii** $t = 6$ **iii** $t = 12$

d **i** At what times is the depth increasing most rapidly?

ii At what times is the depth decreasing most rapidly?

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5I Families of functions

Example 41

Consider the family of functions with rules of the form $f(x) = (x - a)^2(x - b)$, where a and b are positive constants with $b > a$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(a, 0)$ is always a local maximum.

d Find the values of a and b if the stationary points occur where $x = 3$ and $x = 4$.

Solution

a Use a CAS calculator to find that $f'(x) = (x - a)(3x - a - 2b)$.

b The coordinates of the stationary points are $(a, 0)$ and $\left(\dfrac{a+2b}{3}, \dfrac{4(a-b)^3}{27}\right)$.

c Use a CAS calculator to find that $f''(x) = 6x - 4a - 2b$. Thus $f''(a) = 2a - 2b < 0$, as $a < b$, and therefore the stationary point at $(a, 0)$ is a local maximum.

d Since $a < b$, we must have $a = 3$ and $\dfrac{a+2b}{3} = 4$. Therefore $b = \dfrac{9}{2}$.

Example 42

The graph of $y = x^3 - 3x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^3 - 3x^2$.

b Find the coordinates of the turning points of its image.

Solution

a The turning points have coordinates $(0, 0)$ and $(2, -4)$.

b The turning points of the image are (a, b) and $(2 + a, -4 + b)$.

Example 43

A cubic function with rule $f(x) = ax^3 + bx^2 + cx$ has a stationary point at $(1, 6)$.

a Find a and b in terms of c.

b Find the value of c for which the graph has a point of inflection at $x = 2$.

Solution

a Since $f(1) = 6$, we obtain

$$a + b + c = 6 \qquad (1)$$

Since $f'(x) = 3ax^2 + 2bx + c$ and $f'(1) = 0$, we obtain

$$3a + 2b + c = 0 \qquad (2)$$

The solution of equations (1) and (2) is $a = c - 12$ and $b = 18 - 2c$.

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b The rule is

$$f(x) = (c - 12)x^3 + (18 - 2c)x^2 + cx$$
$$\therefore \quad f'(x) = 3(c - 12)x^2 + 2(18 - 2c)x + c$$
$$\therefore \quad f''(x) = 6(c - 12)x + 2(18 - 2c)$$

If $f''(2) = 0$, then

$$12(c - 12) + 2(18 - 2c) = 0$$
$$8c - 108 = 0$$
$$\therefore \quad c = \frac{27}{2}$$

Exercise 5I

Example 41

1 Consider the family of functions with rules $f(x) = (x - 1)^2(x - b)$, where $b > 1$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(1, 0)$ is always a local maximum.

d Find the value of b if the stationary points occur where $x = 1$ and $x = 4$.

Example 42

2 The graph of the function $y = x^4 - 4x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^4 - 4x^2$.

b Find the coordinates of the turning points of its image.

Example 43

3 A cubic function f has rule $f(x) = ax^3 + bx^2 + cx$. The graph has a stationary point at $(1, 10)$.

a Find a and b in terms of c.

b Find the value of c for which the graph has a point of inflection at $x = 3$.

4 Consider the function defined by $f(x) = x^2 - ax^3$, $\{ x \in \mathbb{R} : x \geq 0 \}$, where a is a real number with $a > 0$.

a Determine the intervals on which f is a strictly decreasing function and the intervals on which f is a strictly increasing function.

b Find the equation of the tangent to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

c Find the equation of the normal to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

d What is the range of f?

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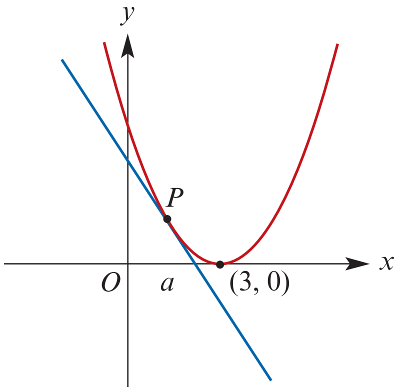
5 A line with equation $y = mx + c$ is a tangent to the curve $y = (x - 3)^2$ at a point $P(a, y)$ where $0 < a < 3$.



- **a** **i** Find the gradient of the curve at $x = a$ for $0 < a < 3$.
  - **ii** Hence express m in terms of a.
- **b** State the coordinates of the point P, expressing your answer in terms of a.
- **c** Find the equation of the tangent where $x = a$.
- **d** Find the x-axis intercept of the tangent.

6
- **a** The graph of $f(x) = x^4$ is translated to the graph of $y = f(x + h)$. Find the possible values of h if $f(1 + h) = 16$.
- **b** The graph of $f(x) = x^3$ is transformed to the graph of $y = f(ax)$. Find the possible value of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.
- **c** The quartic function with equation $y = ax^4 - bx^3$ has a turning point with coordinates $(1, 16)$. Find the values of a and b.

7 Consider the cubic function with rule $f(x) = (x - a)^2(x - 1)$ where $a > 1$.
- **a** Find the coordinates of the turning points of the graph of $y = f(x)$.
- **b** State the nature of each of the turning points.
- **c** Find the equation of the tangent to the curve at the point where:

  i $x = 1$ **ii** $x = a$ **iii** $x = \dfrac{a+1}{2}$

8 Consider the quartic function with rule $f(x) = (x - 1)^2(x - b)^2$ where $b > 1$.
- **a** Find the derivative of f.
- **b** Find the coordinates of the turning points of f.
- **c** Find the value of b such that the graph of $y = f(x)$ has a turning point at $(2, 1)$.

9 A cubic function has rule $y = ax^3 + bx^2 + cx + d$. It passes through the points $(1, 6)$ and $(10, 8)$ and has turning points where $x = -1$ and $x = 1$. Find the values of a, b, c and d.

10 A quartic function f has rule $f(x) = ax^4 + bx^3 + cx^2 + dx$. The graph has a stationary point at $(1, 1)$ and passes through the point $(-1, 4)$.
- **a** Find a, b and c in terms of d.
- **b** Find the value of d for which the graph has a stationary point at $x = 4$.

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Review

Chapter summary

Assignment

Nrich

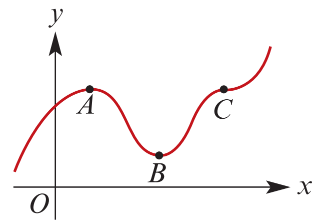
Tangents and normals

Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, then

- the equation of the **tangent** to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$
- the equation of the **normal** to the curve at (x_1, y_1) is given by $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$.

Stationary points

- A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is a **stationary point** if $f'(a) = 0$.
- The graph shown has three stationary points: A, B and C.



A Point A is a **local maximum** point. Notice that immediately to the left of A the gradient is positive, and immediately to the right the gradient is negative.

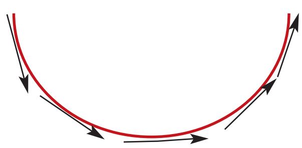
B Point B is a **local minimum** point. Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

C Point C is a **stationary point of inflection**.

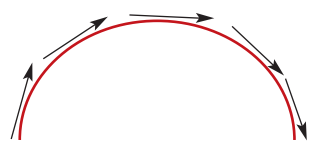
Stationary points of types A and B are referred to as **turning points**.

Using the second derivative in graph sketching

- Concave up: $f''(x) > 0$



- Concave down: $f''(x) < 0$



- A **point of inflection** is where the curve changes from concave up to concave down or from concave down to concave up.
- At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.
- Second derivative test For the graph of $y = f(x)$:
  - If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
  - If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
  - If $f''(a) = 0$, then further investigation is necessary.

Maximum and minimum values

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

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Motion in a straight line

For an object moving in a straight line with position x at time t:

- velocity $v = \frac{dx}{dt}$
- acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Linear approximation using the increments formula: $\delta y \approx \frac{dy}{dx} \times \delta x$

- We can use the tangent to a curve at a point to approximate the curve near that point.
- For a small number h, the change in the value of $y = f(x)$ as x changes from a to $a + h$ can be approximated using $f(a + h) - f(a) \approx hf'(a)$.
- Using Leibniz notation, we can write this as $\delta y \approx \frac{dy}{dx} \times \delta x$, where δx is a small change in x and δy is the corresponding change in y.

Short-answer questions

1 **a** Find the equation of the tangent to the curve $y = x^3 - 8x^2 + 15x$ at the point with coordinates $(4, -4)$.

b Find the coordinates of the point where the tangent meets the curve again.

2 Find the equation of the tangent to the curve $y = 3x^2$ at the point where $x = a$. If this tangent meets the y-axis at P, find the y-coordinate of P in terms of a.

3 **a** Find the equation of the tangent to the curve with equation $y = x^3 - 7x^2 + 14x - 8$ at the point where $x = 1$.

b Find the x-coordinate of a second point on this curve at which the tangent is parallel to the tangent at $x = 1$.

4 Use the formula $A = \pi r^2$ for the area of a circle to find:

a the average rate at which the area of a circle changes with respect to the radius as the radius increases from $r = 2$ to $r = 3$

b the instantaneous rate at which the area changes with respect to r when $r = 3$.

5 For each of the following, find the stationary points of the graph and state their nature:

a $f(x) = 4x^3 - 3x^4$ **b** $g(x) = x^3 - 3x - 2$ **c** $h(x) = x^3 - 9x + 1$

6 Sketch the graph of $y = x^3 - 6x^2 + 9x$.

7 The derivative of the function $y = f(x)$ is $\frac{dy}{dx} = (x - 1)^2(x - 2)$. Find the x-coordinate and state the nature of each stationary point.

8 Find the equation of the tangent to the curve $y = x^3 - 3x^2 - 9x + 11$ at $x = 2$.

9 Let $f(x) = 3 + 6x^2 - 2x^3$. Determine the values of x for which the graph of $y = f(x)$ has a positive gradient.

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10 For what value(s) of x do the graphs of $y = x^3$ and $y = x^3 + x^2 + x - 2$ have the same gradient?

11 For the function with rule $f(x) = (x-1)^{\frac{4}{5}}$:

a State the values for which the function is differentiable, and find the rule for f'.

b Find the equations of the tangents at the points $(2, 1)$ and $(0, 1)$.

c Find the coordinates of the point of intersection of the two tangents.

12 A spherical bubble, initially of radius 1 cm, expands steadily, its radius increases by 1 cm/s and it bursts after 5 seconds.

a Find the rate of increase of volume with respect to the radius when the radius is 4 cm.

b By using the chain rule, find the rate of increase of volume with respect to time when the radius is 4 cm.

13 A vehicle is travelling in a straight line away from a point O. Its distance from O after t seconds is $0.25e^t$ metres. Find the velocity of the vehicle at $t = 0$, $t = 1$, $t = 2$, $t = 4$.

14 The temperature, θ°C, of material inside a nuclear power station at time t seconds after a reaction begins is given by $\theta = \frac{1}{4}e^{100t}$.

a Find the rate of increase of temperature at time t.

b Find the rate of increase of temperature when $t = \frac{1}{20}$.

15 Find the equation of the tangent to $y = e^x$ at $(1, e)$.

16 The diameter of a tree (D cm) t years after 1 January 2010 is given by $D = 50e^{kt}$.

a Prove that $\frac{dD}{dt} = cD$ for some constant c.

b If $k = 0.2$, find the rate of increase of D when $D = 100$.

17 Find the minimum value of $e^{3x} + e^{-3x}$.

18 **a** Find the equation of the tangent to $y = \ln x$ at the point $(e, 1)$.

b Find the equation of the tangent to $y = 2\sin\left(\frac{x}{2}\right)$ at the point $\left(\frac{\pi}{2}, \sqrt{2}\right)$.

c Find the equation of the tangent to $y = \cos x$ at the point $\left(\frac{3\pi}{2}, 0\right)$.

d Find the equation of the tangent to $y = \ln(x^2)$ at the point $(-\sqrt{e}, 1)$.

19 For each of the following, state the coordinates of the point(s) of inflection:

a $y = x^3 - 8x^2$

b $y = \sin\left(x - \frac{\pi}{6}\right),\ 0 \le x \le 2\pi$

c $y = \ln(x) + \frac{1}{x}$

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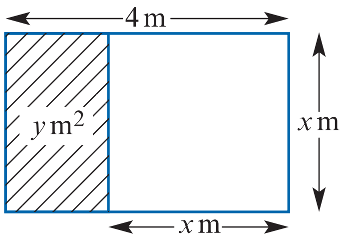
Extended-response questions

1 The diagram shows a rectangle with sides 4 m and x m and a square with side x m. The area of the shaded region is $y\,\text{m}^2$.



a Find an expression for y in terms of x.

b Find the set of possible values for x.

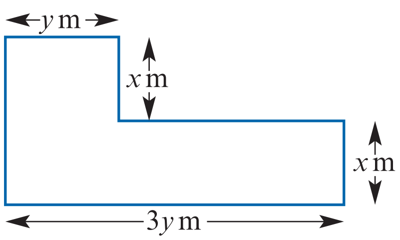
c Find the maximum value of y and the corresponding value of x.

d Explain briefly why this value of y is a maximum.

e Sketch the graph of y against x.

f State the set of possible values for y.

2 A flower bed is to be L-shaped, as shown in the figure, and its perimeter is 48 m.



a Write down an expression for the area, $A\,\text{m}^2$, in terms of y and x.

b Find y in terms of x.

c Write down an expression for A in terms of x.

d Find the values of x and y that give the maximum area.

e Find the maximum area.

3 It costs $(12 + 0.008x)$ dollars per kilometre to operate a truck at x kilometres per hour. In addition it costs \$14.40 per hour to pay the driver.

a What is the total cost per kilometre if the truck is driven at:

i 40 km/h **ii** 64 km/h?

b Write an expression for C, the total cost per kilometre, in terms of x.

c Sketch the graph of C against x for $0 < x < 120$.

d At what speed should the truck be driven to minimise the total cost per kilometre?

4 A box is to be made from a 10 cm by 16 cm sheet of metal by cutting equal squares out of the corners and bending up the flaps to form the box. Let the lengths of the sides of the squares be x cm and let the volume of the box formed be $V\,\text{cm}^3$.

a Show that $V = 4(x^3 - 13x^2 + 40x)$.

b State the set of x-values for which the expression for V in terms of x is valid.

c Find the values of x such that $\dfrac{dV}{dx} = 0$.

d Find the dimensions of the box if the volume is to be a maximum.

e Find the maximum volume of the box.

f Sketch the graph of V against x for the domain established in part **b**.

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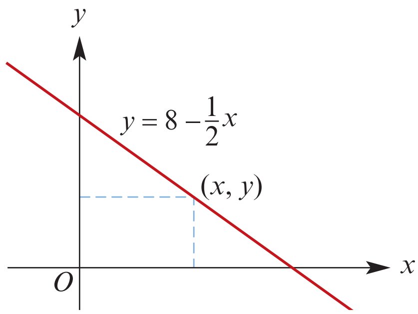
Review

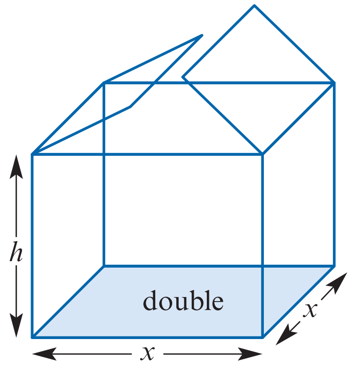
5 A rectangle has one vertex at the origin, another on the positive x-axis, another on the positive y-axis and a fourth on the line $y = 8 - \frac{x}{2}$.
What is the greatest area the rectangle can have?



6 At a factory the time, T seconds, spent in producing a certain size metal component is related to its weight, w kg, by $T = k + 2w^2$, where k is a constant.

- **a** If a 5 kg component takes 75 seconds to produce, find k.
- **b** Sketch the graph of T against w.
- **c** Write down an expression for the average time A (in seconds per kilogram).
- **d** **i** Find the weight that yields the minimum average machining time.
  ii State the minimum average machining time.

7 A manufacturer produces cardboard boxes that have a square base. The top of each box consists of a double flap that opens as shown. The bottom of the box has a double layer of cardboard for strength. Each box must have a volume of 12 cubic metres.



- **a** Show that the area of cardboard required is given by $C = 3x^2 + 4xh$.
- **b** Express C as a function of x only.
- **c** Sketch the graph of C against x for $x > 0$.
- **d** **i** What dimensions of the box will minimise the amount of cardboard used?
  ii What is the minimum area of cardboard used?

8 An open tank is to be constructed with a square base and vertical sides to contain 500 m^3 of water. What must be the area of sheet metal used in its construction if this area is to be a minimum?

9 A piece of wire of length 1 m is bent into the shape of a sector of a circle of radius a cm and sector angle θ. Let the area of the sector be A cm^2.

- **a** Find A in terms of a and θ.
- **b** Find A in terms of θ.
- **c** Find the value of θ for which A is a maximum.
- **d** Find the maximum area of the sector.

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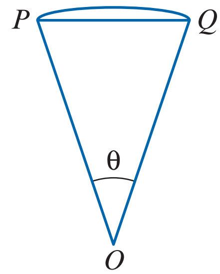
10 A piece of wire of fixed length, L cm, is bent to form the boundary $OPQO$ of a sector of a circle. The circle has centre O and radius r cm. The angle of the sector is θ radians.



a Show that the area, A cm^2, of the sector is given by

$$A = \frac{1}{2}rL - r^2$$

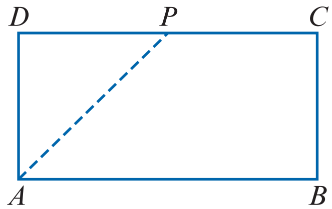
b **i** Find a relationship between r and L for which $\frac{dA}{dr} = 0$.

ii Find the corresponding value of θ.

iii Determine the nature of the stationary point found in part **i**.

c Show that, for the value of θ found in part **b ii**, the area of the triangle OPQ is approximately 45.5% of the area of sector OPQ.

11 A Queensland resort has a large swimming pool as illustrated, with $AB = 75$ m and $AD = 30$ m.



A boy can swim at 1 m/s and run at $1\frac{2}{3}$ m/s. He starts at A, swims to a point P on DC, and runs from P to C. He takes 2 seconds to pull himself out of the pool.

Let $DP = x$ m and the total time taken be T s.

a Show that

$$T = \sqrt{x^2 + 900} + \frac{3}{5}(75 - x) + 2$$

b Find $\frac{dT}{dx}$.

c **i** Find the value of x for which the time taken is a minimum.

ii Find the minimum time.

d Find the time taken if the boy runs from A to D and then from D to C.

12 **a** Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

b Find the equation of the tangent to the curve $y = e^{2x}$ at the point $\left(\frac{1}{2}, e\right)$.

c Find the equation of the tangent to the curve $y = e^{kx}$ at the point $\left(\frac{1}{k}, e\right)$.

d Show that $y = xke$ is the only tangent to the curve $y = e^{kx}$ which passes through the origin.

e Hence determine for what values of k the equation $e^{kx} = x$ has:

i a unique real solution

ii no real solution.

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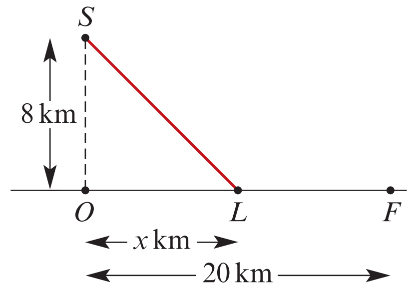
Review

13 The point S is 8 km offshore from the point O, which is located on the straight shore of a lake, as shown in the diagram. The point F is on the shore, 20 km from O. Contestants race from the start, S, to the finish, F, by rowing in a straight line to some point, L, on the shore and then running along the shore to F. A certain contestant rows at 5 km per hour and runs at 15 km per hour.

a Show that, if the distance OL is x km, the time taken by this contestant to complete the course is (in hours):

$$T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

b Show that the time taken by this contestant to complete the course has its minimum value when $x = 2\sqrt{2}$. Find this time.



14 At noon the captain of a ship sees two fishing boats approaching. One of them is 10 km due east and travelling west at 8 km/h. The other is 6 km due north and travelling south at 6 km/h. At what time will the fishing boats be closest together and how far apart will they be?

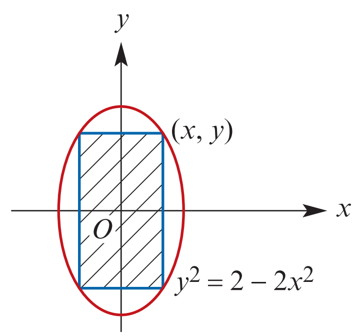
15 A rectangular beam is to be cut from a non-circular tree trunk whose cross-sectional outline can be represented by the equation $y^2 = 2 - 2x^2$.

a Show that the area of the cross-section of the beam is given by $A = 4x\sqrt{2 - 2x^2}$ where x is the half-width of the beam.

b State the possible values for x.

c Find the value of x for which the cross-sectional area of the beam is a maximum and find the corresponding value of y.

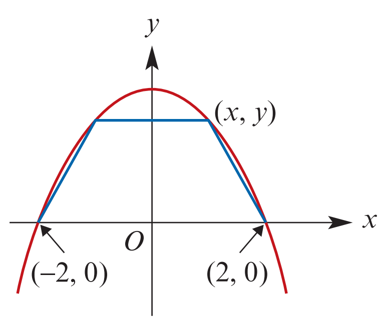
d Find the maximum cross-sectional area of the beam.



16 An isosceles trapezium is inscribed in the parabola $y = 4 - x^2$ as illustrated.

a Show that the area of the trapezium is given by $\frac{1}{2}(4 - x^2)(2x + 4)$.

b Show that the trapezium has its greatest area when $x = \frac{2}{3}$.



c Repeat with the parabola $y = a^2 - x^2$:

i Show that the area, A, of the trapezium is given by $(a^2 - x^2)(a + x)$.

ii Use the product rule to find $\frac{dA}{dx}$.

iii Show that a maximum occurs when $x = \frac{a}{3}$.

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Review

17 Assume that the number of bacteria present in a culture at time t is given by $N(t)$ where $N(t) = 24te^{-0.2t}$. At what time will the population be at a maximum? Find the maximum population.

18 It is believed that, for some time after planting in ideal conditions, the area covered by a particular species of ground-cover plant has a rate of increase of y cm^2/week, given by $y = -t^3 + bt^2 + ct$ where t is the number of weeks after planting.

a Find b and c using the table of observations on the right.

t	1	2
y	10	24

b Assume that the model is accurate for the first 8 weeks after planting. When during this period is:

i the area covered by the plant a maximum

ii the rate of increase in area a maximum?

c According to the model, if the plant covered 100 cm^2 when planted, what area will it cover after 4 weeks?

d Discuss the implications for the future growth of the plant if the model remains accurate for longer than the first 4 weeks.

19 Let $f(x) = x^3 - 3x^2 + 6x - 10$.

a Find the coordinates of the point on the graph of f for which $f'(x) = 3$.

b Express $f'(x)$ in the form $a(x + p)^2 + q$.

c Hence show that the gradient of f is greater than 3 for all points on the curve of f other than the point found in part **a**.

20 A curve with equation of the form $y = ax^3 + bx^2 + cx + d$ has zero gradient at the point $\left(\frac{1}{3}, \frac{4}{27}\right)$ and also touches, but does not cross, the x-axis at the point $(1, 0)$.

a Find a, b, c and d.

b Find the values of x for which the curve has a negative gradient.

c Sketch the curve.

21 The volume of water, V m^3, in a reservoir when the depth indicator shows y metres is given by the formula

$$V = \frac{\pi}{3}[(y + 630)^3 - 630^3]$$

a Find the volume of water in the reservoir when $y = 40$.

b Find the rate of change of volume with respect to depth, y.

c Sketch the graph of V against y for $0 \le y \le 60$.

d If $y = 60$ m is the maximum depth of the reservoir, find the capacity (m^3) of the reservoir.

e If $\frac{dV}{dt} = 20\,000 - 0.005\pi(y + 630)^2$, where t is the time in days from 1 January, sketch the graph of $\frac{dV}{dt}$ against y for $0 \le y \le 60$.

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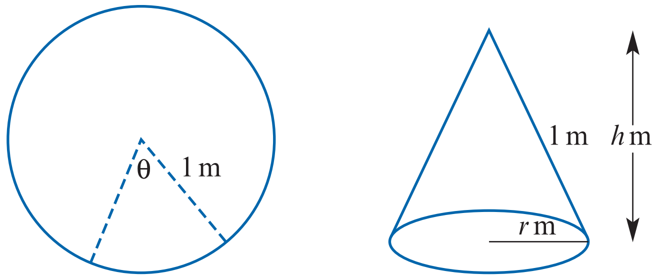
Review

22 A cone is made by cutting out a sector with central angle θ from a circular piece of cardboard of radius 1 m and joining the two cut edges to form a cone of slant height 1 m as shown in the following diagrams.



The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$.

a **i** Find r in terms of θ.

ii Find h in terms of θ.

iii Show that $V = \frac{1}{3}\pi\left(\frac{2\pi - \theta}{2\pi}\right)^2\sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$.

b Find the value of V when $\theta = \frac{\pi}{4}$.

c Find the value(s) of θ for which the volume of the cone is 0.3 m^3.

d **i** Use a calculator to determine the value of θ that maximises the volume of the cone.

ii Find the maximum volume.

e Determine the maximum volume using calculus.

23 **a** For the function with rule $f(x) = x^3 + ax^2 + bx$, plot the graph of each of the following using a calculator. (Give axis intercepts, coordinates of stationary points and the nature of stationary points.)

i $a = 1,\ b = 1$

ii $a = -1,\ b = -1$

iii $a = 1,\ b = -1$

iv $a = -1,\ b = 1$

b **i** Find $f'(x)$.

ii Solve the equation $f'(x) = 0$ for x, giving your answer in terms of a and b.

c **i** Show that the graph of $y = f(x)$ has exactly one stationary point if $a^2 - 3b = 0$.

ii If $b = 3$, find the corresponding value(s) of a which satisfy $a^2 - 3b = 0$. Find the coordinates of the stationary points and state the nature of each.

iii Using a calculator, plot the graph(s) of $y = f(x)$ for these values of a and b.

iv Plot the graphs of the corresponding derivative functions on the same set of axes.

d State the relationship between a and b if no stationary points exist for the graph of $y = f(x)$.

24 For what value of x is $\frac{\ln x}{x}$ a maximum? That is, when is the ratio of the logarithm of a number to the number a maximum?

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25 Consider the function with rule $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$.

a **i** If $x + 1$ is a factor of $f(x)$, find the value of a.

ii Using a calculator, plot the graph of $y = f(x)$ for this value of a.

b Let $g(x) = 6x^4 - x^3 + 21x^2 - 6x + 8$.

i Plot the graph of $y = g(x)$.

ii Find the minimum value of $g(x)$ and the value of x for which this occurs.

iii Find $g'(x)$.

iv Using a calculator, solve the equation $g'(x) = 0$ for x.

v Find $g'(0)$ and $g'(10)$.

vi Find $g''(x)$.

vii Show that the graph of $y = g'(x)$ has no stationary points and thus deduce that $g'(x) = 0$ has only one solution.

26 For the quartic function f with rule $f(x) = (x - a)^2(x - b)^2$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = 2(x - a)(x - b)[2x - (b + a)]$.

b **i** Solve the equation $f'(x) = 0$ for x. **ii** Solve the equation $f(x) = 0$ for x.

c Hence find the coordinates of the stationary points of the graph of $y = f(x)$.

d Plot the graph of $y = f(x)$ on a calculator for several values of a and b.

e **i** If $a = b$, then $f(x) = (x - a)^4$. Sketch the graph of $y = f(x)$.

ii If $a = -b$, find the coordinates of the stationary points.

iii Plot the graph of $y = f(x)$ for several values of a, given that $a = -b$.

27 For the quartic function f with rule $f(x) = (x - a)^3(x - b)$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = (x - a)^2[4x - (3b + a)]$.

b **i** Solve the equation $f'(x) = 0$. **ii** Solve the equation $f(x) = 0$.

c Find the coordinates of the stationary points of the graph of $y = f(x)$ and state the nature of the stationary points.

d Using a calculator, plot the graph of $y = f(x)$ for several values of a and b.

e If $a = -b$, state the coordinates of the stationary points in terms of a.

f **i** State the relationship between b and a if there is a local minimum for $x = 0$.

ii Illustrate this for $b = 1$ and $a = -3$ on a calculator.

g Show that, if there is a turning point for $x = \dfrac{a+b}{2}$, then $b = a$ and $f(x) = (x - a)^4$.

28 A psychologist hypothesised that the ability of a mouse to memorise during the first 6 months of its life can be modelled by the function f given by $f\colon (0, 6) \to \mathbb{R}$, $f(x) = x \ln x + 1$, where $f(x)$ is the ability to memorise at age x months.

a Find $f'(x)$.

b Find the value of x for which $f'(x) = 0$ and hence find when the mouse's ability to memorise is a minimum.

c Sketch the graph of f.

d When is the mouse's ability to memorise a maximum in this period?

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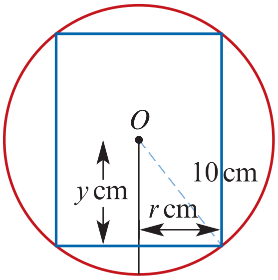
29 A cylinder is to be cut from a sphere. The cross-section through the centre of the sphere is as shown. The radius of the sphere is 10 cm. Let r cm be the radius of the cylinder.



a **i** Find y in terms of r and hence the height, h cm, of the cylinder.

ii The volume of a cylinder is given by $V = \pi r^2 h$. Find V in terms of r.

b **i** Plot the graph of V against r using a calculator.

ii Find the maximum volume of the cylinder and the corresponding values of r and h. (Use a calculator.)

iii Find the two possible values of r if the volume is 2000 cm^3.

c **i** Find $\frac{dV}{dr}$.

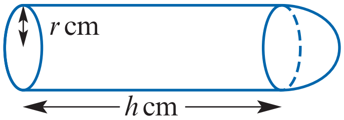
ii Hence find the exact value of the maximum volume and the value of r for which this occurs.

d **i** Plot the graph of the derivative function $\frac{dV}{dr}$ against r, using a calculator.

ii From the calculator, find the values of r for which $\frac{dV}{dr}$ is positive.

iii From the calculator, find the values of r for which $\frac{dV}{dr}$ is increasing.

30 A wooden peg consists of a cylinder of length h cm and a hemispherical cap of radius r cm. The volume, V cm^3, of the peg is given by $V = \pi r^2 h + \frac{2}{3}\pi r^3$.



a If the surface area of the peg is 100π cm^2:

i Find h in terms of r.

ii Find V as a function of r.

iii Find the possible values of r (i.e. find the domain of the function defined in part **ii**).

iv Find $\frac{dV}{dr}$.

v Sketch the graph of V against r.

b If $h = 6$, then $V = 6\pi r^2 + \frac{2}{3}\pi r^3$. For $r = 4$:

i Show that a small increase of k cm in the radius results in an approximate increase of $80\pi k$ cm^3 in the volume.

ii Show that a small increase of $q\%$ in the radius results in an approximate increase of $2.3q\%$ in the volume.

31 **a** Consider the curve with equation $y = (2x^2 - 5x)e^{ax}$. If the curve passes through the point with coordinates $(3, 10)$, find the value of a.

b **i** For the curve with equation $y = (2x^2 - 5x)e^{ax}$, find the x-axis intercepts.

ii Use calculus to find the x-values for which there is a turning point, in terms of a.

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32 The kangaroo population in a certain confined region is given by $f(x) = \dfrac{100\,000}{1 + 100e^{-0.3x}}$, where x is the time in years.

a Find $f'(x)$.

b Find the rate of growth of the kangaroo population when:

i $x = 0$ **ii** $x = 4$

33 Consider the function $f(x) = 8\ln(6 - 0.2x)$, $\{x \in \mathbb{R} : x < a\}$ where a is the largest possible value such that f can be defined in this way.

a What is the value of a?

b Find the exact values for the coordinates of the points where the graph of $y = f(x)$ crosses each axis.

c Find the gradient of the tangent to the graph of $y = f(x)$ at the point where $x = 20$.

d Sketch the graph of $y = f(x)$.

34 **a** Using a calculator, plot the graphs of $f(x) = \sin x$ and $g(x) = e^{\sin x}$ on the one screen.

b Find $g'(x)$ and hence find the coordinates of the stationary points of $y = g(x)$ for $x \in [0, 2\pi]$.

c Give the range of g.

d State the period of g.

35 **a** Show that the tangent to the graph of $y = e^x$ for $x = 0$ has equation $y = x + 1$.

b Plot the graphs of $y = e^x$ and $y = x + 1$ on a calculator.

c Let $f(x) = e^x$ and $g(x) = x + 1$. Use a calculator to investigate functions of the form

$$h(x) = af(x - b) + c \quad \text{and} \quad k(x) = ag(x - b) + c$$

Comment on your observations.

d Use the chain rule and properties of transformations to prove that, if the tangent to the curve $y = f(x)$ at the point (x_1, y_1) has equation $y = mx + c$, then the tangent to the curve $y = af(bx)$ at the point $\left(\dfrac{x_1}{b}, y_1 a\right)$ has equation $y = a(mbx + c)$.

36 A certain chemical starts to dissolve in water at time $t = 0$. It is known that, if x is the number of grams not dissolved after t hours, then

$$x = \frac{60}{5e^{\lambda t} - 3}, \quad \text{where } \lambda = \frac{1}{2}\ln\left(\frac{6}{5}\right)$$

a Find the amount of chemical present when:

i $t = 0$ **ii** $t = 5$

b Find $\dfrac{dx}{dt}$ in terms of t.

c **i** Show that $\dfrac{dx}{dt} = -\lambda x - \dfrac{\lambda x^2}{20}$.

ii Sketch the graph of $\dfrac{dx}{dt}$ against x for $x \geq 0$.

iii Write a short explanation of your result.

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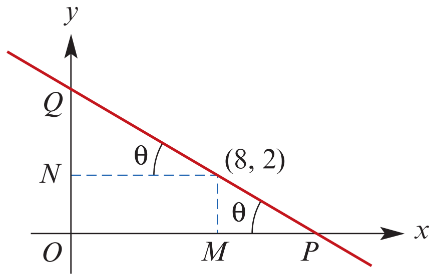
Review

37 A straight line is drawn through the point $(8, 2)$ to intersect the positive y-axis at Q and the positive x-axis at P. (In this problem we will determine the minimum value of $OP + OQ$.)



a Show that $\dfrac{d}{d\theta}\left(\dfrac{1}{\tan\theta}\right) = -\operatorname{cosec}^2\theta$.

Note: $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

b Find MP in terms of θ.

c Find NQ in terms of θ.

d Hence find $OP + OQ$ in terms of θ. Denote $OP + OQ$ by x.

e Find $\dfrac{dx}{d\theta}$.

f Find the minimum value of x and the value of θ for which this occurs.

38 Let $f(x) = e^x - e^{-x}$.

a Find $f'(x)$.

b Find $\{ x : f(x) = 0 \}$.

c Show that $f'(x) > 0$ for all x.

d Sketch the graph of f.

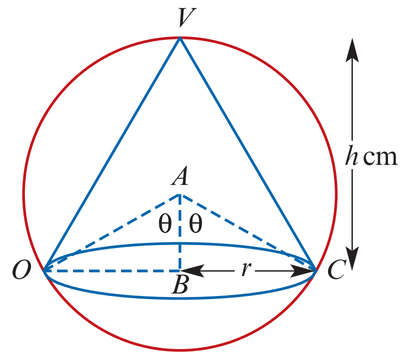
39 **a** Find all values of x for which $(\ln x)^2 = 2\ln x$.

b Find the gradient of each of the curves $y = 2\ln x$ and $y = (\ln x)^2$ at the point $(1, 0)$.

c Use these results to sketch, on one set of axes, the graphs of $y = 2\ln x$ and $y = (\ln x)^2$.

d Find $\{ x : 2\ln x > (\ln x)^2 \}$.

40 A cone is inscribed inside a sphere as illustrated. The radius of the sphere is a cm, and both the angles $\angle OAB$ and $\angle CAB$ have magnitude θ. The height of the cone is h cm and the radius of the cone is r cm.



a Find h, the height of the cone, in terms of a and θ.

b Find r, the radius of the cone, in terms of a and θ.

The volume, V cm^3, of the cone is given by $V = \dfrac{1}{3}\pi r^2 h$.

c Use the results from parts **a** and **b** to show that

$$V = \frac{1}{3}\pi a^3 \sin^2\theta \cdot (1 + \cos\theta)$$

d Find $\dfrac{dV}{d\theta}$ (where a is a constant) and hence find the value of θ for which the volume is a maximum.

e Find the maximum volume of the cone in terms of a.

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41 Some bacteria are introduced into a supply of fresh milk. After t hours there are y grams of bacteria present, where

$$y = \frac{Ae^{bt}}{1 + Ae^{bt}} \qquad (1)$$

and A and b are positive constants.

a Show that $0 < y < 1$ for all values of t.

b Find $\frac{dy}{dt}$ in terms of t.

c From equation (1), show that $Ae^{bt} = \frac{y}{1 - y}$.

d **i** Show that $\frac{dy}{dt} = by(1 - y)$.

ii Hence, or otherwise, show that the maximum value of $\frac{dy}{dt}$ occurs when $y = 0.5$.

e If $A = 0.01$ and $b = 0.7$, find when, to the nearest hour, the bacteria will be increasing at the fastest rate.

42 Let $f(x) = \frac{e^x}{x}$, $\{ x \in \mathbb{R} : x > 0 \}$.

a Find $f'(x)$.

b Find $\{ x : f'(x) = 0 \}$.

c Find the coordinates of the one stationary point and state its nature.

d **i** Find $\frac{f'(x)}{f(x)}$. **ii** Find $\lim_{x \to \infty} \frac{f'(x)}{f(x)}$ and comment.

e Sketch the graph of f.

f Over a period of years, the number of birds (n) in an island colony decreased and increased with time (t years) according to the approximate formula

$$n = \frac{ae^{kt}}{t}$$

where t is measured from 1900 and a and k are constant. If during this period the population was the same in 1965 as it was in 1930, when was it least?

43 A culture contains 1000 bacteria and 5 hours later the number has increased to 10 000. The number, N, of bacteria present at any time, t hours, is given by $N = Ae^{kt}$.

a Find the values of A and k.

b Find the rate of growth at time t.

c Show that, at time t, the rate of growth is proportional to the number of bacteria present.

d Find this rate of growth when:

i $t = 4$ **ii** $t = 50$

44 The populations of two ant colonies, A and B, are increasing according to the rules:

A population $= 2 \times 10^4 e^{0.03t}$

B population $= 10^4 e^{0.05t}$

After how many years will their populations:

a be equal **b** be increasing at the same rate?

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45 A particle on the end of a spring, which is hanging vertically, is oscillating such that its height, h metres, above the floor after t seconds is given by

$$y = 0.5 + 0.2\sin(3\pi t), \quad t \geq 0$$

a Find the greatest height above the floor and the time at which this height is first reached.

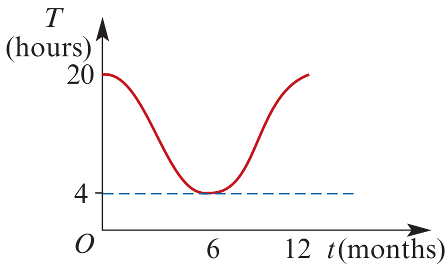
b Find the period of oscillation.

c Find the speed of the particle when $t = \frac{1}{3}, \frac{2}{3}, \frac{1}{6}$.

46 The length of night on Seal Island varies between 20 hours in midwinter and 4 hours in midsummer. The relationship between T, the number of hours of night, and t, the number of months past the longest night in 2010, is given by

$$T(t) = p + q\cos(\pi r t)$$

where p, q and r are constants.
Assume that the year consists of 12 months of equal length.
The graph of T against t is illustrated.



a Find the value of:

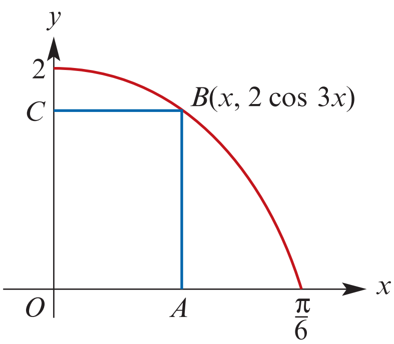
i r **ii** p and q

b Find $T'(3)$ and $T'(9)$ and find the rate of change of hours of night with respect to the number of months.

c Find the average rate of change of hours of night from $t = 0$ to $t = 6$.

d After how many months is the rate of change of hours of night a maximum?

47 A section of the graph of $y = 2\cos(3x)$ is shown in the diagram.



a Show that the area, A, of the rectangle $OABC$ in terms of x is $2x\cos(3x)$.

b **i** Find $\frac{dA}{dx}$.

ii Find $\frac{dA}{dx}$ when $x = 0$ and $x = \frac{\pi}{6}$.

c **i** On a calculator, plot the graph of $A = 2x\cos(3x)$ for $x \in \left[0, \frac{\pi}{6}\right]$.

ii Find the two values of x for which the area of the rectangle is 0.2 square units.

iii Find the maximum area of the rectangle and the value of x for which this occurs.

d **i** Show that $\frac{dA}{dx} = 0$ is equivalent to $\tan(3x) = \frac{1}{3x}$.

ii Using a calculator, plot the graphs of $y = \tan(3x)$ and $y = \frac{1}{3x}$ for $x \in \left(0, \frac{\pi}{6}\right)$ and find the coordinates of the point of intersection.

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48 **a** A population of insects grows according to the model

$$N(t) = 1000 - t + 2e^{\frac{t}{20}} \quad \text{for } t \geq 0$$

where t is the number of days after 1 January 2000.

i Find the rate of growth of the population as a function of t.

ii Find the minimum population size and value of t for which this occurs.

iii Find $N(0)$.

iv Find $N(100)$.

v Sketch the graph of N against t for $0 \leq t \leq 100$.

b It is found that the population of another species is given by

$$N_2(t) = 1000 - t^{\frac{1}{2}} + 2e^{\frac{t^{\frac{1}{2}}}{20}}$$

i Find $N_2(0)$.

ii Find $N_2(100)$.

iii Plot the graph of $y = N_2(t)$ for $t \in [0, 5000]$ on a calculator.

iv Solve the equation $N_2'(t) = 0$ and hence give the minimum population of this species of insects.

c A third model is

$$N_3(t) = 1000 - t^{\frac{3}{2}} + 2e^{\frac{t}{20}}$$

Use a calculator to:

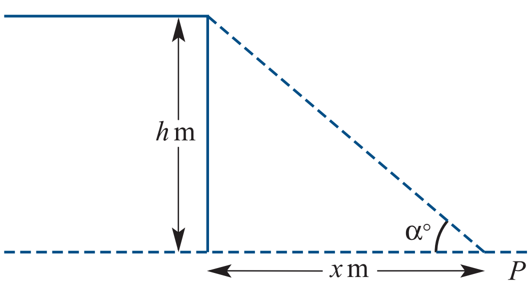
i plot a graph for $0 \leq t \leq 200$

ii find the minimum population and the time at which this occurs.

d **i** For N_3, find $N_3'(t)$.

ii Show that $N_3'(t) = 0$ is equivalent to $t = 20\ln(15\sqrt{t})$.

49 The height, h m, of a cliff is determined by measuring the angle of elevation, $\alpha°$, of the top of the cliff from a point P level with the base of the cliff and at a distance x m from it.



a Show that, if an error of $(\delta\alpha)°$ is made in the measurement of the angle, then the error in the calculated height of the cliff is approximately $\frac{\pi x}{180}\sec^2(\alpha°)\delta\alpha$ metres.

Hint: You will need to express the angle $\alpha°$ in radians.

Now let $x = 50$.

b Using a calculator, plot the graph of $y = \frac{5\pi}{18}\sec^2(\alpha°)$ for $0 < \alpha < 90$.

c If the greatest error in the measurement of the angle is $0.02°$, what is the greatest possible error in the calculated height of the cliff in terms of α?

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6 Integration

In this chapter

Syllabus references

Topics: Anti-differentiation; Definite integrals; Fundamental theorem; Applications of integration

Subtopics: 3.2.1 – 3.2.22

We have used the derivative to find the gradients of tangents to curves, and in turn this has been used in graph sketching. The derivative has also been used to define instantaneous rate of change and to solve problems involving motion in a straight line.

It comes as a surprise that a related idea can be used to determine areas. In this chapter we define an area function A for a given function f on an interval $[a, b]$, and show that the derivative of the area function is the original function f. Hence, you can go from the function f to its area function by a process which can loosely be described as 'undoing' the derivative. This result is so important that it carries the title **fundamental theorem of calculus**.

The result was developed over many centuries: a method for determining areas described in the last section of this chapter is due to Archimedes. The final result was brought together by both Leibniz and Newton in the seventeenth century. The wonder of it is that the two seemingly distinct ideas – calculation of areas and calculation of gradients – were shown to be so closely related.

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6A The area under a graph

Consider a function and its graph $y = f(x)$. We want to find the area under the graph. For now we'll assume that the graph $y = f(x)$ is always *above the x-axis*, and we will determine the area between the graph $y = f(x)$ and the x-axis. We set left and right endpoints and determine the area between those endpoints.

Estimating the area under a graph

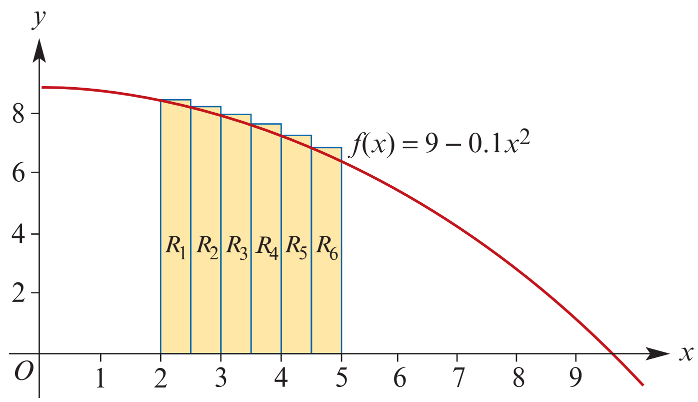
Below is the graph of

$$f(x) = 9 - 0.1x^2$$

We consider two methods for estimating the area under this graph between $x = 2$ and $x = 5$.

The left-endpoint estimate

We first find an approximation for the area under the graph between $x = 2$ and $x = 5$ by dividing the region into rectangles as illustrated. The width of each rectangle is 0.5.



Areas of rectangles:

- Area of $R_1 = 0.5 \times f(2.0) = 0.5 \times 8.60 = 4.30$ square units
- Area of $R_2 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$ square units
- Area of $R_3 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$ square units
- Area of $R_4 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$ square units
- Area of $R_5 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$ square units
- Area of $R_6 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$ square units

The sum of the areas of the rectangles is 23.62 square units.

This is called the **left-endpoint estimate** for the area under the graph.

The left-endpoint estimate will be larger than the actual area for a graph that is decreasing over the interval, and smaller than the actual area for a graph that is increasing.

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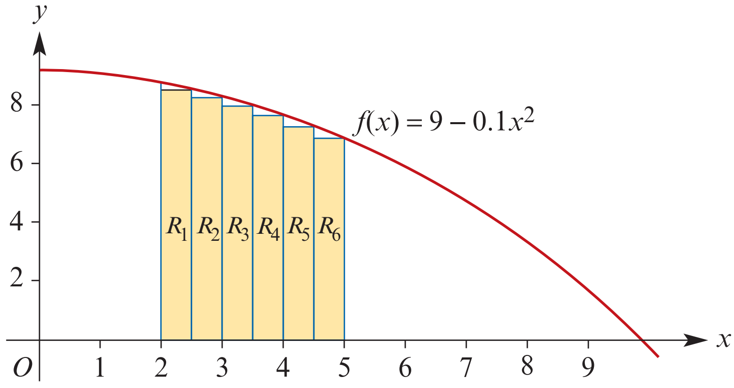
The right-endpoint estimate



Areas of rectangles:

- Area of $R_1 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$ square units
- Area of $R_2 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$ square units
- Area of $R_3 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$ square units
- Area of $R_4 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$ square units
- Area of $R_5 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$ square units
- Area of $R_6 = 0.5 \times f(5.0) = 0.5 \times 6.50 = 3.25$ square units

The sum of the areas of the rectangles is 22.67 square units.

This is called the **right-endpoint estimate** for the area under the graph.

For f decreasing over $[a, b]$: left-endpoint estimate $\geq$ true area $\geq$ right-endpoint estimate

For f increasing over $[a, b]$: left-endpoint estimate $\leq$ true area $\leq$ right-endpoint estimate

It is clear that, if narrower strips are chosen, we obtain an estimate that is closer to the true value. This is time-consuming to do by hand, but a computer program or spreadsheet makes the process quite manageable.

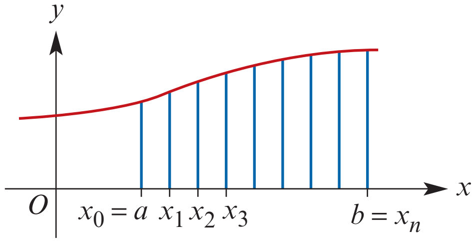
In general, to estimate the area under the graph of $y = f(x)$ between $x = a$ and $x = b$, we divide the interval $[a, b]$ on the x-axis into n equal subintervals $[a, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_{n-1}, b]$ as illustrated.

Left-endpoint estimate

$$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$

Right-endpoint estimate

$$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$



These two methods are not limited to situations in which the graph is either increasing or decreasing for the whole interval. They may be used to determine the area under the curve for any continuous function on an interval $[a, b]$.

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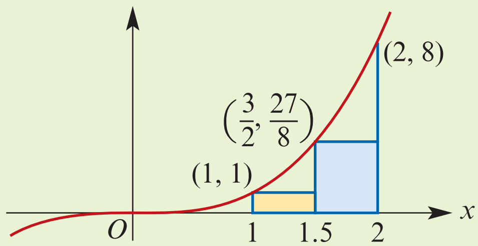
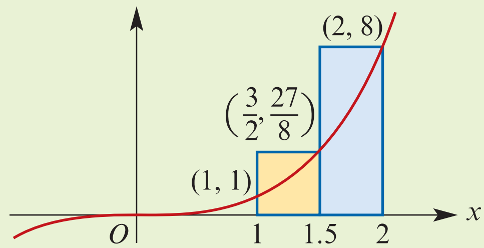
Example 1

Approximate the area under the curve $y = x^3$ between $x = 1$ and $x = 2$ by finding the sum of the areas of the shaded rectangles:

a



b



Solution

a $\text{Area} = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{27}{8}$

$= \frac{1}{2} + \frac{27}{16}$

$= 2\frac{3}{16}$ square units

(This is the left-endpoint method.)

b $\text{Area} = \frac{1}{2} \times \frac{27}{8} + \frac{1}{2} \times 8$

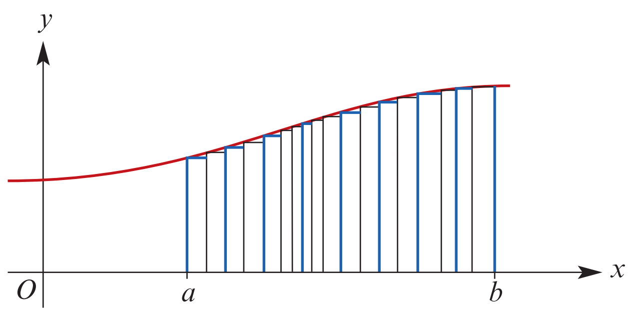
$= \frac{27}{16} + 4$

$= 5\frac{11}{16}$ square units

(This is the right-endpoint method.)

If f is a continuous function such that $f(x)$ is positive for all x in the interval $[a, b]$, and if the interval $[a, b]$ is partitioned into arbitrarily small subintervals, then the **area** under the curve between $x = a$ and $x = b$ can be defined by this limiting process.

The diagram on the right shows rectangles formed from a partition. The rectangles can be of varying width, but in the limit the width of all the rectangles must approach zero.



The definite integral

Suppose that f is a continuous function on a closed interval $[a, b]$ and that $f(x)$ is positive for all x in this interval. Then the area under the graph of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$, and is denoted by

$$\int_a^b f(x)\, dx$$

The function f is called the integrand, and a and b are the lower and upper limits of the integral.

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By using summation notation, this limiting process can be expressed as

$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i)\,\delta x_i$$

where the interval $[a, b]$ is partitioned into n subintervals, with the ith subinterval of length δx_i and containing x_i, and $\delta x = \max\{\, \delta x_i : i = 1, 2, \dots, n \,\}$.

For a linear function or a piecewise-defined function with linear components, the area under the graph may be found using geometric techniques.

Example 2

Evaluate each of the following by using an area formula:

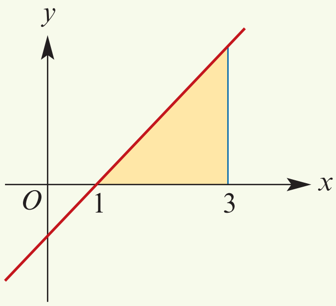
a $\int_1^3 x - 1\,dx$ **b** $\int_1^3 (x-1)\,dx + \int_{-1}^1 (1-x)\,dx$ **c** $\int_1^2 x + 1\,dx$

Solution

a Area of triangle $= \frac{1}{2} \times 2 \times 2$

$= 2$ square units

Therefore $\int_1^3 x - 1\,dx = 2$

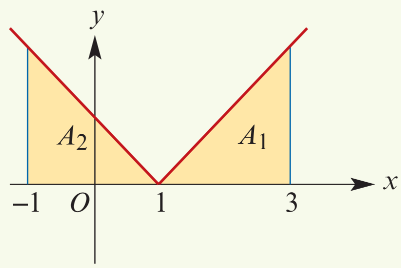


b Area $= A_1 + A_2$

$= 2 + \frac{1}{2} \times 2 \times 2$

$= 4$ square units

Therefore $\int_1^3 (x-1)\,dx + \int_{-1}^1 (1-x)\,dx = 4$

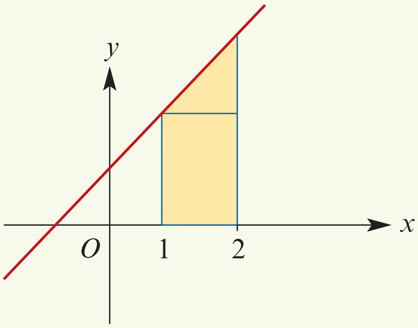


c The required region is a trapezium.

Area $= \frac{1}{2} \times 1 \times (2 + 3)$

$= \frac{5}{2}$ square units

Therefore $\int_1^2 x + 1\,dx = \frac{5}{2}$



A calculus method for determining areas will be introduced in Section 6E.

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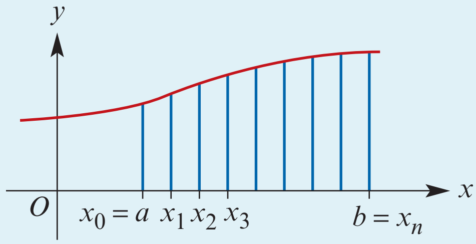
Summary 6A

- **Estimating area**

Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[a, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, b]$ as illustrated.



Estimates for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:

- Left-endpoint estimate

$$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$

- Right-endpoint estimate

$$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$

- **Exact area**

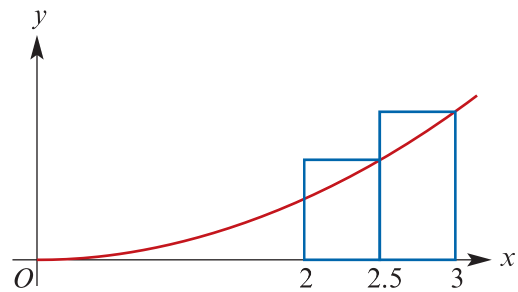
Let f be a continuous function on a closed interval $[a, b]$ such that $f(x)$ is positive for all $x \in [a, b]$. The exact area under the graph of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$, and is denoted by $\int_a^b f(x)\,dx$.

Exercise 6A

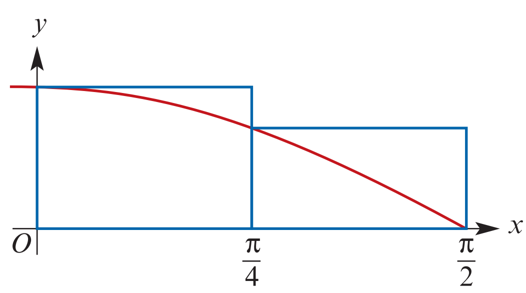
Example 1

1 Use two rectangles to approximate the area contained between the curve and the x-axis. Use the method indicated and give your answer correct to two decimal places.

a $y = \frac{1}{2}x^2$ between $x = 2$ and $x = 3$ using the right-endpoint method



b $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ using the left-endpoint method



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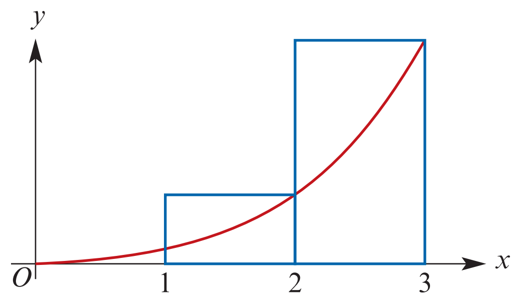
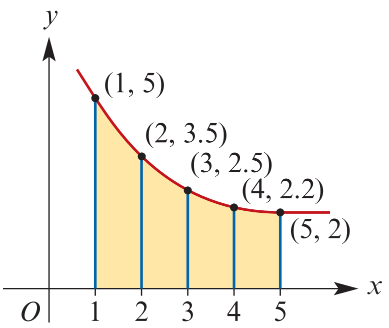
c $y = \frac{1}{2}x^3$ between $x = 1$ and $x = 3$ using the right-endpoint method



2 To approximate the area of the shaded region, use the subintervals shown to calculate:

a the left-endpoint estimate

b the right-endpoint estimate.



3 Calculate an approximation to the area under the graph of $y = x(4 - x)$ between $x = 0$ and $x = 4$ using:

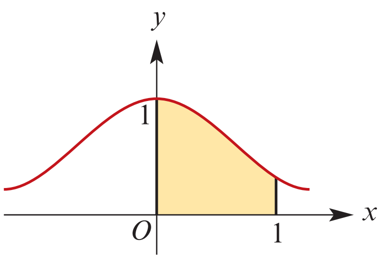
a 4 strips of width 1.0 (Do this by hand.)

b 20 strips of width 0.2 (You can use a calculator program if you wish.)

4 The graph is that of $y = \frac{1}{1 + x^2}$. It is known that the area of the shaded region is $\frac{\pi}{4}$.

a Apply the right-endpoint rule with strips of width 0.25 to estimate the area under the curve.

b Hence find an approximate value for π. How could you improve the approximation?



5 A table of values is given for the rule $y = f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
y	3	3.5	3.7	3.8	3.9	3.9	4.0	4.0	3.7	3.3	2.9

Find the area enclosed by the graph of $y = f(x)$, the lines $x = 0$ and $x = 10$, and the x-axis by using:

a the left-endpoint estimate **b** the right-endpoint estimate.

6 Calculate, by using the right-endpoint estimate, an approximation to the area under the graph of $y = 2^x$ between $x = 0$ and $x = 3$, using strips of width 0.5. Write your answer correct to two decimal places.

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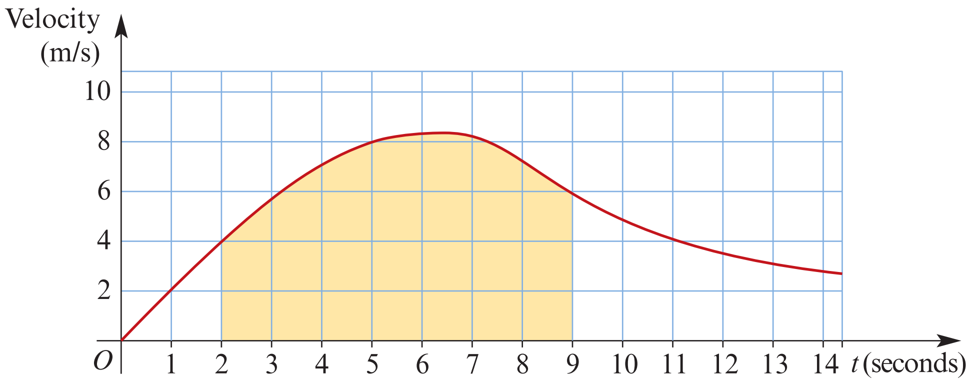
7 The graph shows the velocity (in m/s) of an object at time t seconds.

a Use the left-endpoint rule to estimate the area of the shaded region.

b What does this area represent?



Example 2

8 Evaluate each of the following by using an area formula:

a $\int_2^5 x - 2 \, dx$ **b** $\int_{-1}^2 (2 - x) \, dx + \int_2^5 (x - 2) \, dx$ **c** $\int_1^2 2x + 1 \, dx$

6B Antidifferentiation: indefinite integrals

Later in this chapter, we will see how to find the exact area under a graph using the technique of 'undoing' the derivative. In this section, we formalise the idea of 'undoing' a derivative.

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

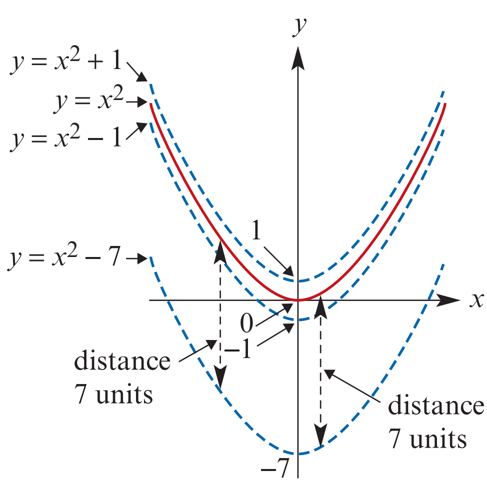
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$.

If two functions have the same derivative function, then they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several antiderivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



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Notation

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x \, dx = x^2 + c$$

This is read as 'the **general antiderivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we could write:

$$\int 2x \, dx = \{ f(x) : f'(x) = 2x \} = \{ x^2 + c : c \in \mathbb{R} \}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.

In general:

> If $F'(x) = f(x)$, then $\int f(x) \, dx = F(x) + c$, where c is an arbitrary real number.

The reason why the symbol is the same as that used for the definite integral in Section 6A will become evident in Section 6E.

The antiderivative of x^r where $r \neq -1$

We know that:

$f(x) = x^3$ implies $f'(x) = 3x^2$ $\qquad$ $f(x) = x^8$ implies $f'(x) = 8x^7$

$f(x) = x^{\frac{3}{2}}$ implies $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$ $\qquad$ $f(x) = x^{-4}$ implies $f'(x) = -4x^{-5}$

Reversing this process gives:

$\int 3x^2 \, dx = x^3 + c$ $\qquad$ where c is an arbitrary constant

$\int 8x^7 \, dx = x^8 + c$ $\qquad$ where c is an arbitrary constant

$\int \frac{3}{2}x^{\frac{1}{2}} \, dx = x^{\frac{3}{2}} + c$ $\qquad$ where c is an arbitrary constant

$\int -4x^{-5} \, dx = x^{-4} + c$ $\qquad$ where c is an arbitrary constant

We also have:

$$\int x^2 \, dx = \frac{1}{3}x^3 + c \qquad \int x^{\frac{1}{2}} \, dx = \frac{2}{3}x^{\frac{3}{2}} + c \qquad \int x^{-5} \, dx = -\frac{1}{4}x^{-4} + c$$

Generalising, it is seen that:

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c, \quad \{ r \in \mathbb{Q} : r \neq -1 \}$$

Note: This result can only be applied for suitable values of x for a given value of r. For example, if $r = \frac{1}{2}$, then $x \in (0, \infty)$ is a suitable restriction. If $r = -2$, we can take $\{ x \in \mathbb{R} : x \neq 0 \}$, and if $r = 3$, we can take $x \in \mathbb{R}$.

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We also record the following results, which follow immediately from the corresponding results for differentiation:

Sum $\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$

Difference $\int f(x) - g(x)\, dx = \int f(x)\, dx - \int g(x)\, dx$

Multiple $\int kf(x)\, dx = k \int f(x)\, dx$, where k is a real number

Example 3

Find the general antiderivative (indefinite integral) of each of the following:

a $3x^5$ **b** $3x^2 + 4x^{-2} + 3$

Solution

a
$$\begin{aligned} &\int 3x^5\, dx \\ &= 3\int x^5\, dx \\ &= 3 \times \frac{x^6}{6} + c \\ &= \frac{x^6}{2} + c \end{aligned}$$

b
$$\begin{aligned} &\int 3x^2 + 4x^{-2} + 3\, dx \\ &= 3\int x^2\, dx + 4\int x^{-2}\, dx + 3\int 1\, dx \\ &= \frac{3x^3}{3} + \frac{4x^{-1}}{-1} + \frac{3x}{1} + c \\ &= x^3 - \frac{4}{x} + 3x + c \end{aligned}$$

Example 4

Find y in terms of x if:

a $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **b** $\dfrac{dy}{dx} = 3\sqrt{x}$ **c** $\dfrac{dy}{dx} = x^{\frac{3}{4}} + x^{-\frac{3}{4}}$

Solution

a
$$\begin{aligned} \int \frac{1}{x^2}\, dx &= \int x^{-2}\, dx \\ &= \frac{x^{-1}}{-1} + c \\ \therefore \quad y &= \frac{-1}{x} + c \end{aligned}$$

b
$$\begin{aligned} \int 3\sqrt{x}\, dx &= 3\int x^{\frac{1}{2}}\, dx \\ &= 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\ \therefore \quad y &= 2x^{\frac{3}{2}} + c \end{aligned}$$

c $y = \dfrac{4}{7}x^{\frac{7}{4}} + 4x^{\frac{1}{4}} + c$

Given extra information, we can find a unique antiderivative.

Example 5

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$$\int x^3 + 4x^2\, dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

$$\therefore \quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

As $f(0) = 0$, we have $c = 0$. Hence $f(x) = \dfrac{x^4}{4} + \dfrac{4x^3}{3}$.

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Example 6

If the gradient of the tangent at a point (x, y) on a curve is given by $2x$ and the curve passes through the point $(-1, 4)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 2x$.

$$\int 2x\,dx = \frac{2x^2}{2} + c = x^2 + c$$

$$\therefore \quad f(x) = x^2 + c$$

But $f(-1) = 4$ and therefore $4 = (-1)^2 + c$.

Hence $c = 3$ and so $f(x) = x^2 + 3$.

Summary 6B

- Antiderivative of x^r, for $\{ r \in \mathbb{Q} : r \neq -1 \}$:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$$

- Properties of antidifferentiation:
  - $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$
  - $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number

Skill-sheet

Exercise 6B

Example 3

1 Find:

a $\int \frac{1}{2}x^3\,dx$ **b** $\int 5x^3 - 2x\,dx$ **c** $\int \frac{4}{5}x^3 - 3x^2\,dx$ **d** $\int (2-z)(3z+1)\,dz$

Example 4

2 Find y in terms of x if:

a $\frac{dy}{dx} = \frac{1}{x^3}$ **b** $\frac{dy}{dx} = 4\sqrt[3]{x}$ **c** $\frac{dy}{dx} = x^{\frac{1}{4}} + x^{-\frac{3}{5}}$

3 Find:

a $\int 3x^{-2}\,dx$ **b** $\int 2x^{-4} + 6x\,dx$ **c** $\int 2x^{-2} + 6x^{-3}\,dx$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}\,dx$ **e** $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}\,dx$ **f** $\int 4x^{\frac{3}{5}} + 12x^{\frac{5}{3}}\,dx$

Example 5

4 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = 2x - 3$ and $y = 1$ when $x = 1$ **b** $\frac{dy}{dx} = x^3$ and $y = 6$ when $x = 0$

c $\frac{dy}{dx} = x^{\frac{1}{2}} + x$ and $y = 6$ when $x = 4$

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5 Find:

a $\int \sqrt{x}(2 + x)\,dx$ **b** $\int \frac{3z^4 + 2z}{z^3}\,dz$ **c** $\int \frac{5x^3 + 2x^2}{x}\,dx$

d $\int \sqrt{x}(2x + x^2)\,dx$ **e** $\int x^2(2 + 3x^2)\,dx$ **f** $\int \sqrt[3]{x}(x + x^4)\,dx$

Example 6 **6** A curve with equation $y = f(x)$ passes through the point $(2, 0)$ and $f'(x) = 3x^2 - \frac{1}{x^2}$. Find $f(x)$.

7 Find s in terms of t if $\frac{ds}{dt} = 3t - \frac{8}{t^2}$ and $s = 1\frac{1}{2}$ when $t = 1$.

8 A curve $y = f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a stationary point at $(2, 1)$. Find:

a the value of k **b** the value of $f(x)$ when $x = 7$.

6C The antiderivative of $(ax + b)^r$

Case 1: $r \neq -1$

For $f(x) = (ax + b)^{r+1}$, where $r \neq -1$, we can use the chain rule to find

$$f'(x) = a(r + 1)(ax + b)^r$$

Thus it follows that:

$$\int (ax + b)^r\,dx = \frac{1}{a(r + 1)}(ax + b)^{r+1} + c, \quad r \neq -1$$

This result does not hold for $r = -1$.

Example 7

Find the general antiderivative of:

a $(3x + 1)^5$ **b** $(2x - 1)^{-2}$

Solution

a $\int (3x + 1)^5\,dx = \frac{1}{3 \times 6}(3x + 1)^6 + c$

$= \frac{1}{18}(3x + 1)^6 + c$

b $\int (2x - 1)^{-2}\,dx = \frac{1}{2 \times (-1)}(2x - 1)^{-1} + c$

$= -\frac{1}{2}(2x - 1)^{-1} + c$

Using the TI-Nspire

- Use menu > **Calculus** > **Integral** to find the integral of $(2x - 1)^{-2}$.

Note: The integral template can also be accessed using the 2D-template palette [template key] or shift +.

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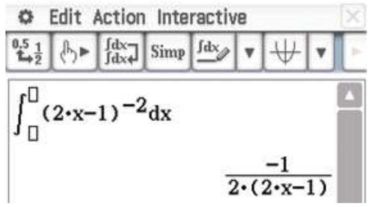
Using the Casio ClassPad

- Enter and highlight the expression $(2x-1)^{-2}$.
- Select **Interactive** > **Calculation** > $\int$.

Note: The two boxes on the integral allow for definite integrals to be evaluated. This is covered later in the chapter.



Case 2: $r = -1$

But what happens when $r = -1$? In other words, what is $\int \frac{1}{ax+b}\,dx$?

Remember that $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Thus $\int \frac{1}{x}\,dx = \ln x + c$ provided that $x > 0$.

More generally:

For $ax + b > 0$,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c$$

For $x < 0$, we have

$$\frac{d}{dx}\big(\ln(-x)\big) = \frac{1}{-x} \times (-1) = \frac{1}{x}$$

and so $\int \frac{1}{x}\,dx = \ln(-x)$.

More generally, for $ax + b < 0$, we have

$$\frac{d}{dx}\big(\ln(-ax-b)\big) = \frac{1}{-ax-b} \times (-a) = \frac{a}{ax+b}$$

and so $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(-ax-b)$.

We can summarise these results as:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c \qquad \text{for } ax+b > 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(-ax-b) + c \qquad \text{for } ax+b < 0$$

Example 8

a Find the general antiderivative of $\frac{2}{3x-2}$ for $x > \frac{2}{3}$.

b Find the general antiderivative of $\frac{2}{3x-2}$ for $x < \frac{2}{3}$.

c Given $\frac{dy}{dx} = \frac{3}{x}$ for $x > 0$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

d Given $\frac{dy}{dx} = \frac{3}{x}$ for $x < 0$ and $y = 10$ when $x = -1$, find an expression for y in terms of x.

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Solution

a For $x > \frac{2}{3}$,

$$\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\ln(3x-2) + c$$
$$= \frac{2}{3}\ln(3x-2) + c$$

b For $x < \frac{2}{3}$,

$$\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\ln(2-3x) + c$$
$$= \frac{2}{3}\ln(2-3x) + c$$

c $y = \int \frac{3}{x}\,dx = 3\ln x + c$

When $x = 1$, $y = 10$ and so

$$10 = 3\ln 1 + c$$
$$10 = 0 + c$$
$$\therefore \quad c = 10$$

Hence $y = 3\ln x + 10$.

d $y = \int \frac{3}{x}\,dx = 3\ln(-x) + c$

When $x = -1$, $y = 10$ and so

$$10 = 3\ln 1 + c$$
$$10 = 0 + c$$
$$\therefore \quad c = 10$$

Hence $y = 3\ln(-x) + 10$.

Summary 6C

- If $\{ r \in \mathbb{Q} : r \neq -1 \}$, then
$$\int (ax+b)^r\,dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c$$
- For $ax + b > 0$,
$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c$$
- For $ax + b < 0$,
$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(-ax-b) + c$$

Skill-sheet

Exercise 6C

Example 7

1 Find:

a $\int (2x-1)^2\,dx$ **b** $\int (2-t)^3\,dt$ **c** $\int (5x-2)^3\,dx$

d $\int (4x-6)^{-2}\,dx$ **e** $\int (6-4x)^{-3}\,dx$ **f** $\int (4x+3)^{-3}\,dx$

g $\int (3x+6)^{\frac{1}{2}}\,dx$ **h** $\int (3x+6)^{-\frac{1}{2}}\,dx$ **i** $\int (2x-4)^{\frac{7}{2}}\,dx$

j $\int (3x+11)^{\frac{4}{3}}\,dx$ **k** $\int \sqrt{2-3x}\,dx$ **l** $\int (5-2x)^4\,dx$

Example 8

2 Find an antiderivative of each of the following:

a $\frac{1}{2x}$, $x > 0$ **b** $\frac{1}{3x+2}$, $x > -\frac{2}{3}$ **c** $\frac{4}{1+4x}$, $x > -\frac{1}{4}$

d $\frac{5}{3x-2}$, $x > \frac{2}{3}$ **e** $\frac{3}{1-4x}$, $x < \frac{1}{4}$ **f** $\frac{3}{2-\frac{x}{2}}$, $x < 4$

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3 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = \frac{1}{2x}$ and $y = 2$ when $x = e^2$

b $\frac{dy}{dx} = \frac{2}{5 - 2x}$ and $y = 10$ when $x = 2$

4 A curve with equation $y = f(x)$ passes through the point $(5 + e, 10)$ and $f'(x) = \frac{10}{x - 5}$. Find the equation of the curve.

5 Given that $\frac{dy}{dx} = \frac{3}{x - 2}$ and $y = 10$ when $x = 0$, find an expression for y in terms of x.

6 Given that $\frac{dy}{dx} = \frac{5}{2 - 4x}$ and $y = 10$ when $x = -2$, find an expression for y in terms of x.

7 Given that $\frac{dy}{dx} = \frac{5}{2 - 4x}$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

6D The antiderivative of e^{kx}

In Chapter 4 we found that, if $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$.

Thus:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

Example 9

Find the general antiderivative of each of the following:

a e^{4x} **b** $e^{5x} + 6x$ **c** $e^{3x} + 2$ **d** $e^{-x} + e^x$

Solution

a $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$

b $\int e^{5x} + 6x\,dx = \frac{1}{5}e^{5x} + 3x^2 + c$

c $\int e^{3x} + 2\,dx = \frac{1}{3}e^{3x} + 2x + c$

d $\int e^{-x} + e^x\,dx = -e^{-x} + e^x + c$

Example 10

If the gradient of the tangent at a point (x, y) on a curve is given by $5e^{2x}$ and the curve passes through the point $(0, 7.5)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 5e^{2x}$.

$$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + c$$

$$\therefore \quad f(x) = \frac{5}{2}e^{2x} + c$$

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But $f(0) = 7.5$ and therefore

$$7.5 = \frac{5}{2}e^0 + c$$
$$= 2.5 + c$$
$$\therefore \quad c = 5$$

Hence $f(x) = \frac{5}{2}e^{2x} + 5$.

Summary 6D

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

Skill-sheet

Exercise 6D

Example 9

1 Find the general antiderivative of each of the following:

a e^{6x} **b** $e^{2x} + 3x$ **c** $e^{-3x} + 2x$ **d** $e^{-2x} + e^{2x}$

2 Find:

a $\int e^{2x} - e^{\frac{x}{2}}\,dx$ **b** $\int \frac{e^{2x} + 1}{e^x}\,dx$ **c** $\int 2e^{3x} - e^{-x}\,dx$

d $\int 5e^{\frac{x}{3}} - 2e^{\frac{x}{5}}\,dx$ **e** $\int 3e^{\frac{2x}{3}} - 3e^{\frac{7x}{5}}\,dx$ **f** $\int 5e^{\frac{4x}{3}} - 3e^{\frac{2x}{3}}\,dx$

Example 10

3 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = e^{2x} - x$ and $y = 5$ when $x = 0$

b $\frac{dy}{dx} = \frac{3 - e^{2x}}{e^x}$ and $y = 4$ when $x = 0$

4 Given that $\frac{dy}{dx} = ae^{-x} + 1$ and that when $x = 0$, $\frac{dy}{dx} = 3$ and $y = 5$, find the value of y when $x = 2$.

5 A curve for which $\frac{dy}{dx} = e^{kx}$, where k is a constant, is such that the tangent at $(1, e^2)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k

b the equation of the curve.

6 A curve for which $\frac{dy}{dx} = -e^{kx}$, where k is a constant, is such that the tangent at $(1, -e^3)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k

b the equation of the curve.

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6E The fundamental theorem of calculus and the definite integral

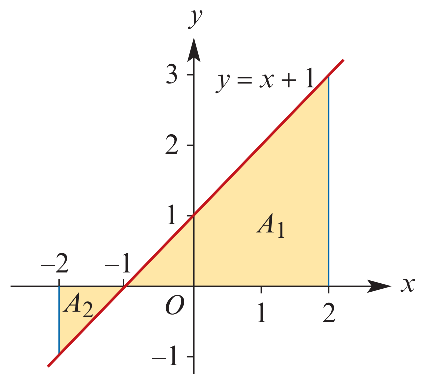
The integrals that you have learned to evaluate in the previous sections are known as **indefinite integrals** because they are only defined to within an arbitrary constant: for example, we have $\int 3x^2 \, dx = x^3 + c$. In general terms, we can write $\int f(x) \, dx = F(x) + c$; that is, the integral of $f(x)$ is $F(x)$ plus a constant, where $F(x)$ is an antiderivative of $f(x)$.

We now resume our consideration of the **definite integral** and investigate its connection with the indefinite integral.

Signed area

We first look at regions below the x-axis as well as those above the x-axis.

Consider the graph of $y = x + 1$ shown to the right.



$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$ (area of a triangle)

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

The total area is $A_1 + A_2 = 5$.

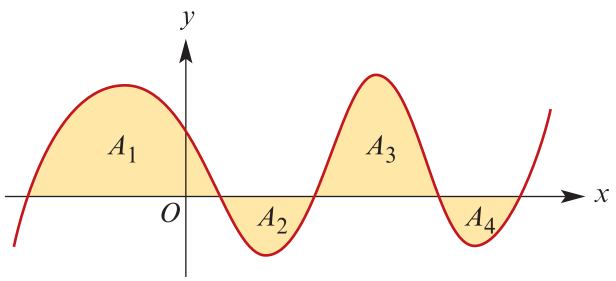
The **signed area** is $A_1 - A_2 = 4$.

Regions above the x-axis have **positive signed area**.

Regions below the x-axis have **negative signed area**.

The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$.

The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.



For any continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x) \, dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

In this more general setting, the definite integral can still be determined by a limiting process as discussed in the first section of this chapter.

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The fundamental theorem of calculus

The fundamental theorem of calculus provides a connection between the area definition of the definite integral and the antiderivatives discussed previously. An outline of the proof is given in the final section of this chapter.

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f.

To facilitate setting out, we sometimes write

$$G(b) - G(a) = [G(x)]_a^b$$

Example 11

Evaluate the definite integral $\int_1^2 x\,dx$.

Solution

We have $\int x\,dx = \frac{1}{2}x^2 + c$ and so

$$\int_1^2 x\,dx = \frac{1}{2} \times 2^2 + c - \left(\frac{1}{2} \times 1^2 + c\right)$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

Note: The arbitrary constant cancels out. Because of this, we ignore it when evaluating definite integrals. We also use the more compact notation $G(b) - G(a) = [G(x)]_a^b$ to help with setting out:

$$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

Example 12

Evaluate each of the following the definite integrals:

a $\int_2^3 x^2\,dx$

b $\int_3^2 x^2\,dx$

c $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

Solution

a $\int_2^3 x^2\,dx$

$$= \left[\frac{x^3}{3}\right]_2^3$$

$$= \frac{27}{3} - \frac{8}{3}$$

$$= \frac{19}{3}$$

b $\int_3^2 x^2\,dx$

$$= \left[\frac{x^3}{3}\right]_3^2$$

$$= \frac{8}{3} - \frac{27}{3}$$

$$= -\frac{19}{3}$$

c $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{5}$$

$$= \frac{16}{15}$$

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Example 13

Evaluate each of the following definite integrals:

a $\int_0^1 2e^{-2x}\,dx$ **b** $\int_0^4 e^{2x} + 1\,dx$ **c** $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx$

Solution

a
$$\begin{aligned}\int_0^1 2e^{-2x}\,dx &= \left[\frac{2}{-2}e^{-2x}\right]_0^1 \\ &= -1(e^{-2\times1} - e^{-2\times0}) \\ &= -1(e^{-2} - 1) \\ &= 1 - e^{-2}\end{aligned}$$

b
$$\begin{aligned}\int_0^4 e^{2x} + 1\,dx &= \left[\frac{1}{2}e^{2x} + x\right]_0^4 \\ &= \frac{1}{2}e^8 + 4 - \left(\frac{1}{2}e^0 + 0\right) \\ &= \frac{1}{2}(e^8 + 7)\end{aligned}$$

c
$$\begin{aligned}\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx &= \left[\frac{4}{3}x^{\frac{3}{2}} + 2e^{\frac{x}{2}}\right]_1^4 \\ &= \frac{4}{3}\times 8 + 2e^2 - \left(\frac{4}{3} + 2e^{\frac{1}{2}}\right) \\ &= \frac{28}{3} + 2e^2 - 2e^{\frac{1}{2}} \\ &= 2\left(\frac{14}{3} + e^2 - e^{\frac{1}{2}}\right)\end{aligned}$$

Example 14

Evaluate each of the following definite integrals:

a $\int_6^8 \frac{1}{x-5}\,dx$ **b** $\int_4^5 \frac{1}{2x-5}\,dx$

Solution

a
$$\begin{aligned}\int_6^8 \frac{1}{x-5}\,dx &= \left[\ln(x-5)\right]_6^8 \\ &= \ln 3 - \ln 1 \\ &= \ln 3\end{aligned}$$

b
$$\begin{aligned}\int_4^5 \frac{1}{2x-5}\,dx &= \frac{1}{2}\left[\ln(2x-5)\right]_4^5 \\ &= \frac{1}{2}(\ln 5 - \ln 3) \\ &= \frac{1}{2}\ln\left(\frac{5}{3}\right)\end{aligned}$$

Important properties of the definite integral are listed in the following summary box.

Summary 6E

- For any continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.
- **Fundamental theorem of calculus**

  If f is a continuous function on the interval $[a, b]$, then

  $$\int_a^b f(x)\,dx = [G(x)]_a^b = G(b) - G(a)$$

  where G is any antiderivative of f.

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■ **Properties of the definite integral**

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
- $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Skill-sheet

Exercise 6E

Example 11
Example 12

1 Evaluate each of the following:

a $\int_1^2 x^2\,dx$ **b** $\int_{-1}^3 x^3\,dx$ **c** $\int_0^1 x^3 - x\,dx$

d $\int_{-1}^2 (x+1)^2\,dx$ **e** $\int_1^2 \frac{1}{x^2}\,dx$ **f** $\int_1^4 x^{\frac{1}{2}} + 2x^2\,dx$

g $\int_0^2 x^3 + 2x^2 + x + 2\,dx$ **h** $\int_1^4 2x^{\frac{3}{2}} + 5x^3\,dx$

2 Evaluate each of the following:

a $\int_0^1 (2x+1)^3\,dx$ **b** $\int_0^2 (4x+1)^{-\frac{1}{2}}\,dx$ **c** $\int_1^2 (1-2x)^2\,dx$

d $\int_0^1 (3-2x)^{-2}\,dx$ **e** $\int_0^2 (3+2x)^{-3}\,dx$ **f** $\int_{-1}^1 (4x+1)^3\,dx$

g $\int_0^1 \sqrt{2-x}\,dx$ **h** $\int_3^4 \frac{1}{\sqrt{2x-4}}\,dx$ **i** $\int_0^1 \frac{1}{(3+2x)^2}\,dx$

Example 13

3 Evaluate each of the following:

a $\int_0^1 e^{2x}\,dx$ **b** $\int_0^1 e^{-2x} + 1\,dx$

c $\int_0^1 2e^{\frac{x}{3}} + 2\,dx$ **d** $\int_{-2}^2 \frac{e^x + e^{-x}}{2}\,dx$

4 Given that $\int_0^4 h(x)\,dx = 5$, evaluate:

a $\int_0^4 2h(x)\,dx$ **b** $\int_0^4 h(x) + 3\,dx$ **c** $\int_4^0 h(x)\,dx$

d $\int_0^4 h(x) + 1\,dx$ **e** $\int_0^4 h(x) - x\,dx$

Example 14

5 **a** Find $\int_0^4 \frac{1}{x-6}\,dx$. **b** Find $\int_2^4 \frac{1}{2x-3}\,dx$. **c** Find $\int_5^6 \frac{3}{2x+7}\,dx$.

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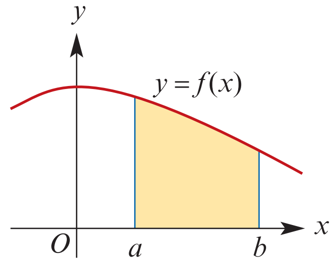
6F Finding the area under a curve

Recall that the definite integral $\int_a^b f(x)\,dx$ gives the net signed area 'under' the curve.

Finding the area of a region

- If $f(x) \geq 0$ for all $x \in [a, b]$, then the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by

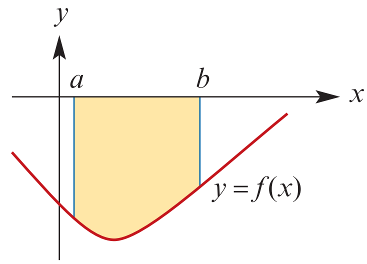
$$A = \int_a^b f(x)\,dx$$



- If $f(x) \leq 0$ for all $x \in [a, b]$, then the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by

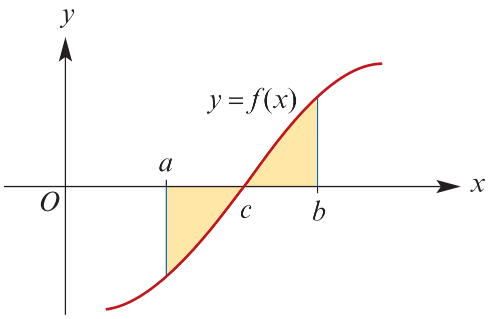
$$A = -\int_a^b f(x)\,dx$$

$$= \int_b^a f(x)\,dx$$



- If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area A of the shaded region is given by

$$A = \int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$$

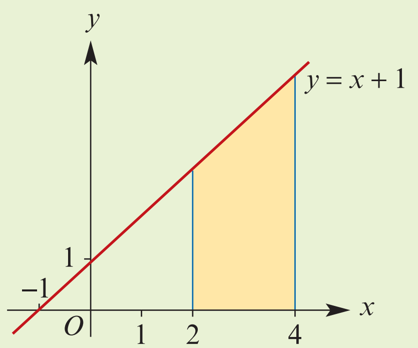


Note: In determining the area 'under' a curve $y = f(x)$, the sign of $f(x)$ in the given interval is the critical factor.

Example 15

a Find the area of the region between the x-axis, the line $y = x + 1$ and the lines $x = 2$ and $x = 4$. Check the answer by working out the area of the trapezium.

b Find the area under the line $y = x + 1$ between $x = -4$ and $x = -2$.



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Solution

a Area $= \int_2^4 x + 1\,dx = \left[\frac{x^2}{2} + x\right]_2^4$

$= \left(\frac{4^2}{2} + 4\right) - \left(\frac{2^2}{2} + 2\right)$

$= 12 - 4 = 8$

The area of the shaded region is 8 square units.

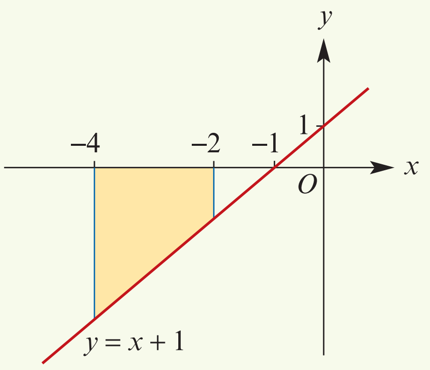
Check: Area of trapezium = average height × base $= \frac{3+5}{2} \times 2 = 8$

b Area $= -\int_{-4}^{-2} x + 1\,dx = -\left[\frac{x^2}{2} + x\right]_{-4}^{-2}$

$= -(0 - 4) = 4$

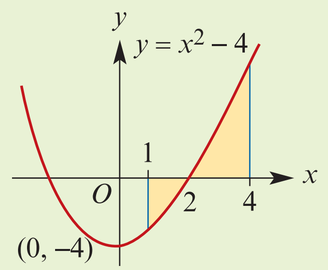
The area of the shaded region is 4 square units.

Note: The negative sign is introduced as the integral gives the *signed area* from −4 to −2, which is negative.



Example 16

Find the exact area of the shaded region.



Solution

Area $= \int_2^4 (x^2 - 4)\,dx + -\int_1^2 (x^2 - 4)\,dx$

$= \left[\frac{x^3}{3} - 4x\right]_2^4 - \left[\frac{x^3}{3} - 4x\right]_1^2$

$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) - \left(\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right)$

$= \frac{56}{3} - 8 - \left(\frac{7}{3} - 4\right) = \frac{37}{3}$

The area is $\frac{37}{3}$ square units.

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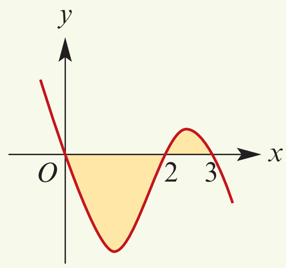
Example 17

Find the exact area of the regions enclosed by the graph of $y = x(2 - x)(x - 3)$ and the x-axis.

Solution

$$y = x(-x^2 + 5x - 6)$$
$$= -x^3 + 5x^2 - 6x$$



$$\text{Area} = \int_2^3 (-x^3 + 5x^2 - 6x)\,dx + -\int_0^2 (-x^3 + 5x^2 - 6x)\,dx$$
$$= \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3 - \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2$$
$$= \left(\frac{-81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right) - \left(-4 + \frac{40}{3} - 12\right)$$
$$= \frac{-81}{4} + 18 + 32 - \frac{80}{3}$$
$$= 50 - \frac{243 + 320}{12} = \frac{37}{12}$$

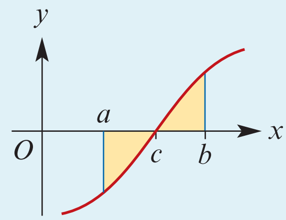
The area is $\frac{37}{12}$ square units.

Note: There is no need to find the coordinates of stationary points.

Summary 6F

Finding areas:

- If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
- If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
- If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



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Exercise 6F

Example 15
Example 16

1 Sketch the graph and find the exact area of the region(s) bounded by the x-axis and the graph of each of the following:

a $y = 3x^2 + 2$ between $x = 0$ and $x = 1$

b $y = x^3 - 8$ between $x = 2$ and $x = 4$

c $y = 4 - x$ between:

i $x = 0$ and $x = 4$ **ii** $x = 0$ and $x = 6$

Example 17

2 Find the exact area bounded by the x-axis and the graph of each of the following:

a $y = x^2 - 2x$

b $y = (4 - x)(3 - x)$

c $y = (x + 2)(7 - x)$

d $y = x^2 - 5x + 6$

e $y = 3 - x^2$

f $y = x^3 - 6x^2$

3 For each of the following, sketch a graph to illustrate the region for which the definite integral gives the area:

a $\int_1^4 2x + 1\, dx$

b $\int_0^3 3 - x\, dx$

c $\int_0^4 x^2\, dx$

d $\int_{-1}^1 4 - 2x^2\, dx$

e $\int_2^4 \sqrt{x}\, dx$

f $\int_0^1 (1 - x)(1 + x)^2\, dx$

4 Find the exact area of the region bounded by the curve $y = 3x + 2x^{-2}$, the lines $x = 2$ and $x = 5$ and the x-axis.

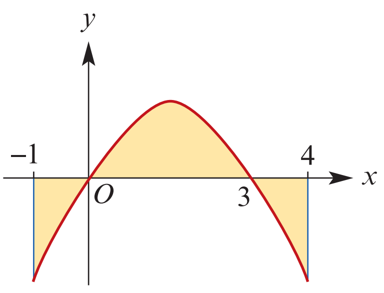
5 Sketch the graph of $f(x) = 1 + x^3$ and find the exact area of the region bounded by the curve and the axes.

6 Sketch the graph of $f(x) = 4e^{2x} + 3$ and find the exact area of the region enclosed by the curve, the axes and the line $x = 1$.

7 Sketch the graph of $y = x(2 - x)(x - 1)$ and find the exact area of the region enclosed by the curve and the x-axis.

8 **a** Evaluate $\int_{-1}^4 x(3 - x)\, dx$.

b Find the exact area of the shaded region in the figure.



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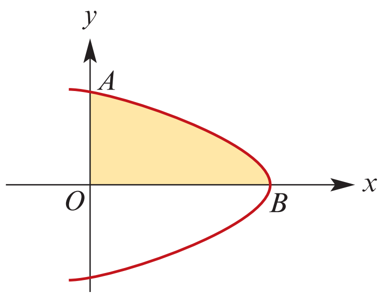
9 **a** In the figure, the graph of $y^2 = 9(1 - x)$ is shown. Find the coordinates of A and B.

b Find the exact area of the shaded region by evaluating

$$\int_0^b 1 - \frac{y^2}{9}\,dy$$

for a suitable choice of b.



10 Sketch the graph of $y = \dfrac{1}{2 - 3x}$ and find the exact area of the region enclosed by the curve, the x-axis and the lines with equations $x = -3$ and $x = -2$.

11 Sketch the graph of $y = 2 + \dfrac{1}{x + 4}$ and find the exact area of the region enclosed by the curve, the axes and the line $x = -2$.

12 Let $a > 0$ with $a \neq 1$.

a Show that $a^x = e^{x(\ln a)}$.

b Hence find the derivative and an antiderivative of a^x.

c Hence, or otherwise, show that the area under the curve $y = a^x$ between the lines $x = 0$ and $x = b$ is $\dfrac{1}{\ln a}(a^b - 1)$.

6G Integration of trigonometric functions

Recall the following results from Chapter 4:

- If $f(x) = \sin(kx + a)$, then $f'(x) = k\cos(kx + a)$.
- If $g(x) = \cos(kx + a)$, then $g'(x) = -k\sin(kx + a)$.

Thus:

$$\int \sin(kx + a)\,dx = -\frac{1}{k}\cos(kx + a) + c$$

$$\int \cos(kx + a)\,dx = \frac{1}{k}\sin(kx + a) + c$$

Example 18

Find an antiderivative of each of the following:

a $\sin\left(3x + \dfrac{\pi}{4}\right)$

b $\dfrac{1}{4}\sin(4x)$

Solution

a $-\dfrac{1}{3}\cos\left(3x + \dfrac{\pi}{4}\right) + c$

b $-\dfrac{1}{16}\cos(4x) + c$

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Example 19

Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin(2x)\, dx$ **b** $\int_0^{\frac{\pi}{2}} 2\cos x + 1\, dx$

Solution

a $\int_0^{\frac{\pi}{4}} \sin(2x)\, dx$

$$= \left[\frac{-1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{-1}{2}\cos\left(\frac{\pi}{2}\right) - \left(\frac{-1}{2}\cos 0\right)$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

b $\int_0^{\frac{\pi}{2}} 2\cos x + 1\, dx$

$$= \left[2\sin x + x\right]_0^{\frac{\pi}{2}}$$

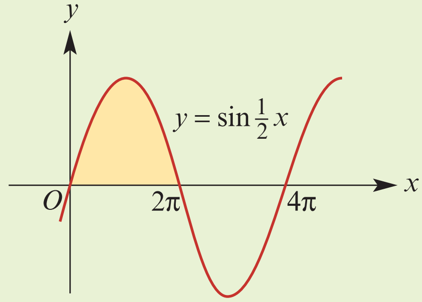
$$= 2\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} - (2\sin 0 + 0)$$

$$= 2 + \frac{\pi}{2}$$

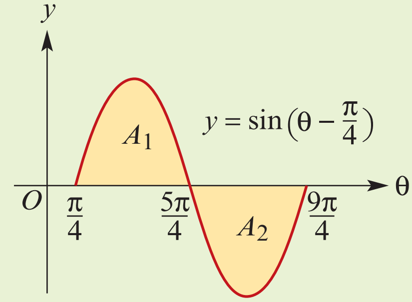
Example 20

Find the exact area of the shaded region for each graph:

a



b



Solution

a Area $= \int_0^{2\pi} \sin\left(\frac{1}{2}x\right) dx$

$$= \left[-2\cos\left(\frac{1}{2}x\right)\right]_0^{2\pi}$$

$$= -2\cos\pi - (-2\cos 0)$$

$$= 4$$

∴ Area of shaded region is 4 square units.

b Regions A_1 and A_2 must be considered separately:

Area $A_1 = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$

$$= \left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -(\cos\pi - \cos 0)$$

$$= 2$$

Area $A_2 = -\int_{\frac{5\pi}{4}}^{\frac{9\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$

$$= -\left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{5\pi}{4}}^{\frac{9\pi}{4}}$$

$$= \cos(2\pi) - \cos\pi$$

$$= 2$$

∴ Total area of shaded region is 4 square units.

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Summary 6G

$$\int \sin(kx + a)\,dx = -\frac{1}{k}\cos(kx + a) + c$$

$$\int \cos(kx + a)\,dx = \frac{1}{k}\sin(kx + a) + c$$

Exercise 6G

Example 18

1 Find an antiderivative of each of the following:

a $\cos(3x)$ **b** $\sin\left(\frac{1}{2}x\right)$ **c** $3\cos(3x)$

d $2\sin\left(\frac{1}{2}x\right)$ **e** $\sin\left(2x - \frac{\pi}{3}\right)$ **f** $\cos(3x) + \sin(2x)$

g $\cos(4x) - \sin(4x)$ **h** $-\frac{1}{2}\sin(2x) + \cos(3x)$ **i** $-\frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)$

j $\sin(\pi x)$

Example 19

2 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin x\,dx$ **b** $\int_0^{\frac{\pi}{4}} \cos(2x)\,dx$

c $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos\theta\,d\theta$ **d** $\int_0^{\frac{\pi}{2}} \sin\theta + \cos\theta\,d\theta$

e $\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta$ **f** $\int_0^{\frac{\pi}{3}} \cos(3\theta) + \sin(3\theta)\,d\theta$

g $\int_0^{\frac{\pi}{3}} \cos(3\theta) + \sin\left(\theta - \frac{\pi}{3}\right)d\theta$ **h** $\int_0^{\pi} \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right)dx$

i $\int_0^{\frac{\pi}{4}} \sin\left(2x - \frac{\pi}{3}\right)dx$ **j** $\int_0^{\pi} \cos(2x) - \sin\left(\frac{x}{2}\right)dx$

Example 20

3 Calculate the exact area of the region bounded by the curve $y = \sin\left(\frac{1}{2}x\right)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

4 For each of the following, draw a graph to illustrate the area given by the definite integral and evaluate the integral:

a $\int_0^{\frac{\pi}{4}} \cos x\,dx$ **b** $\int_0^{\frac{\pi}{3}} \sin(2x)\,dx$ **c** $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos(2x)\,dx$

d $\int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta\,d\theta$ **e** $\int_0^{\frac{\pi}{2}} \sin(2\theta) + 1\,d\theta$ **f** $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1 - \cos(2\theta)\,d\theta$

5 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right)dx$ **b** $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right)dx$

c $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right)dx$ **d** $\int_0^{\frac{\pi}{4}} \cos(3\pi - x)\,dx$

6 Sketch the curve $y = 2 + \sin(3x)$ for the interval $0 \le x \le \frac{2\pi}{3}$ and calculate the exact area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

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6H Further integration techniques

In this section we look at some further integrals to provide additional practice and introduce new approaches.

Example 21

Let $f(x) = \ln(x^2 + 1)$.

a Show that $f'(x) = \dfrac{2x}{x^2 + 1}$.

b Hence evaluate $\displaystyle\int_0^2 \frac{x}{x^2 + 1}\,dx$.

Solution

a Let $y = \ln(x^2 + 1)$ and $u = x^2 + 1$.
Then $y = \ln u$. By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot 2x$$

$$\therefore \quad f'(x) = \frac{2x}{x^2 + 1}$$

b $$\int_0^2 \frac{x}{x^2 + 1}\,dx = \frac{1}{2}\int_0^2 \frac{2x}{x^2 + 1}\,dx$$

$$= \frac{1}{2}\Big[\ln(x^2 + 1)\Big]_0^2$$

$$= \frac{1}{2}(\ln 5 - \ln 1)$$

$$= \frac{1}{2}\ln 5$$

We can generalise the previous example as follows. Let $g(x) = \ln(f(x))$, where $f(x) > 0$. Then the chain rule gives $g'(x) = \dfrac{f'(x)}{f(x)}$, and therefore we have

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c, \quad \text{for } f(x) > 0$$

This result is used in the following example.

Example 22

Find an antiderivative for each of the following:

a $\dfrac{6x}{x^2 + 1}$

b $\dfrac{\sin x}{\cos x}$, for $\cos x > 0$

c $\dfrac{e^x}{e^x + 1}$

Solution

a $$\int \frac{6x}{x^2 + 1}\,dx$$

$$= 3\int \frac{2x}{x^2 + 1}\,dx$$

$$= 3\ln(x^2 + 1) + c$$

b $$\int \frac{\sin x}{\cos x}\,dx$$

$$= -\int \frac{-\sin x}{\cos x}\,dx$$

$$= -\ln(\cos x) + c$$

c $$\int \frac{e^x}{e^x + 1}\,dx$$

$$= \ln(e^x + 1) + c$$

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Example 23

Let $f(x) = \dfrac{\cos x}{\sin x}$.

a Show that $f'(x) = \dfrac{-1}{\sin^2 x}$.

b Hence evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\,dx$.

Solution

a Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \end{aligned}$$

b

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\,dx &= -\left[\frac{\cos x}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= -\frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} \\ &= 1 \end{aligned}$$

Example 24

a If $f(x) = x\ln(kx)$, find $f'(x)$ and hence find $\int \ln(kx)\,dx$, where k is a positive constant.

b If $f(x) = x^2\ln(kx)$, find $f'(x)$ and hence find $\int x\ln(kx)\,dx$, where k is a positive constant.

Solution

a

$$\begin{aligned} f'(x) &= \ln(kx) + x \times \frac{1}{x} \\ &= \ln(kx) + 1 \end{aligned}$$

Antidifferentiate both sides of the equation with respect to x:

$$\begin{aligned} \int f'(x)\,dx &= \int \ln(kx)\,dx + \int 1\,dx \\ x\ln(kx) + c_1 &= \int \ln(kx)\,dx + x + c_2 \end{aligned}$$

Thus

$$\begin{aligned} \int \ln(kx)\,dx &= x\ln(kx) - x + c_1 - c_2 \\ &= x\ln(kx) - x + c \end{aligned}$$

b

$$\begin{aligned} f'(x) &= 2x\ln(kx) + x^2 \times \frac{1}{x} \\ &= 2x\ln(kx) + x \end{aligned}$$

Antidifferentiate both sides of the equation with respect to x:

$$\begin{aligned} \int f'(x)\,dx &= \int 2x\ln(kx)\,dx + \int x\,dx \\ x^2\ln(kx) + c_1 &= \int 2x\ln(kx)\,dx + \frac{x^2}{2} + c_2 \end{aligned}$$

Thus $\displaystyle\int x\ln(kx)\,dx = \frac{1}{2}x^2\ln(kx) - \frac{x^2}{4} + c$

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It is not possible to find rules for antiderivatives of all continuous functions: for example, for e^{-x^2}. However, for these functions we can find approximations of definite integrals.

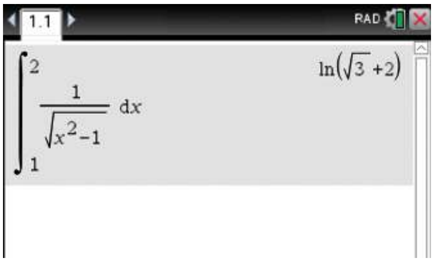
For some functions, a CAS calculator can be used to find exact values of definite integrals where finding an antiderivative by hand is beyond the scope of the course. The following example illustrates this case.

Example 25

Use a CAS calculator to find $\int_1^2 \frac{1}{\sqrt{x^2 - 1}}\, dx$.

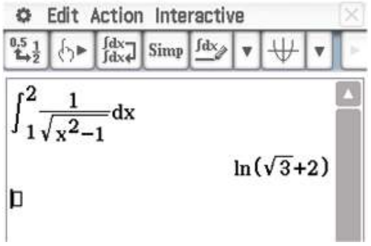
Using the TI-Nspire

Use the **Integral** template from the **Calculus** menu and complete as shown.



Using the Casio ClassPad

- Enter and highlight the expression $\frac{1}{\sqrt{x^2 - 1}}$.
- Go to **Interactive** > **Calculation** > $\int$ and select **Definite**.
- Enter the lower limit and upper limit and tap `OK`.



Exercise 6H

1 Find the exact value of each of the following:

a $\int_1^4 \sqrt{x}\, dx$ **b** $\int_{-1}^1 (1 + x)^2\, dx$ **c** $\int_0^8 \sqrt[3]{x}\, dx$

d $\int_0^{\frac{\pi}{3}} \cos(2x) - \sin\left(\frac{x}{2}\right) dx$ **e** $\int_1^2 e^{2x} + \frac{4}{x}\, dx$ **f** $\int_0^{\pi} \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right) dx$

g $\int_0^{\frac{\pi}{2}} 5x + \sin(2x)\, dx$ **h** $\int_1^4 \left(2 + \frac{1}{x}\right)^2 dx$ **i** $\int_0^1 x^2(1 - x)\, dx$

2 Find the exact area of the region bounded by the graph of $f(x) = \sin x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

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Example 21 **3** Differentiate $\ln(3x^2+7)$ and hence evaluate $\int_0^2 \frac{x}{3x^2+7}\,dx$.

Example 22 **4** Find an antiderivative for each of the following:

a $\frac{x^2}{x^3+3}$, for $x^3+3>0$ **b** $\frac{x+2}{x^2+4x}$, for $x^2+4x>0$ **c** $\frac{e^{2x}}{3+e^{2x}}$

d $\frac{x^2+1}{x^3+3x}$, for $x^3+3x>0$ **e** $\frac{5}{3x-2}$, for $x>\frac{2}{3}$

Example 23 **5** **a** Differentiate $\frac{\sin x}{\cos x}$ and hence find an antiderivative of $\frac{1}{\cos^2 x}$.

b Differentiate $\frac{\cos(2x)}{\sin(2x)}$ and hence find an antiderivative of $\frac{1}{\sin^2(2x)}$.

c Differentiate $x\sin x$ and hence evaluate $\int_0^{\frac{\pi}{4}} x\cos x\,dx$.

Example 24 **6** **a** If $f(x)=x\ln(2x)$, find $f'(x)$ and hence find $\int \ln(2x)\,dx$.

b If $f(x)=x^2\ln(2x)$, find $f'(x)$ and hence find $\int x\ln(2x)\,dx$.

c Find the derivatives of $x+\sqrt{1+x^2}$ and $\ln(x+\sqrt{1+x^2})$.

By simplifying your last result if necessary, evaluate $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$.

7 Find $\frac{d}{dx}(e^{\sqrt{x}})$ and hence evaluate $\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

8 Find $\frac{d}{dx}\left(\sin^3(2x)\right)$ and hence evaluate $\int_0^{\frac{\pi}{4}} \sin^2(2x)\cos(2x)\,dx$.

Example 25 **9** Find the value of each of the following definite integrals, correct to two decimal places:

a $\int_0^{20} 10\cos\left(\frac{\pi x}{40}\right)e^{\frac{x}{80}}\,dx$ **b** $\int_2^5 \frac{e^x}{(x-1)^2}\,dx$ **c** $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos x}{(x-1)^2}\,dx$

10 **a** Show that $\frac{2x+3}{x-1}=2+\frac{5}{x-1}$.

b Hence evaluate $\int_2^4 \frac{2x+3}{x-1}\,dx$.

11 **a** Show that $\frac{5x-4}{x-2}=5+\frac{6}{x-2}$.

b Hence evaluate $\int_3^4 \frac{5x-4}{x-2}\,dx$.

12 Find a function f such that $f'(x)=\sin\left(\frac{1}{2}x\right)$ and $f\left(\frac{4\pi}{3}\right)=2$.

13 For each of the following, find $f(x)$:

a $f'(x)=\cos(2x)$ and $f(\pi)=1$ **b** $f'(x)=\frac{3}{x}$ and $f(1)=6$

c $f'(x)=e^{\frac{x}{2}}$ and $f(0)=1$

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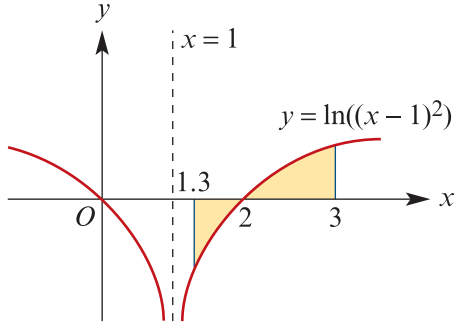
14 Find $\frac{d}{dx}\big(x\sin(3x)\big)$ and hence evaluate $\int_0^{\frac{\pi}{6}} x\cos(3x)\,dx$.

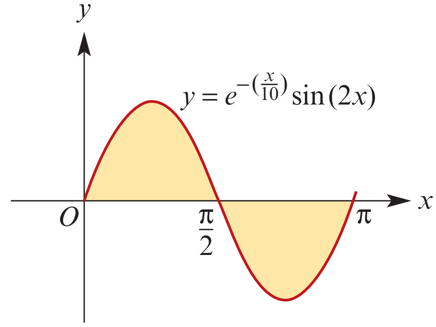
15 The curve with equation $y = a + b\sin\left(\frac{\pi x}{2}\right)$ passes through the points $(0, 1)$ and $(3, 3)$. Find a and b. Find the area of the region enclosed by this curve, the x-axis and the lines $x = 0$ and $x = 1$.

16 For each of the following, find the area of the shaded region correct to three decimal places:

a



b



17 Evaluate $\int_0^{\pi} e^{-\left(\frac{x}{10}\right)}\sin(2x)\,dx$, correct to four decimal places.

18 The gradient of a curve with equation $y = f(x)$ is given by $f'(x) = x + \sin(2x)$ and $f(0) = 1$. Find $f(x)$.

19 Let $f(x) = g'(x)$ and $h(x) = k'(x)$, where $g(x) = (x^2 + 1)^3$ and $k(x) = \sin(x^2)$. Find:

a $\int f(x)\,dx$ **b** $\int h(x)\,dx$ **c** $\int f(x) + h(x)\,dx$

d $\int -f(x)\,dx$ **e** $\int f(x) - 4\,dx$ **f** $\int 3h(x)\,dx$

20 Sketch the graph of $y = \frac{2}{x-1} + 4$ and evaluate $\int_2^3 \frac{2}{x-1} + 4\,dx$.
Indicate on your graph the region for which you have determined the area.

21 Sketch the graph of $y = \sqrt{2x-4} + 1$ and evaluate $\int_2^3 \sqrt{2x-4} + 1\,dx$.
Indicate on your graph the region for which you have determined the area.

22 Evaluate each of the following:

a $\int_3^4 \sqrt{x-2}\,dx$ **b** $\int_0^2 \sqrt{2-x}\,dx$ **c** $\int_0^1 \frac{1}{3x+1}\,dx$

d $\int_1^2 \frac{1}{2x-1} + 3\,dx$ **e** $\int_{2.5}^3 \sqrt{2x-5} - 6\,dx$ **f** $\int_3^4 \frac{1}{\sqrt{x-2}}\,dx$

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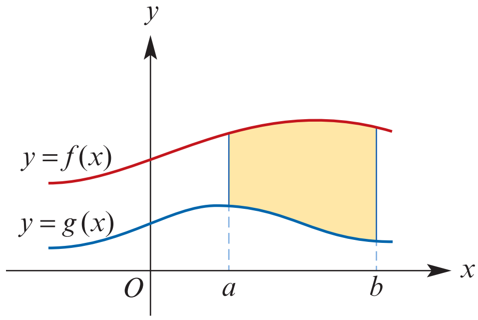
6I The area of a region between two curves

Let f and g be continuous functions on the interval $[a, b]$ such that

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b]$$

Then the area of the region bounded by the two curves and the lines $x = a$ and $x = b$ can be found by evaluating

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$



Example 26

Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

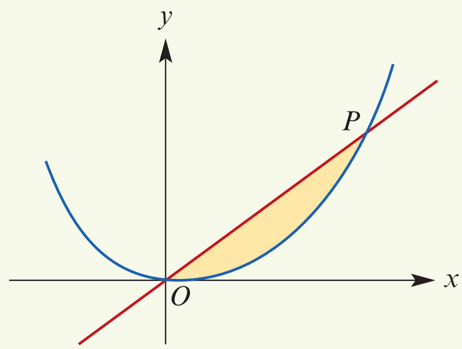
We first find the coordinates of the point P:

$$x^2 = 2x$$
$$x(x-2) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 2$$

Therefore the coordinates of P are $(2, 4)$.

$$\text{Required area} = \int_0^2 2x - x^2\,dx$$
$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$
$$= 4 - \frac{8}{3} = \frac{4}{3}$$

The area is $\frac{4}{3}$ square units.

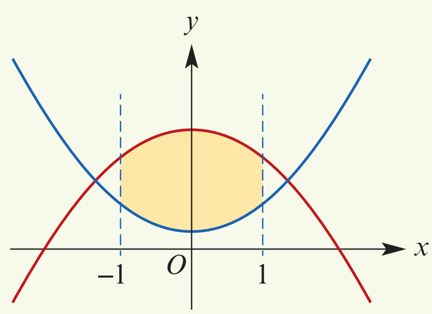


Example 27

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.

Solution

$$\text{Required area} = \int_{-1}^1 4 - x^2 - (x^2 + 1)\,dx$$
$$= \int_{-1}^1 3 - 2x^2\,dx$$
$$= \left[3x - \frac{2x^3}{3}\right]_{-1}^1$$
$$= 3 - \frac{2}{3} - \left(-3 + \frac{2}{3}\right)$$
$$= \frac{14}{3}$$



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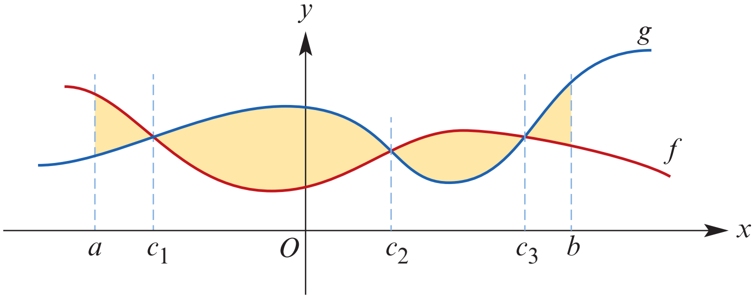
In Examples 26 and 27, the graph of one function is always 'above' the graph of the other for the intervals considered. What happens if the graphs cross?



To find the area of the shaded region, we must consider the intervals $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$ separately. Thus, the shaded area is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$

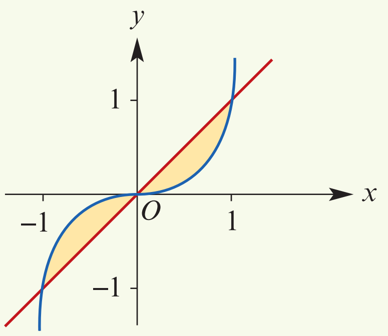
Example 28

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

Solution

The graphs intersect where $f(x) = g(x)$:

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$\therefore \quad x = 0 \text{ or } x = \pm 1$$



We see that:

- $f(x) \geq g(x)$ for $-1 \leq x \leq 0$
- $f(x) \leq g(x)$ for $0 \leq x \leq 1$

Thus the area is given by

$$\int_{-1}^{0} f(x) - g(x)\,dx + \int_0^1 g(x) - f(x)\,dx = \int_{-1}^{0} x^3 - x\,dx + \int_0^1 x - x^3\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= -\left(-\frac{1}{4}\right) + \frac{1}{4}$$

$$= \frac{1}{2}$$

The area is $\frac{1}{2}$ square unit.

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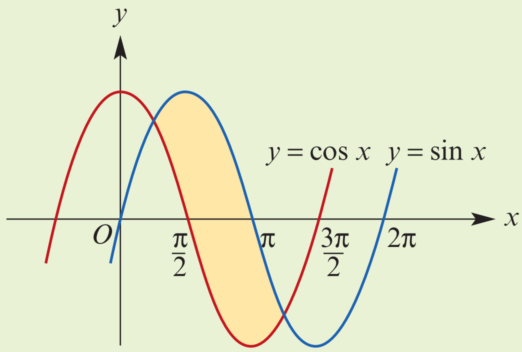
Example 29

Find the area of the shaded region.



Solution

First find the x-coordinates of the two points of intersection.

If $\sin x = \cos x$, then $\tan x = 1$ and so $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx \\
&= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= \frac{4}{\sqrt{2}} = 2\sqrt{2}
\end{aligned}$$

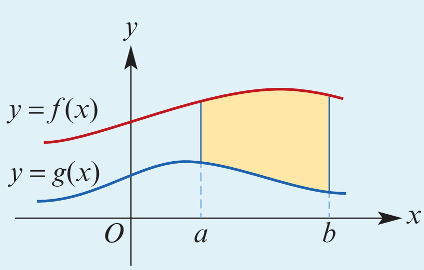
The area is $2\sqrt{2}$ square units.

Summary 6I

To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



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Exercise 6I

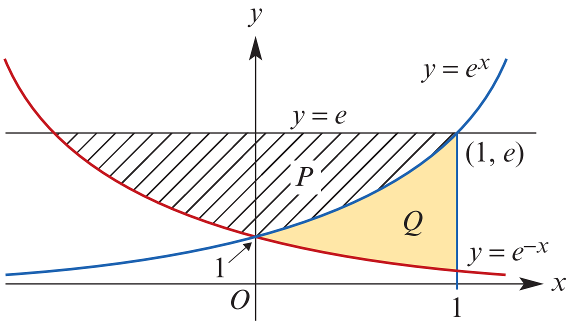
Example 26
1 Find the exact area of the region bounded by the graphs of $y = 12 - x - x^2$ and $y = x + 4$.

Example 27
2 Find the exact area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = (x - 1)^2$.

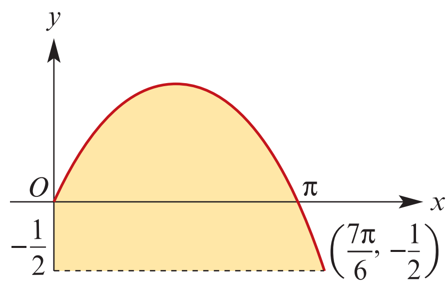
Example 28
3 Find the exact area of the region bounded by the graphs with equations:

a $y = x + 3$ and $y = 12 + x - x^2$
b $y = 3x + 5$ and $y = x^2 + 1$
c $y = 3 - x^2$ and $y = 2x^2$
d $y = x^2$ and $y = 3x$
e $y^2 = x$ and $x - y = 2$

4 **a** Find the area of region P.
b Find the area of region Q.

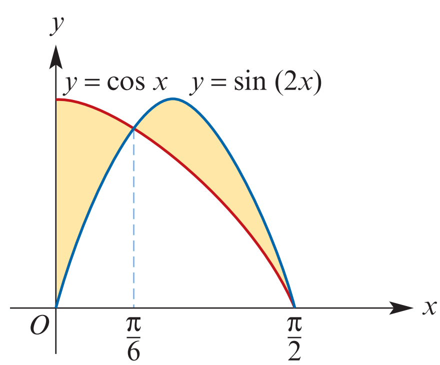


5 The figure shows part of the curve $y = \sin x$. Calculate the area of the shaded region, correct to three decimal places.



Example 29
6 Using the same axes, sketch the curves $y = \sin x$ and $y = \sin(2x)$ for $0 \le x \le \pi$. Calculate the smaller of the two areas enclosed by the curves.

7 Find the area of the shaded region.



8 Find the coordinates of P, the point of intersection of the curves $y = e^x$ and $y = 2 + 3e^{-x}$. If these curves cut the y-axis at points A and B respectively, calculate the area bounded by AB and the arcs AP and BP. Give your answer correct to three decimal places.

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6J Applications of integration

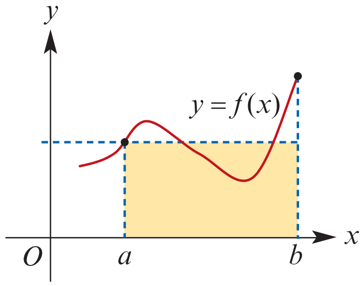
In this section we look at three applications of integration.

Average value of a function

The average value of a function f for an interval $[a, b]$ is defined as:

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

In terms of the graph of $y = f(x)$, the average value is the height of a rectangle having the same area as the area under the graph for the interval $[a, b]$.

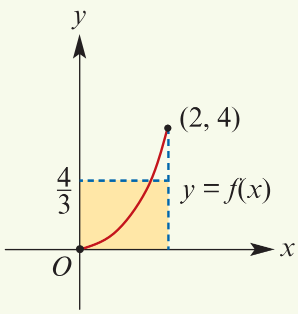


Example 30

Find the average value of $f(x) = x^2$ for the interval $[0, 2]$. Illustrate with a horizontal line determined by this value.

Solution

$$\begin{aligned}\text{Average} &= \frac{1}{2-0}\int_0^2 x^2\,dx \\ &= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 \\ &= \frac{1}{2}\times\frac{8}{3} \\ &= \frac{4}{3}\end{aligned}$$



Note: Area of rectangle $= \int_0^2 f(x)\,dx$

Motion in a straight line

We may be given a rule for acceleration and, by the use of antidifferentiation and some additional information, we can deduce rules for both velocity and position.

Example 31

A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 4$. Find:

a its position x in terms of t

b its position after 3 seconds

c its average velocity in the first 3 seconds

d the distance travelled in the first 3 seconds

e its average speed in the first 3 seconds.

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Solution

a Antidifferentiate v to find the expression for position, x m, at time t seconds:

$$x = t^2 - 4t + c$$

When $t = 0$, $x = 0$, and so $c = 0$.

$$\therefore \quad x = t^2 - 4t$$

b When $t = 3$, $x = -3$. The body is 3 m to the left of O.

c Average velocity $= \dfrac{-3 - 0}{3} = -1$ m/s

d First find when the body is at rest: $v = 0$ implies $2t - 4 = 0$, i.e. $t = 2$.

When $t = 2$, $x = -4$. Therefore the body goes from $x = 0$ to $x = -4$ in the first 2 seconds, and then back to $x = -3$ in the next second.

Thus it has travelled 5 m in the first 3 seconds.

e Average speed $= \dfrac{5}{3}$ m/s

It is useful to observe that, for a time interval $[t_1, t_2]$,

$$\text{Average velocity} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\, dt$$

where $v(t)$ is the velocity at time t.

Example 32

A particle starts from rest 3 metres from a fixed point and moves in a straight line with an acceleration of $a = 6t + 8$. Find its position and velocity at time t seconds.

Solution

We are given the acceleration:

$$a = \frac{dv}{dt} = 6t + 8$$

Find the velocity by antidifferentiating:

$$v = 3t^2 + 8t + c$$

At $t = 0$, $v = 0$, and so $c = 0$.

$$\therefore \quad v = 3t^2 + 8t$$

Find the position by antidifferentiating again:

$$x = t^3 + 4t^2 + d$$

At $t = 0$, $x = 3$, and so $d = 3$.

$$\therefore \quad x = t^3 + 4t^2 + 3$$

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Example 33

A stone is projected vertically upwards from the top of a building 20 m high with an initial velocity of 15 m/s. Find:

a the time taken for the stone to reach its maximum height

b the maximum height reached by the stone

c the time taken for the stone to reach the ground

d the velocity of the stone as it hits the ground.

In this case we only consider the stone's motion in a vertical direction, so we can treat it as motion in a straight line. Also we will assume that the acceleration due to gravity is approximately -10 m/s^2. (Note that downwards is considered the negative direction.)

Solution

Antidifferentiating $a = -10$ gives $v = -10t + c$.

At $t = 0$, $v = 15$ and therefore $v = -10t + 15$.

Antidifferentiating v gives $x = -5t^2 + 15t + d$.

At $t = 0$, $x = 20$ and so $x = -5t^2 + 15t + 20$.

a The stone will reach its maximum height when $v = 0$:

$$-10t + 15 = 0$$
$$\therefore \quad t = 1.5$$

The stone takes 1.5 seconds to reach its maximum height.

b At $t = 1.5$, $x = -5(1.5)^2 + 15(1.5) + 20$

$$= 31.25$$

The maximum height reached by the stone is 31.25 metres.

c The stone reaches the ground when $x = 0$:

$$-5t^2 + 15t + 20 = 0$$
$$-5(t^2 - 3t - 4) = 0$$
$$-5(t - 4)(t + 1) = 0$$
$$\therefore \quad t = 4$$

(The solution $t = -1$ is rejected as $t \geq 0$.)
The stone takes 4 seconds to reach the ground.

d At $t = 4$, $v = -10(4) + 15$

$$= -25$$

The velocity on impact is -25 m/s.

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Rates of change

Given the rate of change of a quantity, we can obtain information about how the quantity varies. For example, we have seen that if the velocity of an object travelling in a straight line is given at time t, then the position of the object at time t can be determined using information about the initial position of the object.

Example 34

The rate of change of temperature with respect to time of a liquid which has been boiled and then allowed to cool is given by

$$\frac{dT}{dt} = -0.5(T - 30)$$

where T is the temperature (°C) at time t (minutes).

a Sketch the graph of $\frac{dT}{dt}$ against T for $T > 30$.

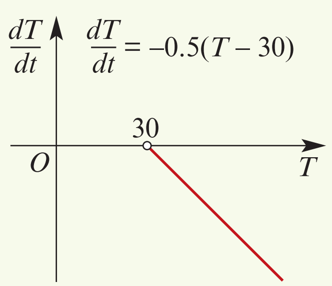
b Sketch the graph of $\frac{dt}{dT}$ against T for $T > 30$.

c **i** Find the area of the region enclosed by the graph of part **b**, the x-axis and the lines $T = 35$ and $T = 120$. Give your answer correct to two decimal places.

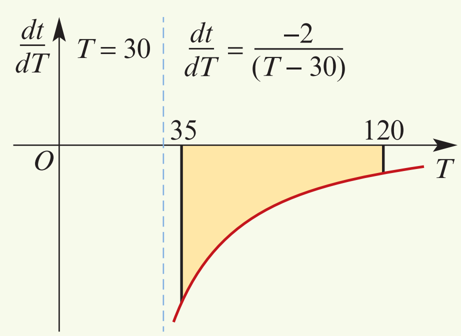
ii What does this area represent?

Solution

a



b



c **i** Area $= -\int_{35}^{120} \frac{-2}{T - 30}\, dT = 5.78$

ii The area represents the time taken for the liquid to cool from 120°C to 35°C.

We can use integration to find the total change in a quantity from the rate of change of the quantity. The result below follows directly from the fundamental theorem of calculus.

Given the rule for $f'(x)$, the **total change** in the value of $f(x)$ between $x = a$ and $x = b$ can be found using

$$f(b) - f(a) = \int_a^b f'(x)\, dx$$

Note: In Example 34, the definite integral $\int_{120}^{35} \frac{dt}{dT}\, dT = \int_{120}^{35} \frac{-2}{T - 30}\, dT = 5.78$ gives the change in t between $T = 120$ and $T = 35$.

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Exercise 6J

Example 30

1 Find the average value of each of the following functions for the stated interval:

a $f(x) = x(2-x),\ x \in [0,2]$

b $f(x) = \sin x,\ x \in [0,\pi]$

c $f(x) = \sin x,\ x \in \left[0, \frac{\pi}{2}\right]$

d $f(x) = \sin(nx),\ x \in \left[0, \frac{2\pi}{n}\right]$

e $f(x) = e^x + e^{-x},\ x \in [-2,2]$

2 An object is cooling and its temperature, T°C, after t minutes is given by $T = 50e^{\frac{-t}{2}}$. What is its average temperature over the first 10 minutes of cooling?

3 Find the average speed over the given interval for each of the following speed functions. For each of them, sketch a graph and mark in the average as a horizontal line. Time is in seconds and speed in metres per second.

a $v = 20t,\ t \in [0,5]$

b $v = 24\sin\left(\frac{1}{4}\pi t\right),\ t \in [0,4]$

c $v = 5(1-e^{-t}),\ t \in [0,5]$

4 An object falls from rest. Its velocity, v m/s, at time t seconds is given by $v = 9.8t$. Find the average velocity of the object over the first 3 seconds of its motion.

5 Find the mean value of $x(a-x)$ from $x = 0$ to $x = a$.

6 A quantity of gas expands according to the law $pv^{0.9} = 300$, where v m^3 is the volume of the gas and p N/m^2 is the pressure.

a What is the average pressure as the volume changes from $\frac{1}{2}$ m^3 to 1 m^3?

b If the change in volume in terms of t is given by $v = 3t + 1$, what is the average pressure over the time interval from $t = 0$ to $t = 1$?

Example 31

7 An object starts from point O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 3$. Find:

a its position x in terms of t

b its position after 3 seconds

c its average velocity in the first 3 seconds

d the distance travelled in the first 3 seconds

e its average speed in the first 3 seconds.

8 The velocity of a particle, v m/s, at time t seconds ($t \geq 0$) is given by $v = 2t^2 - 8t + 6$. It is initially 4 m to the right of a point O. Find:

a its position and acceleration at time t

b its position when the velocity is zero

c its acceleration when the velocity is zero.

9 An object moves in a straight line with an acceleration of 8 m/s^2. If after 1 second it passes through point O and after 3 seconds it is 30 metres from O, find its initial position relative to O.

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Example 32 **10** A particle moves in a straight line so that its acceleration, a m/s^2, after t seconds ($t \geq 0$) is given by $a = 2t - 3$. If the initial position of the object is 2 m to the right of a point O and its initial velocity is 3 m/s, find the particle's position and velocity after 10 seconds.

Example 33 **11** An object is projected vertically upwards with a velocity of 25 m/s. (Its acceleration due to gravity is -10 m/s^2.) Find:

a the object's velocity at time t

b its height above the point of projection at time t

c the time it takes to reach its maximum height

d the maximum height reached

e the time taken to return to the point of projection.

Example 34 **12** Heat escapes from a storage tank such that the rate of heat loss, in kilojoules per day, is given by

$$\frac{dH}{dt} = 1 + \frac{3}{4}\sin\left(\frac{\pi t}{60}\right), \qquad 0 \leq t \leq 200$$

where $H(t)$ is the total accumulated heat loss at time t days after noon on 1 April.

a Sketch the graph of $\frac{dH}{dt}$ against t for $0 \leq t \leq 200$.

b Find the values of t for which the rate of heat loss, i.e. $\frac{dH}{dt}$, is greater than 1.375.

c Find the values of t for which the rate of heat loss reaches its maximum.

d Find the heat lost between:

i $t = 0$ and $t = 120$ **ii** $t = 0$ and $t = 200$

13 The rate of flow of water from a reservoir is given by $\frac{dV}{dt} = 1000 - 30t^2 + 2t^3$ for $0 \leq t \leq 15$, where V is measured in millions of litres and t is the number of hours after the sluice gates are opened.

a Find the rate of flow (in million litres per hour) when $t = 0$ and $t = 2$.

b **i** Find the times when the rate of flow is a maximum.

ii Find the maximum flow.

c Sketch the graph of $\frac{dV}{dt}$ against t for $0 \leq t \leq 15$.

d **i** Find the area beneath the graph between $t = 0$ and $t = 10$.

ii What does this area represent?

14 The population of penguins on an island off the coast of Tasmania is increasing steadily. The rate of growth is given by the function $R(t) = 10\ln(t + 1)$, $\{t \in \mathbb{R} : t \geq 0\}$. The rate is measured in number of penguins per year. The date 1 January 1875 coincides with $t = 0$.

a Find the rate of growth of penguins when $t = 5$, $t = 10$, $t = 100$.

b Sketch the graph of $y = R(t)$.

c Using a CAS calculator, find the area under the graph of $y = R(t)$ between $t = 0$ and $t = 100$. What does this area represent?

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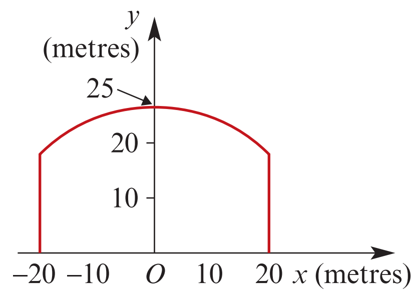
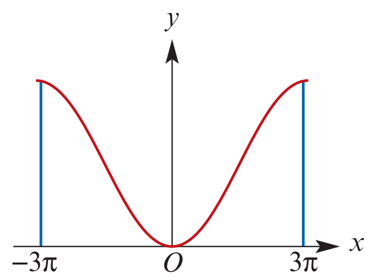
15 The roof of an exhibition hall has the shape of the function $f(x) = 25 - 0.02x^2$, $\{ x \in \mathbb{R} : -20 \le x \le 20 \}$. The hall is 80 metres long. A cross-section of the hall is shown in the figure. An air-conditioning company wishes to find the volume of the hall so that a suitable system may be installed. Find this volume.



16 A long trough with a parabolic cross-section is $1\frac{1}{2}$ metres wide at the top and 2 metres deep. Find the depth of water when the trough is half full.

17 A sculpture has cross-section as shown. The equation of the curve is $y = 3 - 3\cos\left(\frac{x}{3}\right)$ for $x \in [-3\pi, 3\pi]$. All measurements are in metres.



a Find the maximum value of the function and hence the height of the sculpture.

b The sculpture has a flat metal finish on one face, which in the diagram is represented by the region between the curve and the x-axis. Find the area of this region.

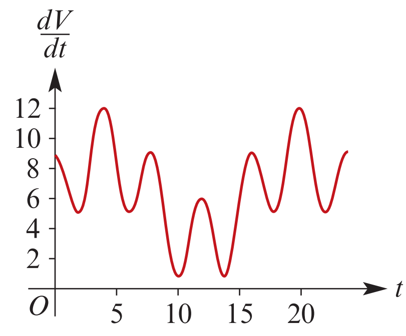
c There is a strut that meets the right side of the curve at right angles and passes through the point $(9, 0)$.

i Find the equation of the normal to the curve where $x = a$.

ii Find, correct to three decimal places, the value of a if the normal passes through $(9, 0)$.

18 The graph shows the number of litres per minute of water flowing through a pipe against the number of minutes since the machine started. The pipe is attached to the machine, which requires the water for cooling.
The curve has equation

$$\frac{dV}{dt} = 3\left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{2}\right) + 2\right]$$



a What is the rate of flow of water when:

i $t = 0$ **ii** $t = 2$ **iii** $t = 4$?

b Find, correct to three decimal places, the maximum and minimum flow through the pipe.

c Find the volume of water which flows through the pipe in the first 8 minutes.

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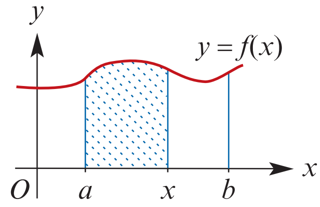
6K The fundamental theorem of calculus

The derivative of the area function

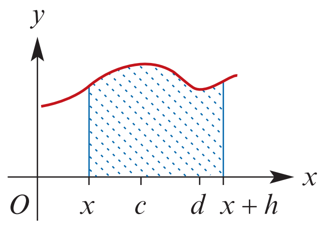
Let $f(x)$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$.



We define the function A geometrically by saying that $A(x)$ is the measure of the area under the curve $y = f(x)$ between a and x. We thus have $A(a) = 0$. We will see that $A'(x) = f(x)$, and thus A is an antiderivative of f.

First consider the quotient $\dfrac{A(x+h) - A(x)}{h}$ for $h > 0$.

By our definition of $A(x)$, it follows that $A(x+h) - A(x)$ is the area between x and $x + h$.



Let c be the point in the interval $[x, x+h]$ such that $f(c) \geq f(z)$ for all $z \in [x, x+h]$, and let d be the point in the same interval such that $f(d) \leq f(z)$ for all $z \in [x, x+h]$.

Thus $f(d) \leq f(z) \leq f(c)$ for all $z \in [x, x+h]$.

Therefore $hf(d) \leq A(x+h) - A(x) \leq hf(c)$.

That is, the shaded region has an area less than the area of the rectangle with base h and height $f(c)$ and an area greater than the area of the rectangle with base h and height $f(d)$.

Dividing by h gives

$$f(d) \leq \frac{A(x+h) - A(x)}{h} \leq f(c)$$

As $h \to 0$, both $f(c)$ and $f(d)$ approach $f(x)$.

Thus we have shown that $A'(x) = f(x)$, and therefore A is an antiderivative of f.

Now let G be any antiderivative of f. Since both A and G are antiderivatives of f, they must differ by a constant. That is,

$$A(x) = G(x) + k$$

where k is a constant. First let $x = a$. We then have

$$0 = A(a) = G(a) + k$$

and so $k = -G(a)$.

Thus $A(x) = G(x) - G(a)$, and letting $x = b$ yields

$$A(b) = G(b) - G(a)$$

The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to $G(b) - G(a)$, where G is any antiderivative of f.

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A similar argument could be used if $f(x) \le 0$ for all $x \in [a, b]$, but in this case we must take $A(x)$ to be the negative of the area under the curve. In general:

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f.

Signed-area functions

We have used the fundamental theorem of calculus to find signed areas using antiderivatives. We can also use the theorem to define antiderivatives using signed-area functions.

For a continuous function f and a constant a, define $A(x)$ to be the signed area under the graph of $y = f(x)$ between a and x. Using a dummy variable t, we can write

$$A(x) = \int_a^x f(t)\,dt$$

We have seen that $A'(x) = f(x)$, and therefore A is an antiderivative of f.

Example 35

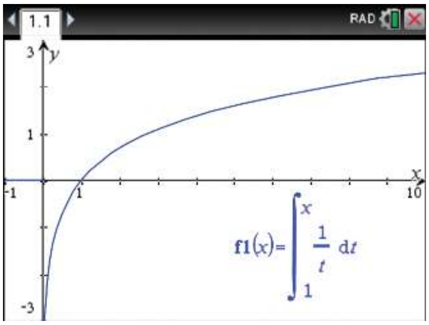
Plot the graph of the signed-area function $F(x) = \int_1^x \frac{1}{t}\,dt$ for $x > 1$.

Using the TI-Nspire

In a **Graphs** page, enter the function

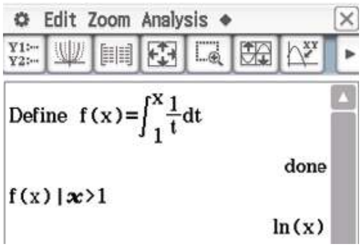
$$f1(x) = \int_1^x \frac{1}{t}\,dt$$

Note: The integral template can be obtained from the 2D-template palette.



Using the Casio ClassPad

- Enter and define the function as shown.



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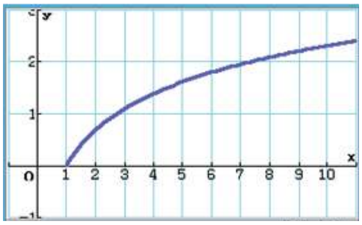
- Graph the function with the restricted domain.



Note: The natural logarithm function can be defined by $\ln(x) = \int_1^x \frac{1}{t}\, dt$.
The number e can then be defined to be the unique real number a such that $\ln(a) = 1$.

The area as the limit of a sum

Finally, we consider the limit of a sum in a special case. This discussion gives an indication of how the limiting process can be undertaken in general.

Notation

We first introduce a notation to help us express sums. We do this through examples:

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$$

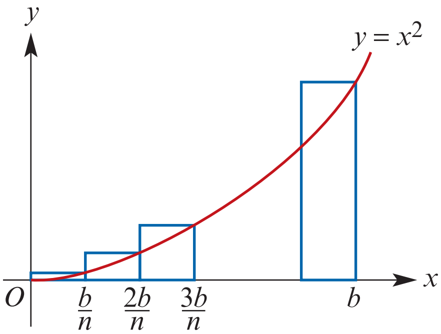
$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$\sum_{i=1}^{n} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_n f(x_n)$$

The symbol $\sum$ is the uppercase Greek letter 'sigma', which is used to denote *sum*.

The area under a parabola

Consider the graph of $y = x^2$. We will find the area under the curve from $x = 0$ to $x = b$ using a technique due to Archimedes.



Divide the interval $[0, b]$ into n equal subintervals:

$$\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \ldots, \left[\frac{(n-1)b}{n}, b\right]$$

Each subinterval is the base of a rectangle with height determined by the right endpoint of the subinterval.

$$\begin{aligned}
\text{Area of rectangles} &= \frac{b}{n}\left[\left(\frac{b}{n}\right)^2 + \left(\frac{2b}{n}\right)^2 + \left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{nb}{n}\right)^2\right] \\
&= \frac{b}{n}\left(\frac{b^2}{n^2} + \frac{4b^2}{n^2} + \frac{9b^2}{n^2} + \cdots + \frac{n^2b^2}{n^2}\right) \\
&= \frac{b^3}{n^3}\left(1 + 4 + 9 + \cdots + n^2\right)
\end{aligned}$$

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There is a rule for working out the sum of the first n square numbers:

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\begin{aligned}\text{Area of rectangles} &= \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{b^3}{n^3} \times \frac{n}{6}(n+1)(2n+1) \\ &= \frac{b^3}{6n^2}(2n^2+3n+1) \\ &= \frac{b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\end{aligned}$$

As n becomes very large, the terms $\frac{3}{n}$ and $\frac{1}{n^2}$ become very small. We write:

$$\lim_{n\to\infty} \frac{b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{b^3}{3}$$

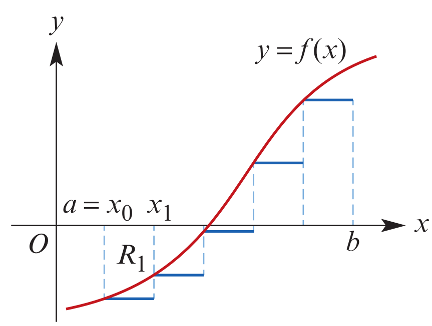
We read this as: the limit of the sum as n approaches infinity is $\frac{b^3}{3}$.

Using n left-endpoint rectangles, and considering the limit as $n \to \infty$, also gives the area $\frac{b^3}{3}$.

The signed area enclosed by a curve

This technique may be applied in general to a continuous function f on an interval $[a, b]$. For convenience, we will consider an increasing function.

Divide the interval $[a, b]$ into n equal subintervals. Each subinterval is the base of a rectangle with its 'height' determined by the left endpoint of the subinterval.



The contribution of rectangle R_1 is $(x_1 - x_0)f(x_0)$. Since $f(x_0) < 0$, the result is negative and so we have found the *signed area* of R_1.

The sum of the signed areas of the rectangles is

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)$$

If the limit as $n \to \infty$ exists, then we can make the following definition:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)\right)$$

We could also have used the right-endpoint estimate: the left- and right-endpoint estimates will converge to the same limit as n approaches infinity. Definite integrals may be defined as the limit of suitable sums, and the fundamental theorem of calculus holds true under this definition.

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Summary 6K

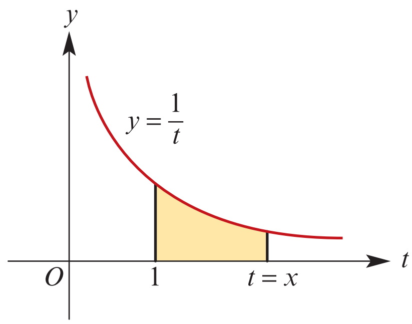
Fundamental theorem of calculus

- If f is a continuous function on an interval $[a, b]$, then $\int_a^b f(x)\,dx = G(b) - G(a)$, where G is any antiderivative of f.
- If f is a continuous function and the function F is defined by $F(x) = \int_a^x f(t)\,dt$, then F is an antiderivative of f.
- $F(x) = \int_a^x f(t)dt$ then $\dfrac{dF(x)}{dx} = f(x)$

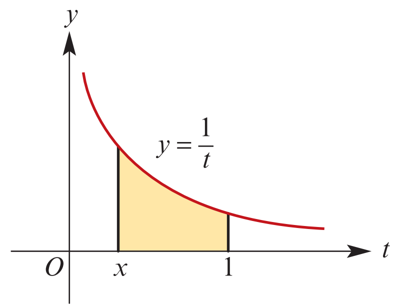
Exercise 6K

1 In each of the following, the rule of the function is defined as an area function. Find $F(x)$ in each case.

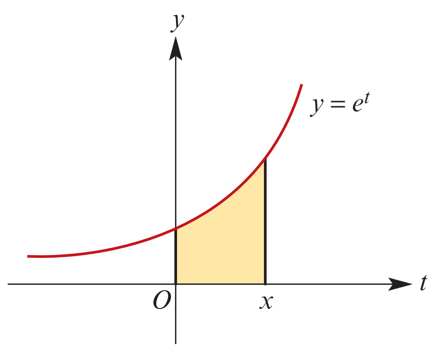
a $F(x) = \int_1^x \frac{1}{t}\,dt$, for $x > 1$



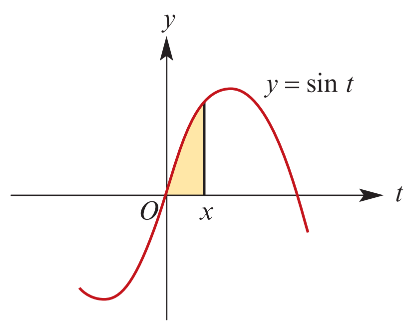
b $F(x) = \int_x^1 \frac{1}{t}\,dt$, for $0 < x < 1$



c $F(x) = \int_0^x e^t\,dt$, for $x \in \mathbb{R}$



d $F(x) = \int_0^x \sin t\,dt$, for $x \in \mathbb{R}$



2 Show that the area under the curve $y = 2x^2$ from $x = 0$ to $x = 1$ is $\frac{2}{3}$ square units by using the right-endpoint method with n rectangles and considering the limit as $n \to \infty$.

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Review

Assign-ment

Nrich

Chapter summary

Antidifferentiation

- To find the general antiderivative:
  If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.
- Basic antiderivatives:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c \qquad \text{where } \{ r \in \mathbb{Q} : r \neq -1 \}$$

$$\int (ax+b)^r\,dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c \qquad \text{where } \{ r \in \mathbb{Q} : r \neq -1 \}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c \qquad \text{for } ax+b > 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(-ax-b) + c \qquad \text{for } ax+b < 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c \qquad \text{where } a \neq 0$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c \qquad \text{where } a \neq 0$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c \qquad \text{where } a \neq 0$$

- Properties of antidifferentiation:
  - $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int k\,f(x)\,dx = k\int f(x)\,dx$, where k is a real number

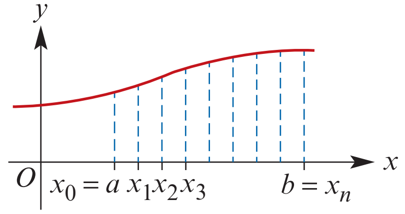
Integration

- Numerical methods for approximating the area under a graph: Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[a, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, b]$.
  - Left-endpoint estimate

$$L_n = \frac{b-a}{n}\big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\big]$$

  - Right-endpoint estimate

$$R_n = \frac{b-a}{n}\big[f(x_1) + f(x_2) + \cdots + f(x_n)\big]$$



- Definite integral The signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$ is denoted by $\int_a^b f(x)\,dx$.
- Fundamental theorem of calculus
  If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = [G(x)]_a^b = G(b) - G(a)$$

  where G is any antiderivative of f.

$$F(x) = \int_a^x f(t)dt \text{ then } \frac{dF(x)}{dx} = f(x)$$

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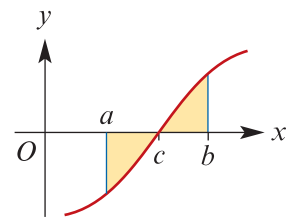
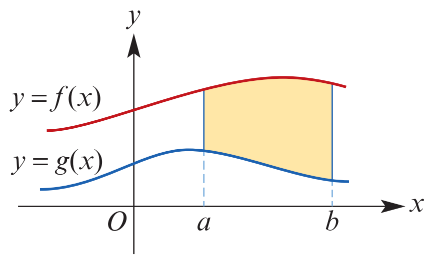
- Finding areas:
  - If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
  - If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
  - If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



  - To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

    $$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

    where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



- Properties of the definite integral:
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
  - $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- The **average value** of a continuous function f for an interval $[a, b]$ is $\dfrac{1}{b-a}\int_a^b f(x)\,dx$.
- The **total change** in the value of f for an interval $[a, b]$ is $f(b) - f(a) = \int_a^b f'(x)\,dx$.

Short-answer questions

1 Evaluate each of the following definite integrals:

a $\int_2^3 x^3\,dx$

b $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta$

c $\int_a^{4a} (a^{\frac{1}{2}} - x^{\frac{1}{2}})\,dx$, where a is a positive constant

d $\int_1^4 \dfrac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}}\,dx$

e $\int_0^{\frac{\pi}{4}} \cos(2\theta)\,d\theta$

f $\int_1^e \dfrac{1}{x}\,dx$

g $\int_0^{\frac{\pi}{2}} \sin 2\left(\theta + \dfrac{\pi}{4}\right)d\theta$

h $\int_0^{\pi} \sin(4\theta)\,d\theta$

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Review

2 Find $\int_{-1}^{2} x + 2f(x)\,dx$ if $\int_{-1}^{2} f(x)\,dx = 5$.

3 Find $\int_{1}^{5} f(x)\,dx$ if $\int_{0}^{1} f(x)\,dx = -2$ and $\int_{0}^{5} f(x)\,dx = 1$.

4 Find $\int_{3}^{-2} f(x)\,dx$ if $\int_{-2}^{1} f(x)\,dx = 2$ and $\int_{1}^{3} f(x)\,dx = -6$.

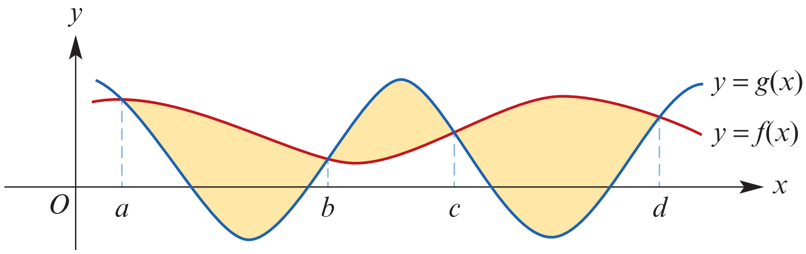
5 Evaluate $\int_{0}^{2} (x+1)^7\,dx$.

6 Evaluate $\int_{0}^{1} (3x+1)^3\,dx$.

7 Find $\int_{0}^{3} f(3x)\,dx$ if $\int_{0}^{9} f(x)\,dx = 5$.

8 Find $\int_{0}^{1} f(3x+1)\,dx$ if $\int_{1}^{4} f(x)\,dx = 5$.

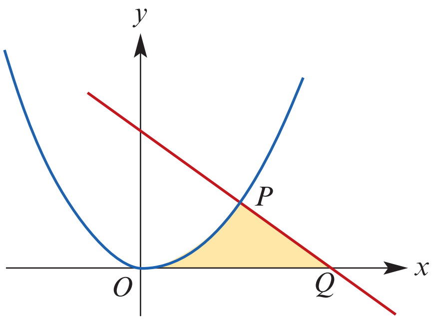
9 Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$.



10 The figure shows the curve $y = x^2$ and the straight line $2x + y = 15$. Find:

a the coordinates of P and Q

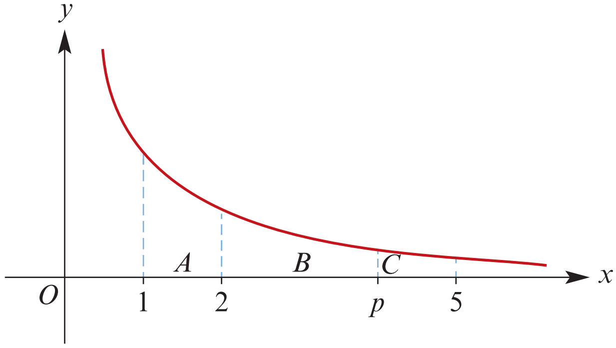
b the area of the shaded region.



11 The figure shows part of the curve $y = \dfrac{10}{x^2}$. Find:

a the area of region A

b the value of p for which the regions B and C are of equal area.



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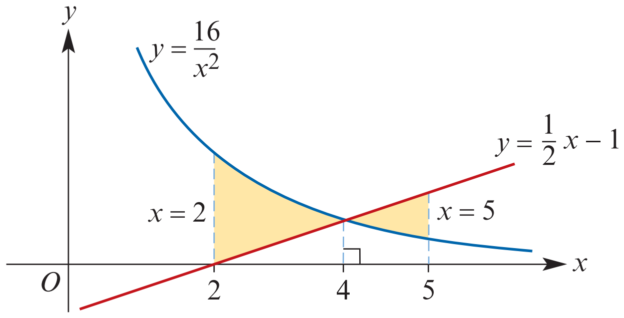
12 Find the area of the shaded region.

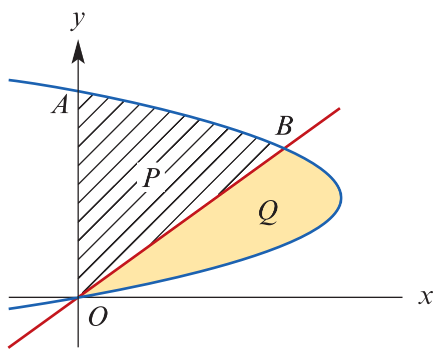


13 The figure shows part of the curve $x = 6y - y^2$ and part of the line $y = x$.

a Find the coordinates of A and B.

b Find the area of region P.

c Find the area of region Q.



14 **a** Sketch the graph of $y = e^x + 1$ and clearly indicate, by shading the region, the area given by the definite integral $\int_0^2 e^x + 1\, dx$.

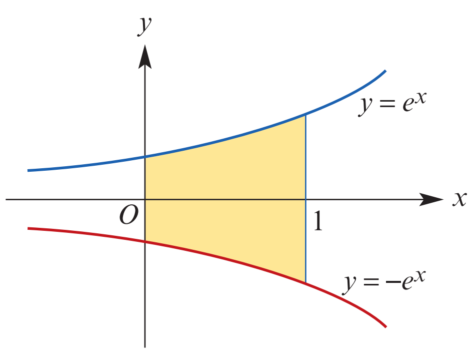
b Evaluate $\int_0^2 e^x + 1\, dx$.

15 **a** Sketch the graphs of $y = e^{-x}$ and $y = e^x$ on the one set of axes and clearly indicate, by shading the region, the area given by $\int_0^2 e^{-x}\, dx + \int_{-2}^0 e^x\, dx$.

b Evaluate $\int_0^2 e^{-x}\, dx + \int_{-2}^0 e^x\, dx$.

16 **a** Evaluate $\int_0^1 e^x\, dx$.

b By symmetry, find the area of the region shaded in the figure.



17 Sketch the graph of $f(x) = 2e^{2x} + 3$ and find the area of the region enclosed between the curve, the axes and the line $x = 1$.

18 Sketch the graph of $y = x(x - 2)(x + 1)$ and find the area of the region contained between the graph and the x-axis. (Do not attempt to find the coordinates of the turning points.)

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19 Evaluate each of the following definite integrals:

a $\int_0^2 e^{-x} + x \, dx$

b $\int_{-2}^{-1} x + \frac{1}{x-1} \, dx$

c $\int_0^{\frac{\pi}{2}} \sin x + x \, dx$

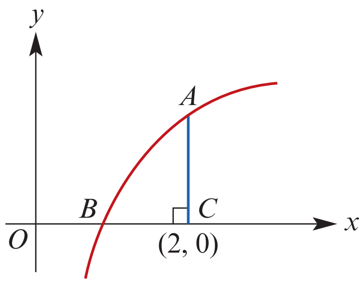
d $\int_{-4}^{-5} e^x + \frac{1}{2-2x} \, dx$

Extended-response questions

1 The diagram shows part of the curve with equation

$$y = x - \frac{1}{x^2}$$

The point C has coordinates $(2, 0)$. Find:



a the equation of the tangent to the curve at point A

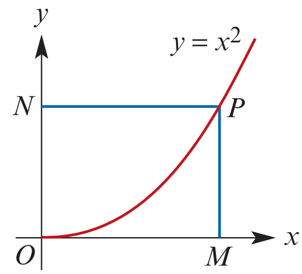
b the coordinates of the point T where this tangent meets the x-axis

c the coordinates of the point B where the curve meets the x-axis

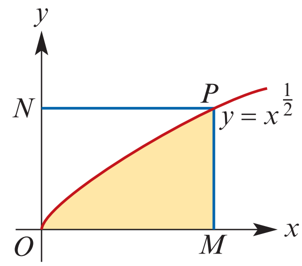
d the area of the region enclosed by the curve and the lines AT and BT

e the ratio of the area found in part **d** to the area of the triangle ATC.

2 **a** In the figure, the point P is on the curve $y = x^2$. Prove that the curve divides the rectangle $OMPN$ into two regions whose areas are in the ratio 2:1.



b In the figure, the point P is on the curve $y = x^{\frac{1}{2}}$. Prove that the area of the shaded region is two-thirds the area of the rectangle $OMPN$.



c Consider a point P on the curve $y = x^n$, with PM and PN the perpendiculars from P to the x-axis and the y-axis respectively. Prove that the area of the region enclosed between PM, the x-axis and the curve is equal to $\frac{1}{n+1}$ of the area of the rectangle $OMPN$.

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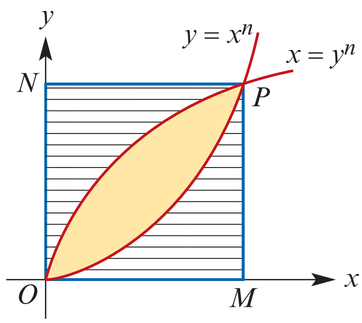
3 **a** Find the area enclosed between the parabolas $y = x^2$ and $y^2 = x$.

b Show that the curves with equations $y = x^n$ and $y^n = x$ intersect at $(1, 1)$, where $n = 1, 2, 3, \dots$.

c Show that the area of the region contained between the curves $y = x^n$ and $y^n = x$ is $\dfrac{n-1}{n+1}$.

d Find the area of the region indicated by horizontal shading in the diagram.



e Use your result from part **c** to find the area of the region between the curves for $n = 10$, $n = 100$ and $n = 1000$.

f Describe the result for n very large.

4 It is thought that the temperature, θ, of a piece of charcoal in a barbecue will increase at a rate $\dfrac{d\theta}{dt}$ given by $\dfrac{d\theta}{dt} = e^{2.6t}$, where θ is in degrees and t is in minutes.

a If the charcoal starts at a temperature of 30°C, find the expected temperature of the charcoal after 3 minutes.

b Sketch the graph of θ against t.

c At what time does the temperature of the charcoal reach 500°C?

d Find the average rate of increase of temperature from $t = 1$ to $t = 2$.

5 It is believed that the velocity of a certain subatomic particle t seconds after a collision will be given by the expression

$$\frac{dx}{dt} = ve^{-t}, \quad v = 5 \times 10^4 \text{ m/s}$$

where x is the distance travelled in metres.

a What is the initial velocity of the particle?

b What happens to the velocity as $t \to \infty$ (i.e. as t becomes very large)?

c How far will the particle travel between $t = 0$ and $t = 20$?

d Find an expression for x in terms of t.

e Sketch the graph of x against t.

6 **a** Differentiate $e^{-3x}\sin(2x)$ and $e^{-3x}\cos(2x)$ with respect to x.

b Hence show that

$$e^{-3x}\sin(2x) + c_1 = -3\int e^{-3x}\sin(2x)\,dx + 2\int e^{-3x}\cos(2x)\,dx$$

and $$e^{-3x}\cos(2x) + c_2 = -3\int e^{-3x}\cos(2x)\,dx - 2\int e^{-3x}\sin(2x)\,dx$$

c Use the two equations from part **b** to determine $\int e^{-3x}\sin(2x)\,dx$.

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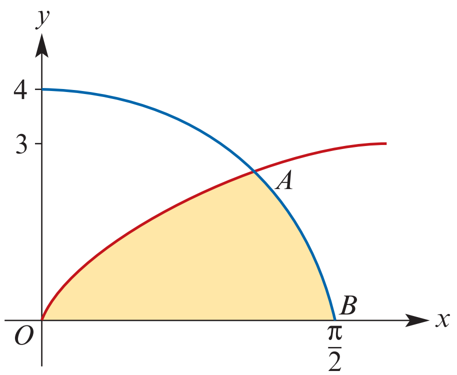
7 The curves $y = 3\sin x$ and $y = 4\cos x$, where $0 \le x \le \frac{\pi}{2}$, intersect at a point A.

a If $x = a$ at the point of intersection of the two curves:

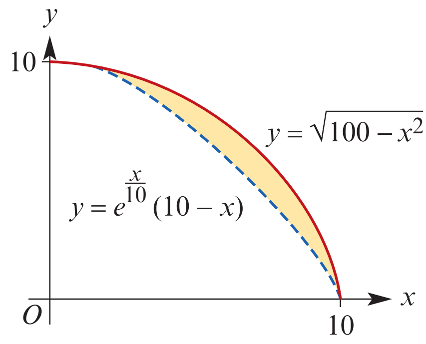
i Find $\tan a$.

ii Hence find $\sin a$ and $\cos a$.

b Hence find the area of the shaded region in the diagram.



8 A teacher attempts to draw a quarter circle of radius 10 on the white board. However, the first attempt results in a curve with equation $y = e^{\frac{x}{10}}(10 - x)$.
The quarter circle has equation $y = \sqrt{100 - x^2}$.



a Find $\frac{dy}{dx}$ for both functions.

b Find the gradient of each of the functions when $x = 0$.

c Find the gradient of $y = e^{\frac{x}{10}}(10 - x)$ when $x = 10$.

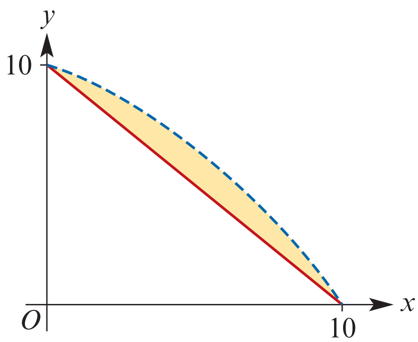
d Find the area of the shaded region correct to two decimal places using a calculator.

e Find the percentage error for the calculation of the area of the quarter circle.

f The teacher draws in a chord from $(0, 10)$ to $(10, 0)$. Find the area of the shaded region using a calculator.

g **i** Use the result that the derivative of $e^{\frac{x}{10}}(10 - x)$ is $-e^{\frac{x}{10}} + \frac{1}{10}e^{\frac{x}{10}}(10 - x)$ to find $\int_0^{10} e^{\frac{x}{10}}(10 - x)\, dx$ by analytic techniques.

ii Find the exact area of the original shaded region and compare it to the answer of part **d**.



9 **a** If $y = x\ln x$, find $\frac{dy}{dx}$. Hence find the value of $\int_1^e \ln x\, dx$.

b If $y = x(\ln x)^n$, where n is a positive integer, find $\frac{dy}{dx}$.

c Let $I_n = \int_1^e (\ln x)^n\, dx$. For $n > 1$, show that $I_n + nI_{n-1} = e$.

d Hence find the value of $\int_1^e (\ln x)^3\, dx$.

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10 The curves $y^2 = ax$ and $x^2 = by$, where a and b are both positive, intersect at the origin and at the point (r, s). Find r and s in terms of a and b. Prove that the two curves divide the rectangle with corners $(0, 0)$, $(0, s)$, (r, s), $(r, 0)$ into three regions of equal area.

11 A water-cooling device has a system of water circulation for the first 30 minutes of its operation. The circulation follows the following sequence:

- For the first 3 minutes water is flowing in.
- For the second 3 minutes water is flowing out.
- For the third 3 minutes water is flowing in.

This pattern is continued for the first 30 minutes. The rate of flow of water is given by the function

$$R(t) = 10e^{\frac{-t}{10}} \sin\left(\frac{\pi t}{3}\right)$$

where $R(t)$ litres per minute is the rate of flow at time t minutes. Initially there are 4 litres of water in the device.

a **i** Find $R(0)$.

ii Find $R(3)$.

b Find $R'(t)$.

c **i** Solve the equation $R'(t) = 0$ for $t \in [0, 12]$.

ii Find the coordinates of the stationary points of $y = R(t)$ for $t \in [0, 12]$.

d Solve the equation $R(t) = 0$ for $t \in [0, 12]$.

e Sketch the graph of $y = R(t)$ for $t \in [0, 12]$.

f **i** How many litres of water flowed into the device for $t \in [0, 3]$?

ii How many litres of water flowed out of the device for $t \in [3, 6]$?

iii How many litres of water are in the device when $t = 6$? (Remember there are initially 4 litres of water.)

g How many litres of water are there in the device when $t = 30$?

12 **a** Use the identities $\cos(2x) = 2\cos^2 x - 1$ and $\cos(2x) = 1 - 2\sin^2 x$ to show that

$$\frac{1 - \cos(2x)}{1 + \cos(2x)} = \sec^2 x - 1$$

b Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1 - \cos(2x)}{1 + \cos(2x)}\, dx$.

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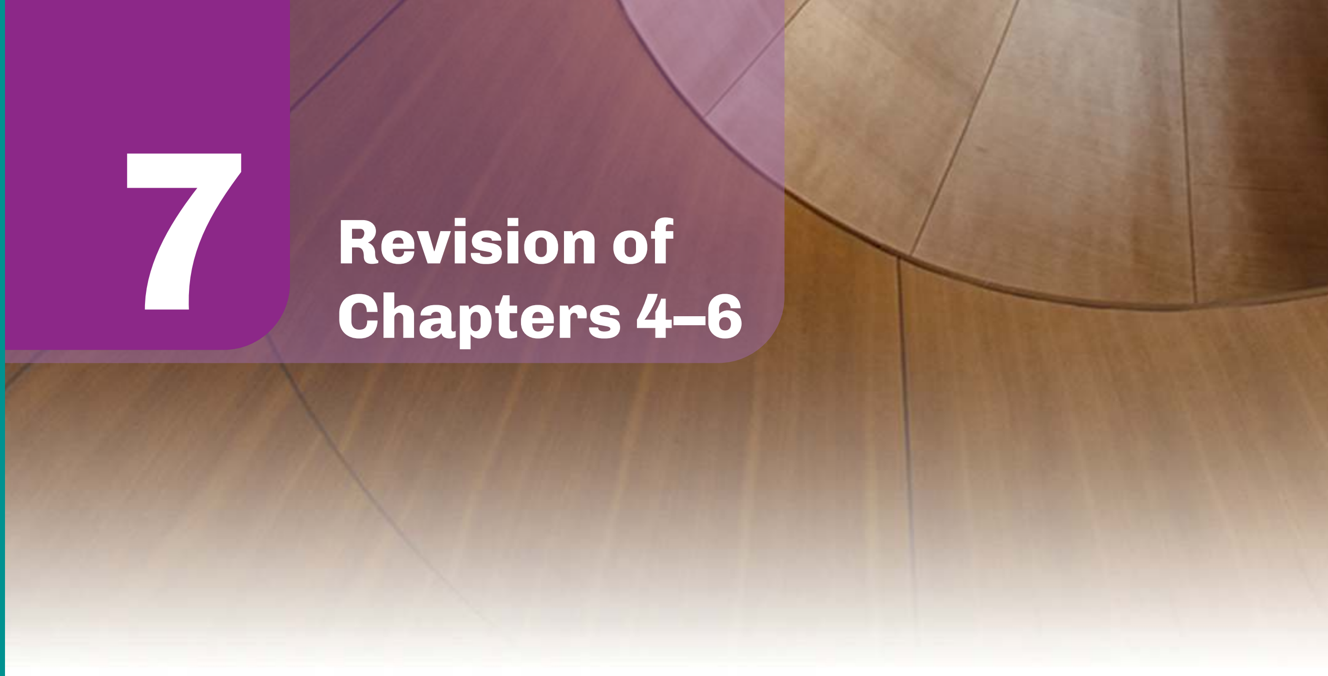
Revision



7 Revision of Chapters 4–6

7A Short-answer questions

1 Let $y = \dfrac{x^2 - 1}{x^4 - 1}$.

a Find $\dfrac{dy}{dx}$.

b Find $\left\{ x : \dfrac{dy}{dx} = 0 \right\}$.

2 Let $y = (3x^2 - 4x)^4$. Find $\dfrac{dy}{dx}$.

3 Let $f(x) = x^2 \ln(2x)$, $\{ x \in \mathbb{R} : x > 0 \}$. Find $f'(x)$.

4 **a** Let $f(x) = e^{2x+1}$. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find b.

b Let $f(x) = e^{2x+1} + k$ where k is a real number. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find k in terms of b.

5 The line $y = mx - 8$ is tangent to the curve $y = x^{\frac{1}{3}} + c$ at the point $(8, a)$. Find the values of a, c and m.

6 Find the average value of the function with rule $f(x) = \dfrac{1}{3x + 1}$ over the interval $[0, 2]$.

7 Find an antiderivative of:

a $\dfrac{3}{5x - 2}$, $x > \dfrac{2}{5}$

b $\dfrac{3}{(5x - 2)^2}$, $x \neq \dfrac{2}{5}$

8 If $f(3) = -2$ and $f'(3) = 5$, find $g'(3)$ where:

a $g(x) = 3x^2 - 5f(x)$

b $g(x) = \dfrac{3x + 1}{f(x)}$

c $g(x) = [f(x)]^2$

9 If $f(4) = 6$ and $f'(4) = 2$, find $g'(4)$ where:

a $g(x) = \sqrt{x}\, f(x)$

b $g(x) = \dfrac{f(x)}{x}$

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10 Given that $f'(x) = \sqrt{3x + 4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.

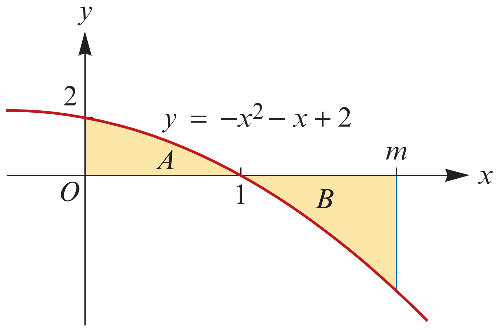
11 If $f(x) = 2x^2 - 3x + 5$, find:

a $f'(x)$ **b** $f'(0)$ **c** $\{x : f'(x) = 1\}$

12 Find the derivative of $\ln(3f(x))$ with respect to x.

13 The tangent to the graph of $y = \sqrt{a - x}$ at $x = 1$ has a gradient of -6. Find the value of a.

14 The graph of $y = -x^2 - x + 2$ is shown. Find the value of m such that regions A and B have the same area.



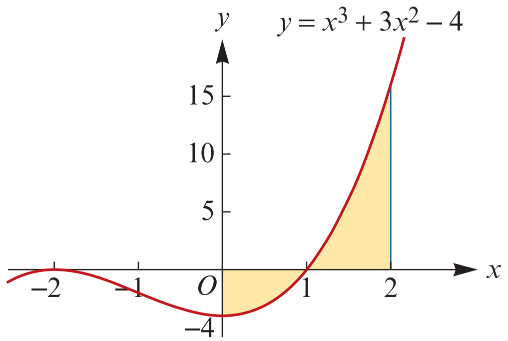
15 Let $f(x) = x^3 + 3x^2 - 4$. The graph of $y = f(x)$ is as shown. Find:

a the coordinates of the stationary points

b $\int_{-2}^{2} f(x)\,dx$

c $\int_{0}^{2} f(x)\,dx$

d the area of the shaded region.



16 If $f(x) = \dfrac{1}{3x - 1}$, find $f'(2)$ and $f''(2)$.

17 If $y = 1 - x^2$, prove that $x\dfrac{dy}{dx} + 2 = 2y$ for all values of x.

18 If $A = 4\pi r^2$, calculate $\dfrac{dA}{dr}$ when $r = 3$.

19 At what point on the graph of $y = 1.8x^2$ is the gradient 1?

20 If $y = 3x^2 - 4x + 7$, find the value of x such that $\dfrac{dy}{dx} = 0$.

21 If $y = \dfrac{x^2 + 2}{x^2 - 2}$, find $\dfrac{dy}{dx}$.

22 If $z = 3y + 4$ and $y = 2x - 1$, find $\dfrac{dz}{dx}$.

23 If $y = (5 - 7x)^9$, calculate $\dfrac{d^2y}{dx^2}$.

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24 If $y = 3x^{\frac{1}{3}}$, find $\frac{dy}{dx}$ when $x = 27$.

25 If $y = \sqrt{5 + x^2}$, find $\frac{dy}{dx}$ when $x = 2$.

26 Find $\frac{dy}{dx}$ when $x = 1$, given that $y = (x^2 + 3)(2 - 4x - 5x^2)$.

27 If $y = \frac{x}{1 + x^2}$, find $\frac{dy}{dx}$ when $x = 1$.

28 If $y = \frac{2 + x}{x^2 + x + 1}$, find $\frac{dy}{dx}$ when $x = 0$.

29 Let $f(x) = \frac{1}{2x + 1}$.

a Use the definition of derivative to find $f'(x)$.

b Find the gradient of the tangent to the graph of f at the point $(0, 1)$.

30 Let $f(x) = x^3 + 3x^2 - 1$. Find:

a $\{x : f'(x) = 0\}$ **b** $\{x : f'(x) > 0\}$ **c** $\{x : f'(x) < 0\}$

31 For $f(x) = x^3 - 3x^2 + 1$, find the values of x for which the graph of f is concave up.

32 Find the x-coordinates of the points of inflection on the graph of $f(x) = x^4 - 8x^2 + 16$.

33 Let $y = \frac{x}{1 - x}$.

a Find $\frac{dy}{dx}$. **b** Write $\frac{dy}{dx}$ in terms of y.

34 If $y = (x^2 + 1)^{-\frac{3}{2}}$, find $\frac{dy}{dx}$.

35 If $y = x^4$, prove that $x\frac{dy}{dx} = 4y$.

36 Show that $f(x) = 2x^5$ is a strictly increasing function for $\mathbb{R}$ by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$, and if $0 > b$, then $f(0) > f(b)$.

37 Evaluate each of the following integrals:

a $\int_0^{\frac{\pi}{2}} 2\sin\left(\frac{x}{2}\right) dx$ **b** $\int_0^{\frac{3}{2}} e^{\frac{x}{2}}\, dx$ **c** $\int_{\frac{1}{2}}^{1} \frac{1}{2x}\, dx$

d $\int_{-1}^{-\frac{1}{2}} \frac{1}{2x}\, dx$ **e** $\int_3^4 \frac{1}{2(x-2)^2}\, dx$ **f** $\int_2^4 \frac{1}{(3x-2)^2}\, dx$

38 Show that $f(x) = -2x^3 + 1$ is a strictly decreasing function for $\mathbb{R}$.

39 Let $f(x) = e^{-mx+2} + 4x$ where m is a positive rational number.

a Find the x-coordinate of the stationary point of the graph of $y = f(x)$ in terms of m.

b Find the values of m for which the x-coordinate of this stationary point is negative.

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40 Use the technique of linear approximation to estimate the increase in the volume of a cube when each of its edges increases from 10 cm to 10.1 cm.

41 The time, T s, for one oscillation of a pendulum of length x cm is given by $T = k\sqrt{x}$, where k is a constant. Using linear approximation, find an estimate for the percentage increase in T if x is increased by 4%.

42 Sketch the graph of $f(x) = x^4 - 10x^2 + 9$, locating the stationary points and the points of inflection.

7B Extended-response questions

1 The amount of salt (s grams) in 100 litres of salt solution at time t minutes is given by $s = 50 + 30e^{-\frac{1}{5}t}$.

- **a** Find the amount of salt in the mixture after 10 minutes.
- **b** Sketch the graph of s against t for $t \geq 0$.
- **c** Find the rate of change of the amount of salt at time t (in terms of t).
- **d** Find the rate of change of the amount of salt at time t (in terms of s).
- **e** Find the concentration (grams per litre) of salt at time $t = 0$.
- **f** Find the value of t for which the salt solution first reaches a concentration of 0.51 grams per litre.

2 A medium is kept at a constant temperature of 20°C. An object is placed in this medium. The temperature, T°C, of the object at time t minutes is given by

$$T = 40e^{-0.36t} + 20, \quad t \geq 0$$

- **a** Find the initial temperature of the object.
- **b** Sketch the graph of T against t for $t \geq 0$.
- **c** Find the rate of change of temperature with respect to time (in terms of t).
- **d** Find the rate of change of temperature with respect to time (in terms of T).

3 A certain food is susceptible to contamination from bacterial spores of two types, F and G. In order to kill the spores, the food is heated to a temperature of 120°C. The number of live spores after t minutes can be approximated by $f(t) = 1000e^{-0.5t}$ for F-type spores and by $g(t) = 1200e^{-0.7t}$ for G-type spores.

- **a** Find the time required to kill 50% of the F-type spores.
- **b** Find the total number of live spores of both types when $t = 0$, and find the percentage of these that are still alive when $t = 5$.
- **c** Find the rate at which the total number of live spores is decreasing when $t = 5$.
- **d** Find the value of t for which the number of live F-type spores and the number of live G-type spores are equal.
- **e** On the same set of axes, sketch the graphs of $y = f(t)$ and $y = g(t)$ for $t \geq 0$.

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4 An object falls from rest in a medium and its velocity, V m/s, after t seconds is given by $V = 100(1 - e^{-0.2t})$.

a Sketch the graph of V against t for $t \geq 0$.

b Express the acceleration at any instant:

i in terms of t **ii** in terms of V.

c Find the value of t for which the velocity of the object is 80 m/s.

5 A manufacturer determines that the total cost, $\$C$ per year, of producing a product is given by $C = 0.05x^2 + 5x + 500$, where x is the number of units produced per year. At what level of output will the average cost per unit be a minimum? (Use a continuous function to model this discrete situation.)

6 An object that is at a higher temperature than its surroundings cools according to Newton's law of cooling: $T = T_0 e^{-kt}$, where T_0 is the original excess of temperature and T is the excess of temperature after time t minutes.

a Prove that $\dfrac{dT}{dt}$ is proportional to T.

b If the original temperature of the object is 100°C, the temperature of its surroundings is 30°C and the object cools to 70°C in 20 minutes, find the value of k correct to three decimal places.

c At what rate is the temperature decreasing after 30 minutes?

7 Suppose that the spread of a cold virus through a population is such that the proportion, $p(t)$, of the population which has had the virus up to time t days after its introduction into the population is given by

$$p(t) = 0.2 - 0.2e^{\frac{-t}{20}} + 0.1e^{\frac{-t}{10}}, \quad \text{for } t \geq 0$$

a **i** Find, correct to four decimal places, the proportion of the population which has had the virus up to 10 days after its introduction.

ii Find the proportion of the population that eventually catches the virus.

b The number of new cases on day t is proportional to $p'(t)$. Find how long after the introduction of the virus the number of new cases per day is at a maximum.

8 A real-estate firm owns the Shantytown Apartments, consisting of 70 garden-type apartments. The firm can find a tenant for all the apartments at \$500 each per month. However, for every \$20 per month increase, there will be two vacancies with no possibility of filling them. What price per apartment will maximise monthly revenue? (Use a continuous function to model this discrete situation.)

9 The amount of liquid, V m^3, in a large pool at time t days is given by $V = \dfrac{5 \times 10^4}{(t+1)^2}$ for $t \geq 0$.

a Find the initial volume of the pool.

b Find the rate of change of volume with respect to time when $t = 1$.

c Find the average rate of change for the interval $t = 1$ to $t = 4$.

d When is the amount of water in the pool less than 1 cubic metre?

e Sketch the graph of V against t for $t \geq 0$.

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10 Each week a factory produced N thousand bottle tops and the cost of production is reckoned to be \$1000C, where $C = (N^3 + 16)^{\frac{1}{4}}$.

a Sketch the graph of C against N. (Use a continuous model.)

b Calculate $\frac{dC}{dN}$. **c** What does $\frac{dC}{dN}$ represent?

11 A company produces items at a cost price of \$2 per item. Market research indicates that the likely number of items sold per month will be $\frac{800}{p^2}$, where p dollars is the selling price of each item. Find the value of p for which the company would expect to maximise its total monthly profit, and the corresponding number of items sold.

12 A curve with equation $y = (ax + b)^{-2}$ has y-axis intercept $\left(0, \frac{1}{4}\right)$ and at this point the gradient is $-\frac{3}{4}$. Find the value(s) of a and b and sketch the graph.

13 The cost of running a ship at a constant speed of V km/h is $160 + \frac{1}{100}V^3$ dollars per hour.

a Find the cost of a journey of 1000 km at a speed of 10 km/h.

b Find the cost, \$$C$, of a journey of 1000 km at a speed of V km/h.

c Sketch the graph of C against V.

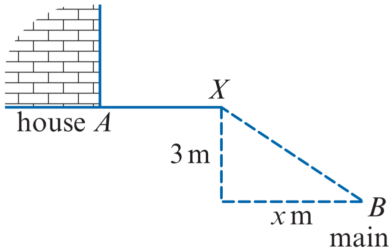
d Find the most economical speed for the journey, and the minimum cost.

e If the ship has a maximum speed of 16 km/h, find the minimum cost.

14 **a** A camper is on an island shore at point A, which is 12 km from the nearest point B on the straight shore of the mainland. He wishes to reach a town C, which is 30 km along the shore from B, in the least possible time. If he can row his boat at 5 km/h and walk at 8 km/h, how far along the shore from B towards C should he land?

b Repeat part **a** if C is only 24 km from B.

15 To connect a house to a gas supply, a pipe must be installed connecting the point A on the house to the point B on the main, where B is 3 m below ground level and at a horizontal distance of 4 m from the building. If it costs \$25 per metre to lay pipe underground and \$10 per metre on the surface, find the length of pipe which should be on the surface to minimise costs.



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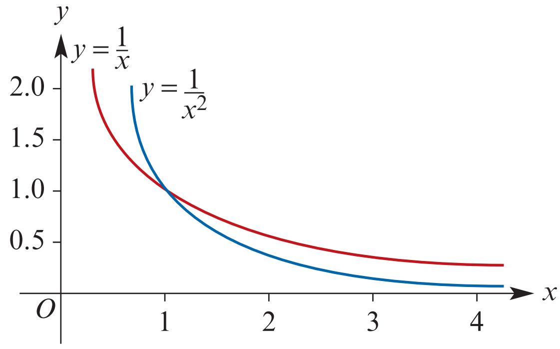
16 Define the functions

$$g(x) = \frac{1}{x}, \quad \{x \in \mathbb{R} : x > 0\}$$

and $$h(x) = \frac{1}{x^2}, \{x \in \mathbb{R} : x > 0\}$$



a Find $\{x : g(x) > h(x)\}$.

b Find $\{x : g'(x) > h'(x)\}$, i.e. find the set of x for which the gradient of g is greater than the gradient of h.

c On one set of axes, sketch the graphs of

$$f(x) = \frac{1}{x^3}, \{x \in \mathbb{R} : x > 0\} \quad \text{and} \quad h(x) = \frac{1}{x^2}, \{x \in \mathbb{R} : x > 0\}$$

Find $\{x : h(x) > f(x)\}$ and $\{x : h'(x) > f'(x)\}$.

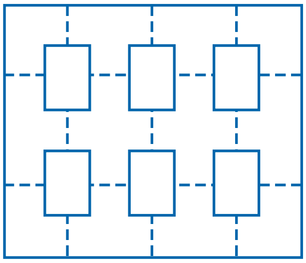
d For $f_1(x) = \frac{1}{x^n}, \{x \in \mathbb{R} : x > 0\}$ and $f_2(x) = \frac{1}{x^{n+1}}, \{x \in \mathbb{R} : x > 0\}$ find $\{x : f_1(x) > f_2(x)\}$ and $\{x : f_1'(x) > f_2'(x)\}$.

17 **a** Find the points $P\left(x, \frac{1}{x}\right)$ on the curve $y = \frac{1}{x}$ for which the distance OP is a minimum, where O is the origin $(0, 0)$.

b Find the points $P\left(x, \frac{1}{x^2}\right)$ on the curve $y = \frac{1}{x^2}$ for which the distance OP is a minimum.

c Find the points $P\left(x, \frac{1}{x^n}\right)$ on the curve $y = \frac{1}{x^n}$ for which the distance OP is a minimum, where n is a positive integer.

18 The figure represents an intended basic design for a workshop wall which is to have six equal windows spaced so that each dashed line has length 2 m.
The total area of window space is to be 36 m^2.



a Express the total area, A m^2, of brickwork as a function of the window height, x m.

b Sketch the graph of A against x.

c Find the dimensions of each window which will give a minimum amount of brickwork.

d If building regulations require that both the height and the width of a window must not be less than 1 m, find the maximum amount of brickwork that could be used.

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19 **a** Sketch the graph of the equation $y = x^2 - a^2$. Label the points A, B at which it cuts the x-axis. Write down the coordinates of A and B.

b Find the area of the region between the x-axis and the graph.

c Draw a rectangle $ABCD$ on your sketch, lying *below* the x-axis, with area equal to the area found in part **b**. What is the length of the side BC?

d If the vertex of the parabola is at point V, calculate the ratio $\dfrac{\text{length of } BC}{\text{length of } OV}$.

20 **a** Calculate $\int_{-3}^{1} (1 - t^2)\, dt$ and illustrate the region of the Cartesian plane for which this integral gives the signed area.

b Show that $\int_{a}^{1} (1 - t^2)\, dt = 0$ implies $a^3 - 3a + 2 = 0$.

c Find the values of a for which $\int_{a}^{1} (1 - t^2)\, dt = 0$.

21 The rate of flow of water into a tank is given by $\dfrac{dV}{dt} = 10e^{-(t+1)}(5 - t)$ for $0 \le t \le 5$, where V litres is the amount of water in the tank at time t minutes. Initially the tank is empty.

a **i** Find the initial rate of flow of water into the tank.

ii Find the value of t for which $\dfrac{dV}{dt} = 0$.

iii Find the time, to the nearest second, when the rate is 1 litre per minute.

iv Find the first time, to the nearest second, when $\dfrac{dV}{dt} < 0.1$.

b Find the amount of water in the tank when $t = 5$.

c Find the time, to the nearest second, when there are 10 litres of water in the tank.

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8 Discrete random variables and their probability distributions

In this chapter

Syllabus references

Topic: General discrete random variables

Subtopics: 3.3.1 – 3.3.8

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We are often required to make decisions based on the chance of a particular occurrence. Some events can be predicted from our present store of knowledge, such as the time of the next high tide. Others, such as whether a head or tail will show when a coin is tossed, are not predictable.

Ideas of uncertainty are pervasive in everyday life, and the use of chance and risk models makes an important impact on many human activities and concerns. Probability is the study of chance and uncertainty.

In this chapter we will extend our knowledge of probability by introducing the concept of the probability distribution for a discrete random variable. Using this distribution we can determine the theoretical values of two important parameters which describe the random variable: the mean and the standard deviation. We will see that together the mean and the standard deviation tell us a lot about the distribution of the variable under consideration.

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8A Sample spaces and probability

In this section we will review the fundamental concepts of probability, the numerical value which we assign to give a measure of the likelihood of an outcome of an experiment. Probability takes a value between 0 and 1, where a probability of 0 means that the outcome is impossible, and a probability of 1 means that it is certain. Generally, the probability of an outcome will be somewhere in between, with a higher value meaning that the outcome is more likely.

Sample spaces and events

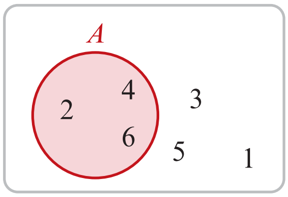
When a six-sided die is rolled, the possible outcomes are the numbers 1, 2, 3, 4, 5, 6. Rolling a six-sided die is an example of a **random experiment**, since while we can list all the possible outcomes, we do not know which one will be observed.

The possible outcomes are generally listed as the elements of a set, and the set of all possible outcomes is called the **sample space** and denoted by the Greek letter ε (epsilon). Thus, for this example:

$$\varepsilon = \{1, 2, 3, 4, 5, 6\}$$

An **event** is a subset of the sample space, usually denoted by a capital letter. If the event A is defined as 'an even number when a six-sided die is rolled', we write

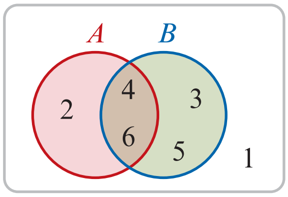
$$A = \{2, 4, 6\}$$



If A and B are two events, then the **union** of A and B, denoted by $A \cup B$, is equivalent to either event A or event B or both occurring.

Thus, if event A is 'an even number when a six-sided die is rolled' and event B is 'a number greater than 2 when a six-sided die is rolled', then $A = \{2, 4, 6\}$, $B = \{3, 4, 5, 6\}$ and

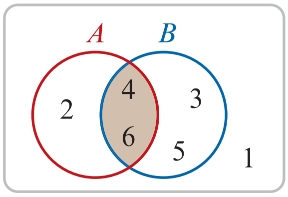
$$A \cup B = \{2, 3, 4, 5, 6\}$$



The **intersection** of A and B, denoted by $A \cap B$, is equivalent to both event A and event B occurring.

Thus, using the events A and B already described:

$$A \cap B = \{4, 6\}$$



In some experiments, it is helpful to list the elements of the sample space systematically by means of a tree diagram.

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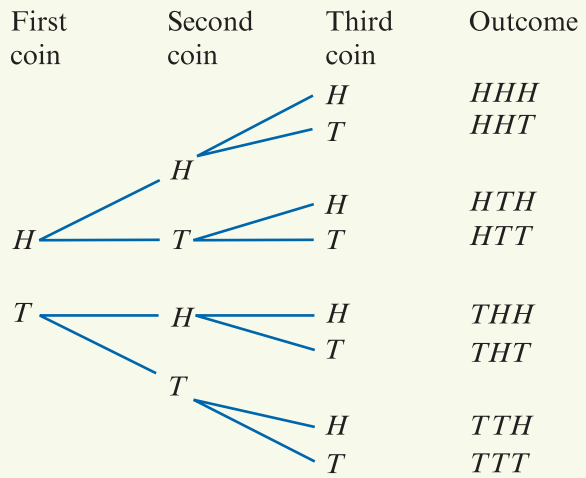
Example 1

Find the sample space when three coins are tossed and the results noted.

Solution

To list the elements of the sample space, construct a tree diagram:



Each path along the branches of the tree identifies an outcome, giving the sample space as

$$\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Determining probabilities for equally likely outcomes

Probability is a numerical measure of the chance of a particular event occurring. There are many approaches to determining probability, but often we assume that all of the possible outcomes are equally likely.

We require that the probabilities of all the outcomes in the sample space sum to 1, and that the probability of each outcome is a non-negative number. This means that the probability of each outcome must lie in the interval [0, 1]. Since six outcomes are possible when rolling a die, we can assign the probability of each outcome to be $\frac{1}{6}$. That is,

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

When the sample space is finite, the **probability of an event** is equal to the sum of the probabilities of the outcomes in that event.

For example, let A be the event that an even number is rolled on the die. Then $A = \{2, 4, 6\}$ and $P(A) = P(2) + P(4) + P(6) = \frac{1}{2}$. Since the outcomes are equally likely, we can calculate this more easily as

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

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Equally likely outcomes

In general, if the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these outcomes.

Then the probability of any event A which contains m of these outcomes is the ratio of the number of elements in A to the number of elements in ε. That is,

$$\mathrm{P}(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

where the notation $n(S)$ is used to represent the number of elements in set S.

We will see that there are other methods of determining probabilities. But whichever method is used, the following rules of probability will hold:

- $\mathrm{P}(A) \geq 0$ for all events $A \subseteq \varepsilon$
- $\mathrm{P}(\varepsilon) = 1$
- The sum of the probabilities of all outcomes of an experiment is 1.
- $\mathrm{P}(\varnothing) = 0$, where $\varnothing$ represents the empty set
- $\mathrm{P}(A') = 1 - \mathrm{P}(A)$, where A' is the complement of A
- $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, the **addition rule**

When two events A and B have no outcomes in common, i.e. when they cannot occur together, they are called **mutually exclusive** events. In this case, we have $\mathrm{P}(A \cap B) = 0$ and so the addition rule becomes:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B), \quad \text{the addition rule when } A \text{ and } B \text{ are mutually exclusive}$$

We illustrate some of these rules in the following example.

Example 2

If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a an ace **b** not a heart **c** an ace or a heart **d** either a king or an ace?

Solution

a Let A be the event 'the card drawn is an ace'. A standard deck of cards contains four aces, so

$$\mathrm{P}(A) = \frac{4}{52} = \frac{1}{13}$$

b Let H be the event 'the card drawn is a heart'. There are 13 cards in each suit, so

$$\mathrm{P}(H) = \frac{13}{52} = \frac{1}{4}$$

and therefore

$$\mathrm{P}(H') = 1 - \mathrm{P}(H) = 1 - \frac{1}{4} = \frac{3}{4}$$

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c Using the addition rule:

$$P(A \cup H) = P(A) + P(H) - P(A \cap H)$$

Now $P(A \cap H) = \frac{1}{52}$, since the event $A \cap H$ corresponds to drawing the ace of hearts. Therefore

$$P(A \cup H) = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

d Let K be the event 'the card drawn is a king'. We observe that $K \cap A = \varnothing$. That is, the events K and A are mutually exclusive. Hence

$$P(K \cup A) = P(K) + P(A) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

Example 3

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected from these 500. Find the probability that:

a the person regularly uses social media

b the person is less than 25 years of age

c the person is less than 25 years of age and does not regularly use social media.

Solution

a $P(\text{Yes}) = \frac{300}{500} = \frac{3}{5}$

b $P(\text{Age} < 25) = \frac{240}{500} = \frac{12}{25}$

c $P(\text{No} \cap \text{Age} < 25) = \frac{40}{500} = \frac{2}{25}$

Explanation

There are 300 out of 500 people who say yes.

There are 240 out of 500 people who are less than 25 years of age.

There are 40 out of 500 people who are less than 25 years of age and say no.

Other methods of determining probabilities

When we are dealing with a random experiment which does not have equally likely outcomes, other methods of determining probability are required.

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Subjective probabilities

Sometimes, the probability is assigned a value on the basis of judgement. For example, a farmer may look at the weather conditions and determine that there is a 70% chance of rain that day, and take appropriate actions. Such probabilities are called **subjective probabilities**.

Probabilities from data

A better way to estimate an unknown probability is by experimentation: by performing the random experiment many times and recording the results. This information can then be used to estimate the chances of the event happening again in the future. The proportion of trials that resulted in this event is called the **relative frequency** of the event. (For most purposes we can consider proportion and relative frequency as interchangeable.) That is,

$$\text{Relative frequency of event } A = \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}}$$

This information can then be used to estimate the probability of the event.

When the number of trials is sufficiently large, the observed relative frequency of an event A becomes close to the probability $\mathrm{P}(A)$. That is,

$$\mathrm{P}(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

If the experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. In many situations, however, experiments are the only way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact, and will in fact vary from sample to sample.

Understanding the variation between estimates is extremely important in the study of statistics, and this is the topic of Chapter 12. At this stage it is valuable to realise that the variation does exist, and that the best estimates of the probabilities will result from using as many trials as possible.

Example 4

Suppose that a die is tossed 1000 times and the following outcomes observed:

Outcome	1	2	3	4	5	6
Frequency	135	159	280	199	133	97

a Use this information to estimate the probability of observing a 6 when this die is rolled.

b What outcome would you predict to be most likely the next time the die is rolled?

Solution

a $\mathrm{P}(6) \approx \dfrac{97}{1000} = 0.097$

b The most likely outcome is 3, since it has the highest relative frequency.

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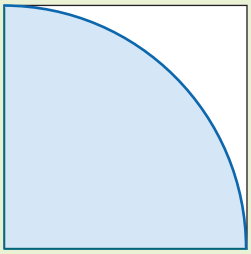
Probabilities from area

When we use the model of equally likely outcomes to determine probabilities, we count both the outcomes in the event and the outcomes in the sample space, and use the ratio to determine the probability of the event.

This idea can be extended to calculate probabilities when areas are involved, by assuming that the probabilities of all points in the region (which can be considered to be the sample space) are equally likely.

Example 5

A dartboard consists of a square of side length 2 metres containing a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.



If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, find the probability of it hitting the blue region.

Solution

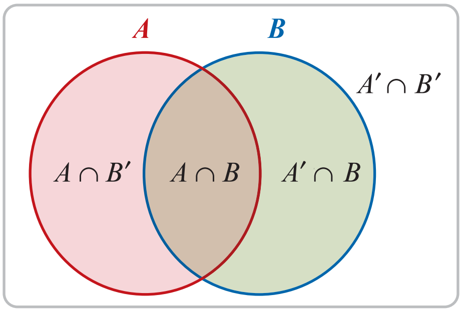
Area of blue region $= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi \times 4 = \pi \text{ m}^2$

Area of dartboard $= 2 \times 2 = 4 \text{ m}^2$

$$\text{P(hitting blue region)} = \frac{\text{area of blue region}}{\text{area of dartboard}} = \frac{\pi}{4}$$

Probability tables

A **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically. Consider the Venn diagram which illustrates two intersecting sets A and B.



From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions: $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$. These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**.

	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

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In a probability table, the entries give the probabilities of each of these events occurring.

	B	B'
A	$P(A \cap B)$	$P(A \cap B')$
A'	$P(A' \cap B)$	$P(A' \cap B')$

Summing the rows and columns, we can complete the table as shown.

	B	B'	
A	$P(A \cap B)$	$P(A \cap B')$	$P(A)$
A'	$P(A' \cap B)$	$P(A' \cap B')$	$P(A')$
	$P(B)$	$P(B')$	1

These tables can be useful when solving problems involving probability, as shown in the next example.

Example 6

Simone visits the dentist every 6 months for a checkup. The probability that she will need her teeth cleaned is 0.35, the probability that she will need a filling is 0.1 and the probability that she will need both is 0.05.

a What is the probability that she will not need her teeth cleaned on a visit, but will need a filling?

b What is the probability that she will not need either of these treatments?

Solution

The information in the question may be entered into a table as shown, where we use C to represent 'cleaning' and F to represent 'filling'.

	F	F'	
C	0.05		0.35
C'			
	0.1		1

All the empty cells in the table may now be filled in by subtraction:

	F	F'	
C	0.05	0.3	0.35
C'	0.05	0.6	0.65
	0.1	0.9	1

a The probability that she will not need her teeth cleaned but will need a filling is given by $P(C' \cap F) = 0.05$.

b The probability that she will not need either of these treatments is $P(C' \cap F') = 0.6$.

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Summary 8A

- The **sample space**, ε, for a random experiment is the set of all possible outcomes.
- An **event** is a subset of the sample space. The probability of an event A occurring is denoted by $\mathrm{P}(A)$.
- Equally likely outcomes If the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each outcome. Then the probability of an event A is given by

$$\mathrm{P}(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{n(A)}{n(\varepsilon)}$$

- Estimates of probability When a probability is unknown, it can be estimated by the relative frequency obtained through repeated trials of the random experiment under consideration. In this case,

$$\mathrm{P}(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

- Whichever method of determining probability is used, the rules of probability hold:
  - $\mathrm{P}(A) \geq 0$ for all events $A \subseteq \varepsilon$
  - $\mathrm{P}(\emptyset) = 0$ and $\mathrm{P}(\varepsilon) = 1$
  - The sum of the probabilities of all outcomes of an experiment is 1.
  - $\mathrm{P}(A') = 1 - \mathrm{P}(A)$, where A' is the complement of A
  - $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, the **addition rule**
- If two events A and B are **mutually exclusive** (i.e. if A and B have no outcomes in common), then $\mathrm{P}(A \cap B) = 0$ and therefore $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

Exercise 8A

Example 1

1 An experiment consists of rolling a die and tossing a coin. Use a tree diagram to list the sample space for the experiment.

2 Two coins are tossed and a die is rolled. Use a tree diagram to show all the possible outcomes.

Example 2

3 If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a a queen
b not a club
c a queen or a heart
d either a king or a queen?

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4 A blank six-sided die is marked with a 1 on two sides, a 2 on one side, and a 3 on the remaining three sides. Find the probability that when the die is rolled:

a a 3 shows **b** a 2 or a 3 shows.

5 Suppose that the probability that a student owns a smartphone is 0.7, the probability that they own a laptop is 0.6, and the probability that they own both is 0.5. What is the probability that a student owns either a smartphone or a laptop or both?

6 At a particular university, the probability that an Arts student studies a language is 0.3, literature is 0.6, and both is 0.25. What is the probability that an Arts student studies either a language or literature or both?

7 A computer manufacturer notes that 5% of their computers are returned owing to faulty disk drives, 2% are returned owing to faulty keyboards, and 0.3% are returned because both disk drives and keyboards are faulty. Find the probability that the next computer manufactured will be returned with:

a a faulty disk drive or a faulty keyboard

b a faulty disk drive and a working keyboard.

8 A new drug has been released and produces some minor side effects: 8% of users suffer only loss of sleep, 12% of users suffer only nausea, and 75% of users have no side effects at all. What percentage of users suffer from both loss of sleep and nausea?

9 In a particular town, the probability that an adult owns a car is 0.7, while the probability that an adult owns a car and is employed is 0.6. If a randomly selected adult is found to own a car, what is the probability that he or she is also employed?

Example 3 **10** An insurance company analysed the records of 500 drivers to determine the relationship between age and accidents in the last year.

	Accidents in the last year				
Age	0	1	2	3	Over 3
Under 20	19	35	25	17	10
20–29	30	45	33	39	17
30–39	40	33	15	6	2
40–49	18	15	10	3	1
Over 49	21	25	17	13	11

What is the probability that a driver chosen from this group at random:

a is under 20 years old and has had three accidents in the last year

b is from 40 to 49 years old and has had no accidents in the last year

c is from 20 to 29 years old

d has had more than three accidents in the last year?

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11 200 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	70	60	130
No	50	20	70
Total	120	80	200

One person is selected at random from these 200.

a What is the probability that the person thinks private individuals should be allowed to carry guns?

b What is the probability that the person is male and thinks private individuals should be allowed to carry guns?

Example 4

12 Use the given data to estimate the probability of the specified event occurring:

a P(head) if a coin is tossed 200 times and 114 heads observed

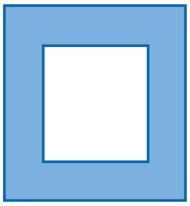
b P(ten) if a spinner is spun 380 times and lands on the 'ten' 40 times

c P(two heads) if two coins are tossed 200 times and two heads are observed on 54 occasions

d P(three sixes) if three dice are rolled 500 times and three sixes observed only twice

Example 5

13 Suppose that a square dartboard consists of a white square of side length 30 cm inside a larger blue square of side length 50 cm, as shown. If a dart thrown at the board has equal chance of landing anywhere on the board, what is the probability it lands in the white area? (Ignore the possibility that it might land on the line or miss the board altogether.)

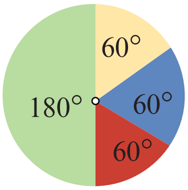


14 A spinner is as shown in the diagram. Find the probability that when spun the pointer will land on:

a the green section

b the yellow section

c any section except the yellow section.



Example 6

15 In a particular country it has been established that the probability that a person drinks tea is 0.45, the probability that a person drinks coffee is 0.65, and the probability that a person drinks neither tea nor coffee is 0.22. Use the information to complete a probability table and hence determine the probability that a randomly selected person in that country:

a drinks tea but not coffee

b drinks tea and coffee.

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16 A chocolate is chosen at random from a box of chocolates. It is known that in this box:

- the probability that the chocolate is dark but not soft-centred is 0.15
- the probability that the chocolate is not dark but is soft-centred is 0.42
- the probability that the chocolate is not dark is 0.60.

Find the probability that the randomly chosen chocolate is:

a dark **b** soft-centred **c** not dark and not soft-centred.

17 Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students are Australian by birth. They also show that 53% of students are Australian by birth and participate in sport. Use this information to find the probability that a student selected at random:

a does not participate in sport

b is Australian by birth and does not participate in sport

c is not Australian by birth and participates in sport

d is not Australian by birth and does not participate in sport.

8B Conditional probability and independence

The probability of an event A occurring when it is known that some event B has occurred is called conditional probability and is written as $\mathrm{P}(A \mid B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Sometimes, the probability of an event is not affected by knowing that another event has occurred. For example, if two coins are tossed, then the probability of the second coin showing a head is independent of whether the first coin shows a head or a tail. Thus,

$$\begin{aligned} &\mathrm{P}(\text{head on second coin} \mid \text{head on first coin}) \\ &= \mathrm{P}(\text{head on second coin} \mid \text{tail on first coin}) \\ &= \mathrm{P}(\text{head on second coin}) \end{aligned}$$

For other situations, however, a previous result may alter the probability. For example, the probability of rain today given that it rained yesterday will generally be different from the probability that it will rain today given that it didn't rain yesterday.

The **conditional probability** of an event A, given that event B has already occurred, is given by

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \qquad \text{if } \mathrm{P}(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times \mathrm{P}(B)$$

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The probabilities associated with a multi-stage experiment can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).

Example 7

In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, find the probability that it rains:

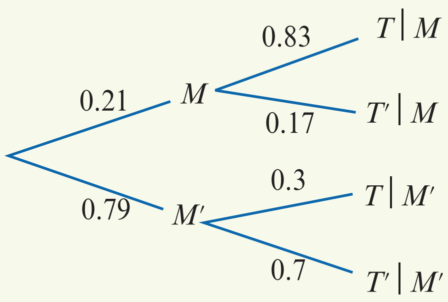
a on both Monday and Tuesday

b on Tuesday.

Solution

Let M represent the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



$P(T \cap M) = 0.21 \times 0.83 = 0.1743$

$P(T' \cap M) = 0.21 \times 0.17 = 0.0357$

$P(T \cap M') = 0.79 \times 0.3 = 0.237$

$P(T' \cap M') = 0.79 \times 0.7 = 0.553$

a The probability that it rains on both Monday and Tuesday is given by

$$\begin{aligned} P(T \cap M) &= 0.21 \times 0.83 \\ &= 0.1743 \end{aligned}$$

b The probability that it rains on Tuesday is given by

$$\begin{aligned} P(T) &= P(T \cap M) + P(T \cap M') \\ &= 0.1743 + 0.237 \\ &= 0.4113 \end{aligned}$$

The solution to part **b** of Example 7 is an application of a rule known as the law of total probability. This can be expressed in general terms as follows:

The **law of total probability** states that, in the case of two events A and B,

$$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$

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Example 8

Adrienne, Regan and Michael are doing the dishes. Since Adrienne is the oldest, she washes the dishes 40% of the time. Regan and Michael each wash 30% of the time. When Adrienne washes the probability of at least one dish being broken is 0.01, when Regan washes the probability is 0.02, and when Michael washes the probability is 0.03. Their parents don't know who is washing the dishes one particular night.

a What is the probability that at least one dish will be broken?

b Given that at least one dish is broken, what is the probability that the person washing was Michael?

Solution

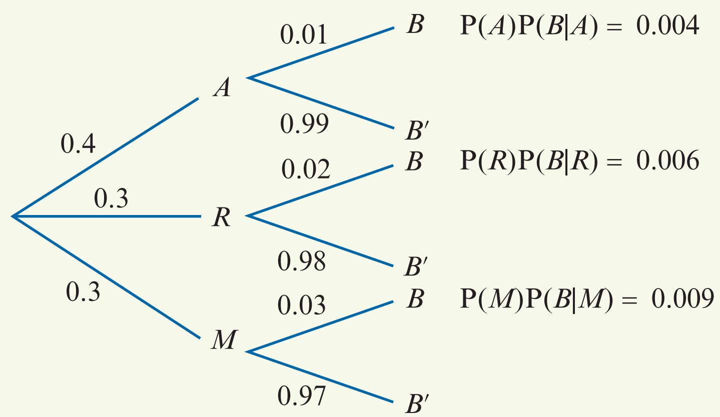
Let A be the event 'Adrienne washes the dishes', let R be the event 'Regan washes the dishes' and let M be the event 'Michael washes the dishes'. Then

$$\mathrm{P}(A) = 0.4, \qquad \mathrm{P}(R) = 0.3, \qquad \mathrm{P}(M) = 0.3$$

Let B be the event 'at least one dish is broken'. Then

$$\mathrm{P}(B \mid A) = 0.01, \qquad \mathrm{P}(B \mid R) = 0.02, \qquad \mathrm{P}(B \mid M) = 0.03$$

This information can be summarised in a tree diagram as shown:



a The probability of at least one dish being broken is

$$\begin{aligned}\mathrm{P}(B) &= \mathrm{P}(B \cap A) + \mathrm{P}(B \cap R) + \mathrm{P}(B \cap M)\\ &= \mathrm{P}(A)\mathrm{P}(B \mid A) + \mathrm{P}(R)\mathrm{P}(B \mid R) + \mathrm{P}(M)\mathrm{P}(B \mid M)\\ &= 0.004 + 0.006 + 0.009\\ &= 0.019\end{aligned}$$

b The required probability is

$$\begin{aligned}\mathrm{P}(M \mid B) &= \frac{\mathrm{P}(M \cap B)}{\mathrm{P}(B)}\\ &= \frac{0.009}{0.019} = \frac{9}{19}\end{aligned}$$

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Example 9

As part of an evaluation of the school canteen, all students at a Senior Secondary College (Years 10–12) were asked to rate the canteen as poor, good or excellent. The results are shown in the table.

	Year			
Rating	10	11	12	Total
Poor	30	20	10	60
Good	80	65	35	180
Excellent	60	65	35	160
Total	170	150	80	400

What is the probability that a student chosen at random from this college:

a is in Year 12

b is in Year 12 and rates the canteen as excellent

c is in Year 12, given that they rate the canteen as excellent

d rates the canteen as excellent, given that they are in Year 12?

Solution

Let T be the event 'the student is in Year 12' and let E be the event 'the rating is excellent'.

a $P(T) = \frac{80}{400} = \frac{1}{5}$

b $P(T \cap E) = \frac{35}{400} = \frac{7}{80}$

c $P(T \mid E) = \frac{35}{160} = \frac{7}{32}$

d $P(E \mid T) = \frac{35}{80} = \frac{7}{16}$

Explanation

a From the table, we can see that there are 80 students in Year 12 and 400 students altogether.

b From the table, there are 35 students who are in Year 12 and also rate the canteen as excellent.

c From the table, a total of 160 students rate the canteen as excellent, and of these 35 are in Year 12.

d From the table, there are 80 students in Year 12, and of these 35 rate the canteen as excellent.

Note: The answers to parts **c** and **d** could also have been found using the rule for conditional probability, but here it is easier to determine the probability directly from the table.

Independent events

Two events A and B are **independent** if the probability of A occurring is the same, whether or not B has occurred.

Independent events

For events A and B with $P(A) \neq 0$ and $P(B) \neq 0$, the following three conditions are all equivalent conditions for the independence of A and B:

- $P(A \mid B) = P(A)$
- $P(B \mid A) = P(B)$
- $P(A \cap B) = P(A) \times P(B)$

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Example 10

The probability that Monica remembers to do her homework is 0.7, while the probability that Patrick remembers to do his homework is 0.4. If these events are independent, then what is the probability that:

a both will do their homework

b Monica will do her homework but Patrick forgets?

Solution

Let M be the event 'Monica does her homework' and let P be the event 'Patrick does his homework'. Since these events are independent:

a
$$\begin{aligned} P(M \cap P) &= P(M) \times P(P) \\ &= 0.7 \times 0.4 \\ &= 0.28 \end{aligned}$$

b
$$\begin{aligned} P(M \cap P') &= P(M) \times P(P') \\ &= 0.7 \times 0.6 \\ &= 0.42 \end{aligned}$$

Summary 8B

- **Conditional probability**
  - The probability of an event A occurring when it is known that some event B has already occurred is called conditional probability and is written $P(A \mid B)$.
  - In general, the **conditional probability** of an event A, given that event B has already occurred, is given by

    $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0$$

    This formula may be rearranged to give the **multiplication rule of probability**:

    $$P(A \cap B) = P(A \mid B) \times P(B)$$

- **Law of total probability**

  The **law of total probability** states that, in the case of two events A and B,

  $$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$

- **Independence**

  Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

  $$P(A \mid B) = P(A)$$

  Events A and B are independent if and only if

  $$P(A \cap B) = P(A) \times P(B)$$

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Skill-sheet

Exercise 8B

Example 7

1 In a certain town, the probability that it rains on any Saturday is 0.25. If it rains on Saturday, then the probability of rain on Sunday is 0.8. If it does not rain on Saturday, then the probability of rain on Sunday is 0.1. For a given week, find the probability that:

a it rains on both Saturday and Sunday

b it rains on neither day

c it rains on Sunday.

2 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.1$, find:

a $\Pr(B \mid A)$

b $\Pr(A \mid B)$

3 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(B \mid A) = 0.1$, find:

a $\Pr(A \cap B)$

b $\Pr(A \mid B)$

4 In Alia's school, the probability that a student studies French is 0.5, and the probability that they study both French and Chinese is 0.3. Find the probability that a student studies Chinese, given that they study French.

Example 8

5 The chance that a harvest is poorer than average is 0.5, but if it is known that a certain disease D is present, this probability increases to 0.8. The disease D is present in 30% of harvests. Find the probability that, when a harvest is observed to be poorer than average, the disease D is present.

Example 9

6 A group of 1000 eligible voters were asked their age and their preference in an upcoming election, with the following results.

	Age			
Preference	18–25	26–40	Over 40	Total
Candidate A	200	100	85	385
Candidate B	250	230	50	530
No preference	50	20	15	85
Total	500	350	150	1000

What is the probability that a person chosen from this group at random:

a is 18–25 years of age

b prefers candidate A

c is 18–25 years of age, given that they prefer candidate A

d prefers candidate A, given that they are 18–25 years of age?

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7 The following data was derived from accident records on a highway noted for its above-average accident rate.

	Probable cause			
Type of accident	Speed	Alcohol	Other	Total
Fatal	42	61	12	115
Non-fatal	88	185	60	333
Total	130	246	72	448

Use the table to find:

a the probability that speed is the cause of the accident
b the probability that the accident is fatal
c the probability that the accident is fatal, given that speed is the cause
d the probability that the accident is fatal, given that alcohol is the cause.

Example 10

8 The probability of James winning a particular tennis match is independent of Sally winning another particular tennis match. If the probability of James winning is 0.8 and the probability of Sally winning is 0.3, find:

a the probability that they both win
b the probability that either or both of them win.

9 An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent?
b Are A and C independent?
c Are B and C independent?

10 If A and B are independent events such that $\mathrm{P}(A) = 0.5$ and $\mathrm{P}(B) = 0.4$, find:

a $\mathrm{P}(A \mid B)$ **b** $\mathrm{P}(A \cap B)$ **c** $\mathrm{P}(A \cup B)$

11 Nathan knows that his probability of kicking more than four goals on a wet day is 0.3, while on a dry day it is 0.6. The probability that it will be wet on the day of the next game is 0.7. Calculate the probability that Nathan will kick more than four goals in the next game.

12 Find the probability that, in three tosses of a fair coin, there are three heads, given that there is at least one head.

13 The test used to determine if a person suffers from a particular disease is not perfect. The probability of a person with the disease returning a positive result is 0.95, while the probability of a person without the disease returning a positive result is 0.02. The probability that a randomly selected person has the disease is 0.03. What is the probability that a randomly selected person will return a positive result?

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14 Anya goes through three sets of traffic lights when she cycles to school each morning. The probability she stops at the first set is 0.6. If she stops at any one set, the probability that she has to stop at the next is 0.9. If she doesn't have to stop at any one set, the probability that she doesn't have to stop at the next is 0.7. Use a tree diagram to find the probability that:

a she stops at all three sets of lights
b she stops only at the second set of lights
c she stops at exactly one set of lights.

15 There are four red socks and two blue socks in a drawer. Two socks are removed at random. What is the probability of obtaining:

a two red socks
b two blue socks
c one of each colour?

16 A car salesperson was interested in the relationship between the size of the car a customer purchased and their marital status. From the sales records, the table on the right was constructed.

	Marital status		
Size of car	Married	Single	Total
Large	60	20	80
Medium	100	60	160
Small	90	70	160
Total	250	150	400

What is the probability that a person chosen at random from this group:

a drives a small car
b is single and drives a small car
c is single, given that they drive a small car
d drives a small car, given that they are single?

17 Jenny has two boxes of chocolates. Box A contains three white chocolates and four dark chocolates. Box B contains two white chocolates and five dark chocolates. Jenny first chooses a box at random and then selects a chocolate at random from it. Find the probability that:

a Jenny selects a white chocolate
b given that Jenny selects a white chocolate, it was chosen from box A.

18 At a particular petrol station, 30% of customers buy premium unleaded, 60% buy standard unleaded and 10% buy diesel. When a customer buys premium unleaded, there is a 25% chance they will fill the tank. Of the customers buying standard unleaded, 20% fill their tank. Of those buying diesel, 70% fill their tank.

a What is the probability that, when a car leaves the petrol station, it will not have a full tank?
b Given that a car leaving the petrol station has a full tank, what is the probability that the tank contains standard unleaded petrol?

19 A bag contains three red, four white and five black balls. If three balls are taken without replacement, what is the probability that they are all the same colour?

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8C Discrete random variables

Suppose that three balls are drawn at random from a jar containing four white and six black balls, with replacement (i.e. each selected ball is replaced before the next draw). The sample space for this random experiment is as follows:

$$\varepsilon = \{WWW, WWB, WBW, BWW, WBB, BWB, BBW, BBB\}$$

Suppose the variable of interest is the number of white balls in the sample. This corresponds to a simpler sample space whose outcomes are numbers.

If X represents the number of white balls in the sample, then the possible values of X are 0, 1, 2 and 3. Since the actual value that X will take is the result of a random experiment, we say that X is a random variable.

A **random variable** is a function that assigns a number to each outcome in the sample space ε.

A random variable can be discrete or continuous:

- A **discrete random variable** is one that can take only a countable number of values. For example, the number of white balls in a sample of size three is a discrete random variable which may take one of the values 0, 1, 2, 3. Other examples include the number of children in a family, and a person's shoe size. (Note that discrete random variables do not have to take only whole-number values.)
- A **continuous random variable** is one that can take any value in an interval of the real number line, and is usually (but not always) generated by measuring. Height, weight, and the time taken to complete a puzzle are all examples of continuous random variables.

In this chapter we are interested in understanding more about discrete random variables.

Consider again the sample space for the random experiment described above. Each outcome in the sample space is associated with a value of X:

Experiment outcome	Value of X
WWW	$X = 3$
WWB	$X = 2$
WBW	$X = 2$
BWW	$X = 2$
WBB	$X = 1$
BWB	$X = 1$
BBW	$X = 1$
BBB	$X = 0$

Associated with each event is a probability. Since the individual draws of the ball from the jar are independent events, we can determine the probabilities by multiplying and adding appropriate terms.

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Example 11

A jar contains four white and six black balls. What is the probability that, if three balls are drawn at random from the jar, with replacement, a white ball will be drawn exactly once (i.e. the situations where $X = 1$ in the table)?

Solution

$X = 1$ corresponds to the outcomes WBB, BWB and BBW.

Since there are 10 balls in total, $\Pr(W) = \dfrac{4}{10} = 0.4$ and $\Pr(B) = \dfrac{6}{10} = 0.6$.

$$\begin{aligned} \text{Thus } \Pr(X = 1) &= \Pr(WBB) + \Pr(BWB) + \Pr(BBW) \\ &= (0.4 \times 0.6 \times 0.6) + (0.6 \times 0.4 \times 0.6) + (0.6 \times 0.6 \times 0.4) \\ &= 0.432 \end{aligned}$$

Discrete probability distributions

The **probability distribution** for a discrete random variable consists of all the values that the random variable can take, together with the probability of each of these values. For example, if a fair die is rolled, then the probability distribution is:

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probability distribution of a discrete random variable X is described by a function

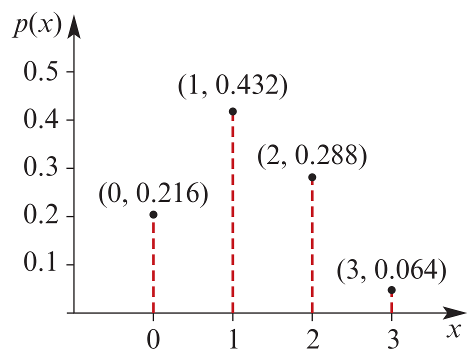
$$p(x) = \Pr(X = x)$$

This function is called a **discrete probability function**.

Consider again the black and white balls from Example 11. The probability distribution for X, the number of white balls in the sample, is given by the following table:

x	0	1	2	3
$p(x)$	0.216	0.432	0.288	0.064

The probability distribution may also be given graphically, as shown on the right.



Note that the probabilities in the table sum to 1, which must occur if all values of the random variable have been listed.

We will use the following notation:

- the sum of all the values of $p(x)$ is written as $\displaystyle\sum_x p(x)$
- the sum of the values of $p(x)$ for x between a and b inclusive is written as $\displaystyle\sum_{a \le x \le b} p(x)$

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For *any* discrete probability function $p(x)$, the following two conditions must hold:

1 Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

$$0 \le p(x) \le 1 \quad \text{for all } x$$

2 The sum of all the values of $p(x)$ must be 1. That is,

$$\sum_x p(x) = 1$$

To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$P(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

Example 12

Consider the table shown.

x	0	1	2	3
$p(x)$	0.2	0.3	0.1	0.4

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $P(X \le 2)$.

Solution

a Yes, each value of $p(x)$ is between 0 and 1, and the values add to 1.

b
$$\begin{aligned} P(X \le 2) &= p(0) + p(1) + p(2) \\ &= 0.2 + 0.3 + 0.1 \\ &= 0.6 \end{aligned}$$

Example 13

Let X be the number of heads showing when a fair coin is tossed three times.

a Find the probability distribution of X and show that all the probabilities sum to 1.

b Find the probability that one or more heads show.

c Find the probability that more than one head shows.

Solution

a The sample space is $\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Now
$$\begin{aligned} p(0) &= P(X = 0) = P(\{TTT\}) &&= \frac{1}{8} \\ p(1) &= P(X = 1) = P(\{HTT, THT, TTH\}) &&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} \\ p(2) &= P(X = 2) = P(\{HHT, HTH, THH\}) &&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} \\ p(3) &= P(X = 3) = P(\{HHH\}) &&= \frac{1}{8} \end{aligned}$$

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Thus the probability distribution of X is:

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b The probability that one or more heads shows is

$$P(X \geq 1) = p(1) + p(2) + p(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

c The probability that more than one head shows is

$$P(X > 1) = P(X \geq 2) = p(2) + p(3) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Example 14

The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6	7
$p(x)$	0.01	0.25	0.40	0.30	0.02	0.02

Find:

a $P(X \geq 4)$ **b** $P(X \geq 4 \mid X > 2)$ **c** $P(X < 5 \mid X > 2)$

Solution

a
$$\begin{aligned} P(X \geq 4) &= P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) \\ &= 0.4 + 0.3 + 0.02 + 0.02 \\ &= 0.74 \end{aligned}$$

b
$$\begin{aligned} P(X \geq 4 \mid X > 2) &= \frac{P(X \geq 4)}{P(X > 2)} \\ &= \frac{0.74}{0.99} \qquad \text{since } P(X > 2) = 1 - 0.01 = 0.99 \\ &= \frac{74}{99} \end{aligned}$$

c
$$\begin{aligned} P(X < 5 \mid X > 2) &= \frac{P(2 < X < 5)}{P(X > 2)} \\ &= \frac{P(X = 3) + P(X = 4)}{P(X > 2)} \\ &= \frac{0.65}{0.99} \\ &= \frac{65}{99} \end{aligned}$$

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Summary 8C

- For *any* discrete probability function $p(x)$, the following two conditions must hold:
  1 Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,
  $$0 \le p(x) \le 1 \quad \text{for all } x$$
  2 The sum of all the values of $p(x)$ must be 1. That is,
  $$\sum_x p(x) = 1$$
- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:
  $$\text{P}(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

Skill-sheet

Exercise 8C

1 Which of the following random variables are discrete?

a the number of people in your family
b waist measurement
c shirt size
d the number of times a die is rolled before obtaining a six

2 Which of the following random variables are discrete?

a your age
b your height to the nearest centimetre
c the time you will wait to be served at the bank
d the number of people in the queue at the bank

Example 11

3 A fair coin is tossed three times and the number of heads noted.

a List the sample space.
b List the possible values of the random variable X, the number of heads, together with the corresponding outcomes.
c Find $\text{P}(X \ge 2)$.

Example 12

4 Consider the following table:

x	0	1	2	3	4
$p(x)$	0.1	0.2	0.1	0.4	0.2

a Does this meet the conditions to be a discrete probability distribution?
b Use the table to find $\text{P}(X \le 3)$.

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Example 13

5 A jar contains four red and five blue balls. A ball is withdrawn, its colour is observed, and it is then replaced. This is repeated three times. Let X be the number of red balls among the three balls withdrawn.

a Find the probability distribution of X and show that all the probabilities sum to 1.

b Find the probability that one or more red balls are obtained.

c Find the probability that more than one red ball is obtained.

Example 14

6 Two dice are rolled and the numbers noted.

a List the sample space.

b A random variable Y is defined as the total of the numbers showing on the two dice. List the possible values of Y, together with the corresponding outcomes.

c Find:

i $\mathrm{P}(Y < 5)$ **ii** $\mathrm{P}(Y = 3 \mid Y < 5)$ **iii** $\mathrm{P}(Y \leq 3 \mid Y < 7)$

iv $\mathrm{P}(Y \geq 7 \mid Y > 4)$ **v** $\mathrm{P}(Y = 7 \mid Y > 4)$ **vi** $\mathrm{P}(Y = 7 \mid Y < 8)$

7 A die is weighted as follows:

$$\mathrm{P}(2) = \mathrm{P}(3) = \mathrm{P}(4) = \mathrm{P}(5) = 0.2, \qquad \mathrm{P}(1) = \mathrm{P}(6) = 0.1$$

The die is rolled twice, and the smaller of the numbers showing is noted. Let Y represent this value.

a List the sample space.

b List the possible values of Y.

c Find $\mathrm{P}(Y = 1)$.

8 Suppose that three balls are selected at random, with replacement, from a jar containing four white and six black balls. If X is the number of white balls in the sample, find:

a $\mathrm{P}(X = 2)$ **b** $\mathrm{P}(X = 3)$

c $\mathrm{P}(X \geq 2)$ **d** $\mathrm{P}(X = 3 \mid X \geq 2)$

9 A fair die is rolled twice and the numbers noted. Define the following events:

A = 'a four on the first roll'

B = 'a four on the second roll'

C = 'the sum of the two numbers is at least eight'

D = 'the sum of the two numbers is at least 10'

a List the sample space obtained.

b Find $\mathrm{P}(A)$, $\mathrm{P}(B)$, $\mathrm{P}(C)$ and $\mathrm{P}(D)$.

c Find $\mathrm{P}(A \mid B)$, $\mathrm{P}(A \mid C)$ and $\mathrm{P}(A \mid D)$.

d Which of the following pairs of events are independent?

i A and B **ii** A and C **iii** A and D

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10 Consider the table shown on the right.

x	0	1	2	3
$p(x)$	0.1	0.4	0.2	0.3

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\mathrm{P}(X \geq 2)$.

11 Which of the following is *not* a probability distribution?

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.5	0.7

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0.25	0.5	0.75	1.0
$p(x)$	−0.5	−0.25	0.25	0.5

d

x	10	20	30	40
$p(x)$	10%	20%	30%	40%

12 Three balls are selected from a jar containing four black and six red balls. Find the probability distribution of the number of black balls in the sample:

a if the ball chosen is replaced after each selection

b if the ball chosen is not replaced after each selection.

13 A coin is known to be biased such that the probability of obtaining a head on any toss is 0.4. Find the probability distribution of X, the number of heads observed when the coin is tossed twice.

14 A spinner is numbered from 1 to 5, and each of the five numbers is equally likely to come up. Find:

a the probability distribution of X, the number showing on the spinner

b $\mathrm{P}(X \geq 3)$, the probability that the number showing on the spinner is three or more

c $\mathrm{P}(X \leq 3 \mid X \geq 3)$

15 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of X, the sum of the numbers showing on the two dice.

c Draw a graph of the probability distribution of X.

d Find $\mathrm{P}(X \geq 9)$, the probability that the sum of the two numbers showing is nine or more.

e Find $\mathrm{P}(X \leq 10 \mid X \geq 9)$.

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16 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of Y, the remainder when the larger number showing is divided by the smaller number. (Note that, if the two numbers are the same, then $Y = 0$.)

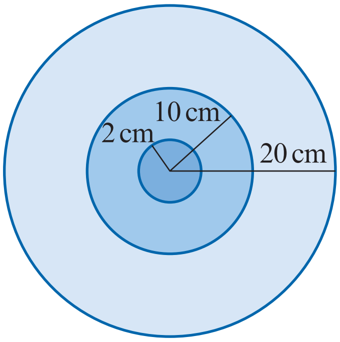
c Draw a graph of the probability distribution of Y.

17 Suppose that two socks are drawn without replacement from a drawer containing four red and six black socks. Let X represent the number of red socks obtained.

a Find the probability distribution for X.

b From the probability distribution, determine the probability that a pair of socks is obtained.

18 A dartboard consists of three circular sections, with radii of 2 cm, 10 cm and 20 cm respectively, as shown in the diagram.
When a dart lands in the centre circle the score is 100 points, in the middle circular section the score is 20 points and in the outer circular section the score is 10 points. Assume that all darts thrown hit the board, each dart is equally likely to land at any point on the dartboard, and none lands on the lines.



a Find the probability distribution for X, the number of points scored on one throw.

b Find the probability distribution for Y, the total score when two darts are thrown.

19 Erin and Nick are going to play a tennis match. Suppose that they each have an equal chance of winning any set (0.5) and that they plan to play until one player has won three sets. Let X be the number of sets played until the match is complete.

a Find $P(X = 3)$.

b List the outcomes that correspond to $X = 4$, and use this to find $P(X = 4)$.

c Hence, or otherwise, find $P(X = 5)$.

8D Expected value (mean), variance and standard deviation

From your studies of statistics, you may already be familiar with the mean as a measure of centre and with the variance and the standard deviation as measures of spread. When these are calculated from a set of data, they are termed ‘sample statistics’. It is also possible to use the probability distribution to determine the theoretically ‘true’ values of the mean, variance and standard deviation. When they are calculated from the probability distribution, they are called ‘population parameters’. Determining the values of these parameters is the topic for this section.

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Expected value (mean)

When the mean of a random variable is determined from the probability distribution, it is generally called the **expected value** of the random variable. Expected value has a wide variety of applications. The concept of expected value first arose in gambling problems, where gamblers wished to know how much they could expect to win or lose in the long run, in order to decide whether or not a particular game was a good investment.

Example 15

A person may buy a lucky ticket for \$1. They have a 20% chance of winning \$2, a 5% chance of winning \$11, and otherwise they lose. Is this a good game to play?

Solution

Let P be the amount the person will profit from each game. As it costs \$1 to play, the person can lose \$1 ($P = -1$), win \$1 ($P = 1$) or win \$10 ($P = 10$). Thus the amount that the person may win, \$$P$, has a probability distribution given by:

p	-1	1	10
$\mathrm{P}(P = p)$	0.75	0.20	0.05

Suppose you played the game 1000 times. You would expect to lose \$1 about 750 times, to win \$1 about 200 times and to win \$10 about 50 times. Thus, you would win about

$$\frac{-1 \times 750 + 1 \times 200 + 10 \times 50}{1000} = -\$0.05 \text{ per game}$$

Thus your 'expectation' is to lose 5 cents per game, and we write this as

$$\mathrm{E}(P) = -0.05$$

Note: This value gives an indication of the worth of the game: in the long run, you would expect to lose about 5 cents per game. This is called the **expected value** of P (or the **mean** of P). It is not the amount we expect to profit on any one game. (You cannot lose 5 cents in one game!) It is the amount that we expect to win on average per game in the long run.

Example 15 demonstrates how the expected value of a random variable X is determined.

The **expected value** of a discrete random variable X is determined by summing the products of each value of X and the probability that X takes that value.

That is,

$$\begin{aligned} \mathrm{E}(X) &= \sum_x x \cdot \mathrm{P}(X = x) \\ &= \sum_x x \cdot p(x) \end{aligned}$$

The expected value $\mathrm{E}(X)$ may be considered as the long-run average value of X. It is generally denoted by the Greek letter μ (mu), and is also called the **mean** of X.

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Example 16

A coin is biased in favour of heads such that the probability of obtaining a head on any single toss is 0.6. The coin is tossed three times and the results noted. If X is the number of heads obtained on the three tosses, find E(X), the expected value of X.

Solution

The following probability distribution can be found by listing the outcomes in the sample space and determining the value of X and the associated probability for each outcome.

x	0	1	2	3
$p(x)$	0.064	0.288	0.432	0.216

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot p(x)\\ &= (0 \times 0.064) + (1 \times 0.288) + (2 \times 0.432) + (3 \times 0.216)\\ &= 0.288 + 0.864 + 0.648\\ &= 1.8\end{aligned}$$

Note: This means that, if the experiment were repeated many times, then an average of 1.8 heads per three tosses would be observed.

Sometimes we wish to find the expected value of a function of X. This is determined by calculating the value of the function for each value of X, and then summing the products of these values and the associated probabilities.

The expected value of $g(X)$ is given by

$$E[g(X)] = \sum_x g(x) \cdot p(x)$$

Example 17

For the random variable X defined in Example 16, find:

a $\mathrm{E}(3X + 1)$ **b** $\mathrm{E}(X^2)$

Solution

a
$$\begin{aligned}\mathrm{E}(3X + 1) &= \sum_x (3x + 1) \cdot p(x)\\ &= (1 \times 0.064) + (4 \times 0.288) + (7 \times 0.432) + (10 \times 0.216)\\ &= 6.4\end{aligned}$$

b
$$\begin{aligned}\mathrm{E}(X^2) &= \sum_x x^2 \cdot p(x)\\ &= (0^2 \times 0.064) + (1^2 \times 0.288) + (2^2 \times 0.432) + (3^2 \times 0.216)\\ &= 3.96\end{aligned}$$

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Let us compare the values found in Example 17 with the value of E(X) found in Example 16. In part **a**, we found that $\mathrm{E}(3X + 1) = 6.4$. Since $3\mathrm{E}(X) + 1 = 3 \times 1.8 + 1 = 6.4$, we see that

$$\mathrm{E}(3X + 1) = 3\mathrm{E}(X) + 1$$

In part **b**, we found that $\mathrm{E}(X^2) = 3.96$. Since $[\mathrm{E}(X)]^2 = 1.8^2 = 3.24$, we see that

$$\mathrm{E}(X^2) \neq [\mathrm{E}(X)]^2$$

These two examples illustrate an important point concerning expected values.

In general, the expected value of a function of X is not equal to that function of the expected value of X. That is,

$$\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$$

An exception is when the function is linear:

Expected value of $aX + b$

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad \text{(for } a, b \text{ constant)}$$

This is illustrated in Example 18.

Example 18

For a \$5 monthly fee, a TV repair company guarantees customers a complete service. The company estimates the probability that a customer will require one service call in a month as 0.05, the probability of two calls as 0.01 and the probability of three or more calls as 0.00. Each call costs the repair company \$40. What is the TV repair company's expected monthly gain from such a contract?

Solution

We may summarise the given information in the following table.

Calls	0	1	2	≥ 3
Gain, g	5	-35	-75	
$\mathrm{P}(G = g)$	0.94	0.05	0.01	0.00

$$\begin{aligned}\mathrm{E}(G) &= \sum_{g=0}^{2} g \cdot \mathrm{P}(G = g)\\ &= 5 \times 0.94 - 35 \times 0.05 - 75 \times 0.01\\ &= 2.20\end{aligned}$$

Thus, the company can expect to gain \$2.20 per month on each contract sold.

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Alternative solution

An alternative method of solution uses the formula for the expected value of $aX + b$, as follows.

Let X be the number of calls received. Then

$$G = 5 - 40X$$

and so $\quad \mathrm{E}(G) = 5 - 40 \times \mathrm{E}(X)$

Since $\quad \mathrm{E}(X) = 1 \times 0.05 + 2 \times 0.01 = 0.07$

we have $\quad \mathrm{E}(G) = 5 - 40 \times 0.07 = 2.20$ as previously determined.

Another useful property of expectation is that the expected value of the sum of two random variables is equal to the sum of their expected values. That is, if X and Y are two random variables, then

$$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$

Measures of variability: variance and standard deviation

As well as knowing the long-run average value of a random variable (the mean), it is also useful to have a measure of how close to this mean are the possible values of the random variable — that is, a measure of how spread out the probability distribution is. The most useful measures of variability for a discrete random variable are the variance and the standard deviation.

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

and may be considered as the long-run average value of the square of the distance from X to μ. The variance is usually denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

From the definition,

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2] \\ &= \sum_x (x - \mu)^2 \cdot \mathrm{P}(X = x)\end{aligned}$$

Since the variance is determined by squaring the distance from X to μ, it is no longer in the units of measurement of the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

The **standard deviation** of X is defined as

$$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

The standard deviation is usually denoted by σ.

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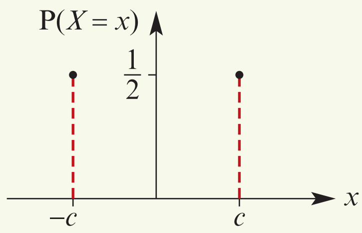
Example 19

Suppose that a discrete random variable X has the probability distribution shown in the following table, where $c > 0$.

x	$-c$	c
$P(X = x)$	0.5	0.5

Find the standard deviation of X.

Solution



$$\mu = E(X) = \sum_x x \cdot P(X = x)$$
$$= (-c \times 0.5) + (c \times 0.5)$$
$$= 0$$

$$\sigma^2 = \text{Var}(X) = E[(X - \mu)^2]$$
$$= E(X^2) \qquad \text{since } \mu = 0$$
$$= \sum_x x^2 \cdot P(X = x)$$
$$= (-c)^2 \times 0.5 + c^2 \times 0.5$$
$$= c^2$$

which is the average of (the distance from X to $\mu)^2$.

Therefore $\sigma = \text{sd}(X) = c$.

Using the definition is not always the easiest way to calculate the variance.

An alternative (computational) formula for variance is

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

Proof We already know that

$$E(aX + b) = aE(X) + b \qquad (1)$$

and

$$E(X + Y) = E(X) + E(Y) \qquad (2)$$

Hence

$$\text{Var}(X) = E[(X - \mu)^2]$$
$$= E(X^2 - 2\mu X + \mu^2)$$
$$= E(X^2) + E(-2\mu X + \mu^2) \qquad \text{using (2)}$$
$$= E(X^2) - 2\mu E(X) + \mu^2 \qquad \text{using (1)}$$
$$= E(X^2) - 2\mu^2 + \mu^2 \qquad \text{since } \mu = E(X)$$
$$= E(X^2) - \mu^2$$

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Example 20

For the probability distribution shown, find $\mathrm{E}(X^2)$ and $[\mathrm{E}(X)]^2$ and hence find the variance of X.

x	0	1	2	3
$\mathrm{P}(X = x)$	0.08	0.18	0.4	0.34

Solution

We have

$$\mathrm{E}(X) = 1 \times 0.18 + 2 \times 0.4 + 3 \times 0.34 = 2$$

$$[\mathrm{E}(X)]^2 = \mu^2 = 4$$

$$\mathrm{E}(X^2) = 1 \times 0.18 + 4 \times 0.4 + 9 \times 0.34 = 4.84$$

Hence

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 = 4.84 - 4 = 0.84$$

Variance of $aX + b$

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) \qquad \text{(for } a, b \text{ constant)}$$

Example 21

If X is a random variable such that $\mathrm{Var}(X) = 9$, find:

a $\mathrm{Var}(3X + 2)$

b $\mathrm{Var}(-X)$

Solution

a $\mathrm{Var}(3X + 2) = 3^2\mathrm{Var}(X)$
$= 9 \times 9$
$= 81$

b $\mathrm{Var}(-X) = \mathrm{Var}(-1 \times X)$
$= (-1)^2\mathrm{Var}(X)$
$= \mathrm{Var}(X)$
$= 9$

Interpretation of standard deviation

We can make the standard deviation more meaningful by giving it an interpretation that relates to the probability distribution.

Example 22

The number of chocolate bars, X, sold by a manufacturer in any month has the following distribution:

x	100	150	200	250	300	400
$p(x)$	0.05	0.15	0.35	0.25	0.15	0.05

What is the probability that X takes a value in the interval $\mu - 2\sigma$ to $\mu + 2\sigma$?

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Solution

First we must find the values of μ and σ.

$$\mu = \mathrm{E}(X) = \sum_x x \cdot p(x)$$
$$= 5 + 22.5 + 70 + 62.5 + 45 + 20$$
$$= 225$$

Before determining the standard deviation σ, we need to find the variance σ^2.

Now $\quad \mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

and $\quad \mathrm{E}(X^2) = \sum_x x^2 \cdot p(x)$

$$= 500 + 3375 + 14\,000 + 15\,625 + 13\,500 + 8000$$
$$= 55\,000$$

Thus $\quad \mathrm{Var}(X) = 55\,000 - (225)^2$

$$= 4375$$

and so $\quad \sigma = \mathrm{sd}(X)$

$$= \sqrt{4375}$$
$$= 66.14 \quad \text{(correct to two decimal places)}$$

Hence $\quad \mathrm{P}(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$= \mathrm{P}(92.72 \le X \le 357.28)$

$= \mathrm{P}(100 \le X \le 300)$ $\qquad$ since X only takes the values in the table

$= 0.95$ $\qquad$ from the probability distribution of X

In this example, 95% of the distribution lies within two standard deviations either side of the mean. While this is not always true, in many circumstances it is approximately true.

For many random variables X,

$$\mathrm{P}(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Example 23

A manufacturer knows that the mean number of faulty light bulbs in a batch of 10 000 is 12, with a standard deviation of 3. He wishes to claim to his clients that 95% of batches will contain between c_1 and c_2 faulty light bulbs. What are two possible values of c_1 and c_2?

Solution

Since $\mathrm{P}(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$, we can say

$$c_1 = \mu - 2\sigma = 6 \quad \text{and} \quad c_2 = \mu + 2\sigma = 18$$

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Summary 8D

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\mu = \mathrm{E}(X) = \sum_x x \cdot \mathrm{P}(X = x)$$
$$= \sum_x x \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- In general, for many random variables X,

$$\mathrm{P}(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Exercise 8D

Example 15

1 Tickets in a game of chance can be purchased for \$2. Each ticket has a 30% chance of winning \$2, a 10% chance of winning \$20, and otherwise loses. How much might you expect to win or lose if you play the game 100 times?

Example 16

2 For each of the following probability distributions, find the mean (expected value):

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.3	0.3

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0	1	2	3	4	5	6	7
$p(x)$	0.09	0.22	0.26	0.21	0.13	0.06	0.02	0.01

d

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$p(x)$	0.08	0.13	0.09	0.19	0.20	0.03	0.10	0.18

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e

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

f

x	−3	−2	−1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

3 A business consultant evaluates a proposed venture as follows. A company stands to make a profit of \$10 000 with probability 0.15, to make a profit of \$5000 with probability 0.45, to break even with probability 0.25, and to lose \$5000 with probability 0.15. Find the expected profit.

4 A spinner is numbered from 0 to 5, and each of the six numbers has an equal chance of coming up. A player who bets \$1 on any number wins \$5 if that number comes up; otherwise, the \$1 is lost. What is the player's expected profit on the game?

5 Suppose that the probability of having a female child is not as high as that of having a male child, and that the following frequency data for the number of male children in a three-child family have been determined from past records.

Number of males	0	1	2	3
Frequency	18	54	57	21

a Using the data, construct a table giving the probability distribution of X, the number of male children in a three-child family.

b What is the mean number of male children in a three-child family?

6 A player throws a die with faces numbered from 1 to 6 inclusive. If the player obtains a 6, she throws the die a second time, and in this case her score is the sum of 6 and the second number; otherwise, her score is the number first obtained. The player has no more than two throws.

Let X be the random variable denoting the player's score. Write down the probability distribution of X, and determine the mean of X.

Example 17

7 The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6
$p(x)$	0.01	0.25	0.40	0.30	0.04

Calculate:

a $E(X)$ **b** $E(X^3)$ **c** $E(5X - 4)$ **d** $E\left(\frac{1}{X}\right)$

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Example 18 **8** Manuel is a car salesperson. In any week his probability of making sales is as follows:

Number of cars sold, x	2	3	4	5	6
$P(X = x)$	0.45	0.25	0.20	0.08	0.02

If he is paid \$2000 commission on each car sold, what is his expected weekly income?

Example 20 **9** A discrete random variable X takes values 0, 1, 2, 4, 8 with probabilities as shown in the table.

x	0	1	2	4	8
$P(X = x)$	p	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

a Find p. **b** Find $E(X)$. **c** Find $Var(X)$.

10 A biased die is such that the probability of any face landing uppermost is proportional to the number on that face. Thus, if X denotes the score obtained in one throw of this die, then $P(X = r) = kr$ for $r = 1, 2, 3, 4, 5, 6$, where k is a constant.

a Find the value of k. **b** Find $E(X)$. **c** Find $Var(X)$.

11 An unbiased die is in the form of a regular tetrahedron and has its faces numbered 1, 2, 3 and 4. When the die is thrown onto a horizontal table, the number on the face in contact with the table is noted. Two such dice are thrown and the score, X, is found by multiplying these numbers together.

a Give the probability distribution of X.

b Determine the values of:

i $P(X > 8)$ **ii** $E(X)$ **iii** $Var(X)$

12 A coin and a six-sided die are thrown simultaneously. The random variable X is defined as follows: If the coin shows a head, then X is the score on the die; if the coin shows a tail, then X is twice the score on the die.

a Find the expected value, μ, of X.

b Find $P(X < \mu)$.

c Find $Var(X)$.

Example 21 **13** If $Var(X) = 16$, find:

a $Var(2X)$ **b** $Var(X + 2)$ **c** $Var(1 - X)$ **d** $sd(3X)$

Example 22 **14** A random variable X has the probability distribution shown. Find:

x	1	2	3	4	5
$P(X = x)$	c	0.3	0.1	0.2	0.05

a the constant c

b $E(X)$, the mean of X

c $Var(X)$, the variance of X, and hence the standard deviation of X

d $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$

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15 A random variable X has the probability distribution shown.

x	1	2	3	4	5
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$

Find:

a the constant k

b $E(X)$, the expectation of X

c $Var(X)$, the variance of X

d $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$

16 Two dice are rolled. If X is the sum of the numbers showing on the two dice, find:

a $E(X)$, the mean of X

b $Var(X)$, the variance of X

c $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$

17 The number of heads, X, obtained when a fair coin is tossed six times has the following probability distribution.

x	0	1	2	3	4	5	6
$p(x)$	0.0156	0.0937	0.2344	0.3126	0.2344	0.0937	0.0156

Find:

a $E(X)$, the mean of X

b $Var(X)$, the variance of X

c $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$

Example 23 **18** The random variable X, the number of heads observed when a fair coin is tossed 100 times, has a mean of 50 and a standard deviation of 5. If $P(c_1 \le X \le c_2) \approx 0.95$, give possible values of c_1 and c_2.

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Review

Chapter summary

Assignment

Nrich

Probability

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - $0 \leq P(A) \leq 1$ for all events $A \subseteq \varepsilon$
  - $P(\varnothing) = 0$ and $P(\varepsilon) = 1$
  - $P(A') = 1 - P(A)$, where A' is the complement of A
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, the **addition rule**.
- Probabilities associated with combined events are sometimes able to be calculated more easily from a probability table.
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$. In this case, we have $P(A \cap B) = 0$ and therefore $P(A \cup B) = P(A) + P(B)$.
- The **conditional probability** of event A occurring, given that event B has already occurred, is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0$$

giving $\quad P(A \cap B) = P(A \mid B) \times P(B) \quad$ (the **multiplication rule**)

- The probabilities associated with multi-stage experiments can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).
- The **law of total probability** states that, in the case of two events A and B,

$$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$

- Two events A and B are **independent** if

$$P(A \mid B) = P(A)$$

so whether or not B has occurred has no effect on the probability of A occurring.

- Events A and B are independent if and only if $P(A \cap B) = P(A) \times P(B)$.

Discrete random variables

- A **discrete** random variable X is one which can take only a countable number of values. Often these values are whole numbers, but not necessarily.
- The **probability distribution** of X is a function $p(x) = P(X = x)$ that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.
- For *any* discrete probability distribution, the following two conditions must hold:

1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

$$0 \leq p(x) \leq 1 \quad \text{for all } x$$

2. The sum of all the values of $p(x)$ must be 1. That is,

$$\sum_x p(x) = 1$$

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Review

- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$\mathrm{P}(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot \mathrm{P}(X = x)\\ &= \sum_x x \cdot p(x)\end{aligned}$$

- The expected value of a function of X is given by

$$\mathrm{E}[g(X)] = \sum_x g(x) \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- Linear function of a discrete random variable:

$$\begin{aligned}\mathrm{E}(aX + b) &= a\mathrm{E}(X) + b\\ \mathrm{Var}(aX + b) &= a^2\mathrm{Var}(X)\end{aligned}$$

- In general, for many random variables X,

$$\mathrm{P}(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Short-answer questions

1 If $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.2$ and $\mathrm{P}(A \cup B) = 0.7$, are the events A and B mutually exclusive? Explain.

2 Show, using a diagram or otherwise, that $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(A' \cap B')$. How would you describe this relationship in words?

3 A box contains five black and four white balls. Find the probability that two balls drawn at random are of different colours if:

a the first ball drawn is replaced before the second is drawn

b the balls are drawn without replacement.

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4 A gambler has two coins, A and B; the probabilities of their turning up heads are 0.8 and 0.4 respectively. One coin is selected at random and tossed twice, and a head and a tail are observed. Find the probability that the coin selected was A.

5 The probability distribution of a discrete random variable X is given by the following table. Show that $p = 0.5$ or $p = 1$.

x	0	1	2	3
$\Pr(X = x)$	$0.4p^2$	0.1	0.1	$1 - 0.6p$

6 A random variable X has the following probability distribution.

x	−1	0	1	2	3	4
$\Pr(X = x)$	k	$2k$	$3k$	$2k$	k	k

Find:

a the constant k **b** $E(X)$, the mean of X **c** $\text{Var}(X)$, the variance of X

7 If X has a probability function given by

$$p(x) = \frac{1}{4}, \quad x = 2, 4, 16, 64$$

find:

a $E(X)$ **b** $E\left(\frac{1}{X}\right)$ **c** $\text{Var}(X)$ **d** $\text{sd}(X)$

8 A manufacturer sells cylinders for \$$x$ each; the cost of the manufacture of each cylinder is \$2. If a cylinder is defective, it is returned and the purchase money refunded. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is $\frac{1}{5}$.

a Let P be the profit per cylinder. Find the probability function of P.

b Find the mean of P in terms of x.

c How much should the manufacturer sell the cylinders for in order to make a profit in the long term?

9 A group of 1000 drivers were classified according to their age and the number of accidents they had been involved in during the previous year. The results are shown in the table.

	Age < 30	Age ≥ 30
At most one accident	130	170
More than one accident	470	230

a Calculate the probability that, if a driver is chosen at random from this group, the driver is aged less than 30 and has had more than one accident.

b Calculate the probability that a randomly chosen driver is aged less than 30, given that he or she has had more than one accident.

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10 This year, 70% of the population have been immunised against a certain disease. Records indicate that an immunised person has a 5% chance of contracting the disease, whereas for a non-immunised person the chance is 60%. Calculate the overall percentage of the population who are expected to contract the disease.

11 Given $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{4}$ and $\Pr(A \mid B) = \frac{1}{6}$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$ **c** $\Pr(A' \mid B)$ **d** $\Pr(A \mid B')$

Extended-response questions

1 Given the following probability function:

x	1	2	3	4	5	6	7
$\Pr(X = x)$	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

a Find c.

b Evaluate $\Pr(X \geq 5)$.

c If $\Pr(X \leq k) > 0.5$, find the minimum value of k.

2 Janet and Alan are going to play a tennis match. The probability of Janet winning the first set is 0.3. After that, Janet's probability of winning a set is 0.6 if she has won the previous set, but only 0.4 if she has lost it. The match will continue until either Janet or Alan has won two sets.

a Construct a tree diagram to show the possible course of the match.

b Find the probability that:

i Janet will win

ii Alan will win.

c Let X be the number of sets played until the match is complete.

i Find the probability distribution of X.

ii Find the expected number of sets that the match will take, $E(X)$.

d Given that the match lasted three sets, find the probability that Alan won.

3 Five identical cards are placed face down on the table. Three of the cards are marked \$5 and the remaining two are marked \$10. A player picks two cards at random (without replacement) and is paid an amount equal to the sum of the values on the two cards. How much should the player pay to play if this is to be a fair game? (A fair game is considered to be one for which $E(X) = 0$, where X is the profit from the game.)

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4 A manufacturing company has three assembly lines: A, B and C. It has been found that 95% of the products produced on assembly line A will be free from faults, 98% from assembly line B will be free from faults and 99% from assembly line C will be free from faults. Assembly line A produces 50% of the day's output, assembly line B produces 30% of the day's output, and the rest is produced on assembly line C. If an item is chosen at random from the company's stock, find the probability that it:

a was produced on assembly line A

b is defective, given that it came from assembly line A

c is defective

d was produced on assembly line A, given that it was found to be defective.

5 A recent study found that P, the number of passengers per car entering a city on the freeway on a workday morning, is given by:

p	0	1	2	3	4	5
$\Pr(P = p)$	0.39	0.27	0.16	0.12	0.04	0.02

a **i** Compute $E(P)$, the mean number of passengers per car.

ii Compute $\text{Var}(P)$ and hence find the standard deviation of P.

iii Find $\Pr(\mu - 2\sigma \le P \le \mu + 2\sigma)$.

b The fees for cars at a toll booth on the freeway are as follows:

- Cars carrying no passengers pay \$1 toll.
- Cars carrying one passenger pay \$0.40 toll.
- Cars carrying two or more passengers pay no toll.

Let T be the toll paid by a randomly selected car on the freeway.

i Construct the probability distribution of T.

ii Find $E(T)$, the mean toll paid per car.

iii Find $\Pr(\mu - 2\sigma \le T \le \mu + 2\sigma)$.

6 The random variable Y, the number of cars sold in a week by a car salesperson, has the following probability distribution:

y	0	1	2	3	4	5	6	7	8
$\Pr(Y = y)$	0.135	0.271	0.271	0.180	0.090	0.036	0.012	0.003	0.002

a Compute $E(Y)$, the mean number of sales per week.

b Compute $\text{Var}(Y)$ and hence find the standard deviation of Y.

c The car salesperson is given a bonus as follows: If fewer than three cars are sold in the week, no bonus is given; if three or four cars are sold, a \$100 bonus is given; for more than four cars, the bonus is \$200. Let B be the bonus paid to the salesperson.

i Construct the probability distribution for B.

ii Find $E(B)$, the mean bonus paid.

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7 A given investment scheme is such that there is a 10% chance of receiving a profit of 40% of the amount invested, a 15% chance of a 30% profit, a 25% chance of a 20% profit, a 20% chance of a 10% profit, a 15% chance of breaking even, a 10% chance of a 10% loss and a 5% chance of 20% loss.

a Find the mean and standard deviation of the percentage return on the amount invested.

b Find the probability that the percentage return on the amount invested is within two standard deviations of the mean.

c An investor investing in the scheme pays a brokerage fee of 2% on the amount invested and a tax of 40% on the return (= profit − brokerage) of the investment. (Assume that a loss results in a tax refund for this investment.)
Express the percentage gain in terms of the percentage return on the amount invested, and hence find the mean and standard deviation of the percentage gain.

8 A concert featuring a popular singer is scheduled to be held in a large open-air theatre. The promoter is concerned that rain will cause people to stay away. A weather forecaster predicts that the probability of rain on any day at that particular time of the year is 0.33. If it does not rain, the promoter will make a profit of \$250 000 on the concert. If it does rain, the profit will be reduced to \$20 000. An insurance company agrees to insure the concert for \$250 000 against rain for a premium of \$60 000. Should the promoter buy the insurance?

9 A game is devised as follows: On two rolls of a single die, you will lose \$10 if the sum showing is 7, and win \$11 if the sum showing is either 11 or 12. How much should you win or lose if any other sum comes up in order for the game to be fair?

10 A new machine is to be developed by a manufacturing company. Prototypes are to be made until one satisfies the specifications of the company. Only then will it go into production. However, if after three prototypes are made none is satisfactory, then the project will be abandoned.
It is estimated that the probability a prototype will fail to produce a satisfactory model is 0.35, independent of any other already tested.

a Find the probability that:

- **i** the first prototype is successful
- **ii** the first is not successful but the second is
- **iii** the first two are not successful but the third is
- **iv** the project is abandoned.

b It is estimated that the cost of developing and testing the first prototype is \$7 million and that each subsequent prototype developed costs half of the one before. Find the expected cost of the project.

c If a machine is developed, then it is estimated that the income will be \$20 million. (If the project is abandoned, there is no income.) Find the expected profit.

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Review

11 A die is loaded such that the chance of throwing a 1 is $\frac{x}{4}$, the chance of a 2 is $\frac{1}{4}$ and the chance of a 6 is $\frac{1}{4}(1 - x)$. The chance of a 3, 4 or 5 is $\frac{1}{6}$. The die is thrown twice.

a Prove that the chance of throwing a total of 7 is $\dfrac{9x - 9x^2 + 10}{72}$.

b Find the value of x which will make this chance a maximum and find this maximum probability.

12 A game of chance consists of rolling a disc of diameter 2 cm on a horizontal square board. The board is divided into 25 small squares, each of side length 4 cm. A player wins a prize if, when the disc settles, it lies entirely within any one small square. There is a ridge around the outside edge of the board so that the disc always bounces back, cannot fall off and lies entirely within the boundary of the large square.
Prizes are awarded as follows:

Centre	(the middle square)	50c
Inner	(the eight squares surrounding the centre)	25c
Corner	(the four corner squares)	12c
Outer	(any other smaller square)	5c

When no skill is involved, the centre of the disc may be assumed to be randomly distributed over the accessible region.

a Calculate the probability in any one throw of winning:

i 50c **ii** 25c **iii** 12c **iv** 5c **v** no prize

b The proprietor wishes to make a profit in the long run, but is anxious to charge as little as possible to attract customers. He charges C cents, where C is an integer. Find the lowest value of C that will yield a profit.

13 Discrete uniform distributions

a Let X represent the number appearing on the uppermost face when a fair die is rolled. The probability distribution of X is shown in the table below.

i Find $\mathrm{E}(X)$.

ii Find $\mathrm{Var}(X)$.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b In general, a random variable X with values $1, 2, 3, \dots, n$ has a **uniform distribution** if each value of X is equally likely, and therefore

$$\mathrm{P}(X = x) = \frac{1}{n}, \quad \text{for } x = 1, 2, 3, \dots, n$$

i Find $\mathrm{E}(X)$. Hint: Use the result $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.

ii Find $\mathrm{Var}(X)$. Hint: Use the result $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

c A number is chosen randomly from the set $\{1, 2, 3, \dots, 10\}$. Let X represent the number chosen. Find:

i $\mathrm{P}(X = 4)$ **ii** $\mathrm{P}(X \le 4)$ **iii** $\mathrm{E}(X)$ **iv** $\mathrm{Var}(X)$

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9 The binomial distribution

In this chapter

Syllabus references

Topics: Bernoulli distributions; Binomial distributions

Subtopics: 3.3.9 – 3.3.16

The binomial distribution is important because it has very wide application. It is concerned with situations where there are two possible outcomes, and many ‘real life’ scenarios of interest fall into this category.

For example:

- A political poll of voters is carried out. Each polled voter is asked whether or not they would vote for the present government.
- A poll of Year 12 students in Australia is carried out. Each student is asked whether or not they watch the ABC on a regular basis.
- The effectiveness of a medical procedure is tested by selecting a group of patients and recording whether or not it is successful for each patient in the group.
- Components for an electronic device are tested to see if they are defective or not.

The binomial distribution has application in each of these examples.

We will use the binomial distribution again in Chapter 12, where we further develop our understanding of sampling.

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9A Bernoulli sequences and the binomial probability distribution

Bernoulli sequences

An experiment often consists of repeated trials, each of which may be considered as having only two possible outcomes. For example, when a coin is tossed, the two possible outcomes are 'head' and 'tail'. When a die is rolled, the two possible outcomes are determined by the random variable of interest for the experiment. If the event of interest is a 'six', then the two outcomes are 'six' and 'not a six'.

A **Bernoulli sequence** is the name used to describe a sequence of repeated trials with the following properties:

- Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
- The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

Example 1

Suppose that a netball player has a probability of $\frac{1}{3}$ of scoring a goal each time she attempts to goal. She repeatedly has shots for goal. Is this a Bernoulli sequence?

Solution

In this example:

- Each trial results in one of two outcomes, goal or miss.
- The probability of scoring a goal $\left(\frac{1}{3}\right)$ is constant for all attempts, as is the probability of a miss $\left(\frac{2}{3}\right)$.
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

Thus, the player's shots at goal can be considered a Bernoulli sequence.

Bernoulli random variables

The outcome from a Bernoulli trial is represented by a **Bernoulli random variable**, which is a random variable that takes only the values 1 (indicating a success) and 0 (indicating a failure).

Thus a Bernoulli random variable Y has a probability distribution of the following form:

y	0	1
$\Pr(Y = y)$	$1 - p$	p

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The binomial probability distribution

The number of successes in a Bernoulli sequence of n trials is called a **binomial random variable** and is said to have a **binomial probability distribution**.

For example, consider rolling a fair six-sided die three times. Let the random variable X be the number of 3s observed.

Let T represent a 3, and let N represent not a 3. Each roll meets the conditions of a Bernoulli trial. Thus X is a binomial random variable.

Now consider all the possible outcomes from the three rolls and their probabilities.

Outcome	Number of 3s	Probability	
TTT	$X = 3$	$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$	$\text{P}(X = 3) = \left(\frac{1}{6}\right)^3$
TTN	$X = 2$	$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$	
TNT	$X = 2$	$\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$	$\text{P}(X = 2) = 3 \times \left(\frac{1}{6}\right)^2 \times \frac{5}{6}$
NTT	$X = 2$	$\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$	
TNN	$X = 1$	$\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$	
NTN	$X = 1$	$\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}$	$\text{P}(X = 1) = 3 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^2$
NNT	$X = 1$	$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$	
NNN	$X = 0$	$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$	$\text{P}(X = 0) = \left(\frac{5}{6}\right)^3$

Thus the probability distribution of X is given by the following table.

x	0	1	2	3
$\text{P}(X = x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Instead of listing all the outcomes to find the probability distribution, we can use our knowledge of selections from Mathematical Methods Units 1 & 2.

Consider the probability that $X = 1$; that is, when exactly one 3 is observed. We can see from the table that there are three ways this can occur. Since the 3 could occur on the first, second or third roll of the die, we can consider this as selecting one object from a group of three, which can be done in $\binom{3}{1}$ ways.

Consider the probability that $X = 2$; that is, when exactly two 3s are observed. Again from the table there are three ways this can occur. Since the two 3s could occur on any two of the three rolls of the die, we can consider this as selecting two objects from a group of three, which can be done in $\binom{3}{2}$ ways.

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This leads us to a general formula for this probability distribution:

$$\Pr(X = x) = \binom{3}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{3-x} \qquad x = 0, 1, 2, 3$$

This is an example of the binomial distribution.

If the random variable X is the number of successes in n independent trials, each with probability of success p, then X has a **binomial distribution** and the rule is

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

Example 2

Find the probability of obtaining exactly three heads when a fair coin is tossed seven times, correct to four decimal places.

Solution

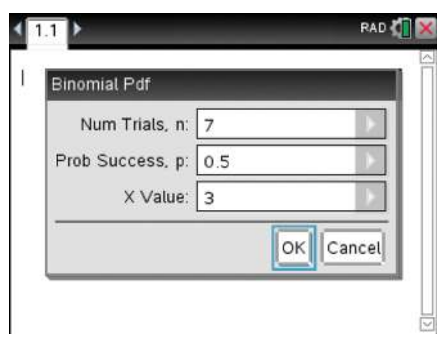
Obtaining a head is considered a success here, and the probability of success on each of the seven independent trials is 0.5.

Let X be the number of heads obtained. In this case, the parameters are $n = 7$ and $p = 0.5$.

$$\Pr(X = 3) = \binom{7}{3}(0.5)^3(1 - 0.5)^{7-3}$$
$$= 35 \times (0.5)^7 = 0.2734$$

Using the TI-Nspire

Use (menu) > **Probability** > **Distributions** > **Binomial Pdf** and complete as shown.



Use (tab) or ▼ to move between cells.

The result is shown below.

Note: You can also type in the command and the parameter values directly if preferred.

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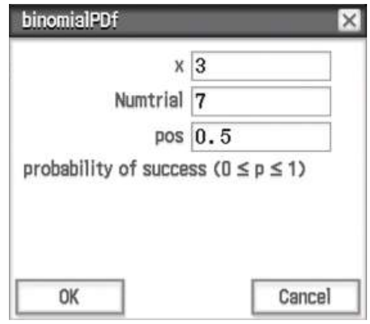
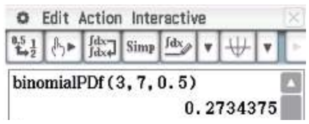
Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, go to **Interactive** > **Distribution** > **Discrete** > **binomialPDf**.
- Enter the number of successes and the parameters as shown. Tap OK.





Example 3

The probability that a person currently in prison has ever been imprisoned before is 0.72. Find the probability that of five prisoners chosen at random at least three have been imprisoned before, correct to four decimal places.

Solution

If X is the number of prisoners who have been imprisoned before, then

$$\Pr(X = x) = \binom{5}{x}(0.72)^x(0.28)^{5-x} \qquad x = 0, 1, \dots, 5$$

and so

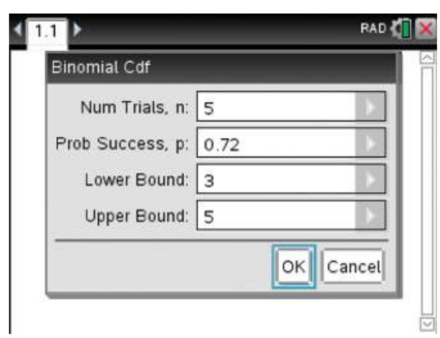
$$\begin{aligned}\Pr(X \geq 3) &= \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)\\ &= \binom{5}{3}(0.72)^3(0.28)^2 + \binom{5}{4}(0.72)^4(0.28)^1 + \binom{5}{5}(0.72)^5(0.28)^0\\ &= 0.8624\end{aligned}$$

Using the TI-Nspire

Use (menu) > **Probability** > **Distributions** > **Binomial Cdf** and complete as shown.

Use (tab) or ▼ to move between cells.

The result is shown below.



Note: You can also type in the command and the parameter values directly if preferred.

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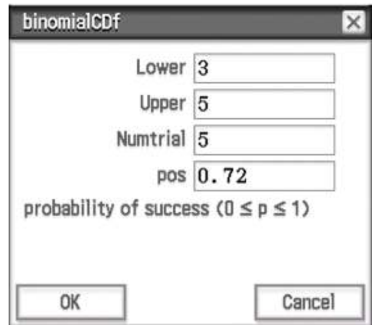
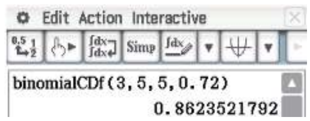
Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, go to **Interactive** > **Distribution** > **Discrete** > **binomialCDf**.
- Enter lower and upper bounds for the number of successes and the parameters as shown. Tap OK.





The binomial distribution and conditional probability

We can use the binomial distribution to solve problems involving conditional probabilities.

Example 4

The probability of a netballer scoring a goal is 0.3. Find the probability that out of six attempts the netballer scores a goal:

a four times **b** four times, given that she scores at least one goal.

Solution

Let X be the number of goals scored.
Then X has a binomial distribution with $n = 6$ and $p = 0.3$.

a
$$\begin{aligned}\mathrm{P}(X = 4) &= \binom{6}{4}(0.3)^4(0.7)^2\\ &= 15 \times 0.0081 \times 0.49\\ &= 0.059\,535\end{aligned}$$

b
$$\begin{aligned}\mathrm{P}(X = 4 \mid X \geq 1) &= \frac{\mathrm{P}(X = 4 \cap X \geq 1)}{\mathrm{P}(X \geq 1)}\\ &= \frac{\mathrm{P}(X = 4)}{\mathrm{P}(X \geq 1)}\\ &= \frac{0.059\,535}{1 - 0.7^6} \qquad \text{since } \mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0)\\ &= 0.0675\end{aligned}$$

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Summary 9A

- A **Bernoulli sequence** is a sequence of trials with the following properties:
  - Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
  - The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
  - The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).
- A **Bernoulli random variable** describes the outcome from a Bernoulli trial; it has a probability distribution of the form $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$.
- The number of successes, X, in a Bernoulli sequence of n trials is called a **binomial random variable** and has a **binomial probability distribution**:

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \ldots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

Skill-sheet

Exercise 9A

Example 1

1 Which of the following describes a Bernoulli sequence?

a tossing a fair coin many times

b drawing balls from an urn containing five red and three black balls, replacing the chosen ball each time

c selecting people at random from the population and noting their age

d selecting people at random from the population and noting their sex, male or female

Example 2

2 Find the probability of obtaining exactly four heads when a fair coin is tossed seven times, correct to four decimal places.

3 For a binomial distribution with $n = 4$ and $p = 0.2$, find the probability of:

a three successes

b four successes.

4 For a binomial distribution with $n = 5$ and $p = 0.4$, find the probability of:

a no successes

b three successes

c five successes.

5 Suppose that a fair coin is tossed three times, and the number of heads observed.

a Write down a general rule for the probability distribution of the number of heads.

b Use the rule to calculate the probability of observing two heads.

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6 Suppose that X is the number of male children born into a family of six children. Assume that the distribution of X is binomial, with probability of success 0.48.

a Write down a general rule for the probability distribution of the number of male children.

b Use the rule to calculate the probability that a family with six children will have exactly two male children.

Example 3

7 A fair die is rolled six times and the number of 2s noted. Find the probability of:

a exactly three 2s **b** more than three 2s **c** at least three 2s.

8 Jo knows that each ticket has a probability of 0.1 of winning a prize in a lucky ticket competition. Suppose she buys 10 tickets.

a Write down a general rule for the probability distribution of the number of winning tickets.

b Use the rule to calculate the probability that Jo has:

i no wins **ii** at least one win.

9 Suppose that the probability that a person selected at random is left-handed is always 0.2. If 11 people are selected at random for the cricket team:

a Write down a general rule for the probability distribution of the number left-handed people on the team.

b Use the rule to calculate the probability of selecting:

i exactly two left-handers **ii** no left-handers **iii** at least one left-hander.

10 In a particular city, the probability of rain falling on any given day is $\frac{1}{5}$.

a Write down a general rule for the probability distribution of the number of days of rain in a week.

b Use the rule to calculate the probability that in a particular week rain will fall:

i every day **ii** not at all **iii** on two or three days.

11 The probability of a particular drug causing side effects in a person is 0.2. What is the probability that at least two people in a random sample of 10 people will experience side effects?

12 Records show that $x\%$ of people will pass their driver's licence on the first attempt. If six students attempt their driver's licence, write down in terms of x the probability that:

a all six students pass **b** only one fails **c** no more than two fail.

13 A supermarket has four checkouts. A customer in a hurry decides to leave without making a purchase if all the checkouts are busy. At that time of day the probability of each checkout being free is 0.25. Assuming that whether or not a checkout is busy is independent of any other checkout, calculate the probability that the customer will make a purchase.

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14 A fair die is rolled 50 times. Find the probability of observing:

a exactly 10 sixes **b** no more than 10 sixes **c** at least 10 sixes.

15 Find the probability of getting at least nine successes in 100 trials for which the probability of success is $p = 0.1$.

16 A fair coin is tossed 50 times. If X is the number of heads observed, find:

a $\mathrm{P}(X = 25)$ **b** $\mathrm{P}(X \leq 25)$ **c** $\mathrm{P}(X \leq 10)$ **d** $\mathrm{P}(X \geq 40)$

17 A survey of the population in a particular city found that 40% of people regularly participate in sport. What is the probability that fewer than half of a random sample of six people regularly participate in sport?

18 An examination consists of six multiple-choice questions. Each question has four possible answers. At least three correct answers are required to pass the examination. Suppose that a student guesses the answer to each question.

a What is the probability the student guesses every question correctly?

b What is the probability the student will pass the examination?

Example 4 **19** The manager of a shop knows from experience that 60% of her customers will use a credit card to pay for their purchases. Find the probability that:

a the next three customers will use a credit card, and the three after that will not

b three of the next six customers will use a credit card

c at least three of the next six customers will use a credit card

d exactly three of the next six customers will use a credit card, given that at least three of the next six customers use a credit card.

20 A multiple-choice test has eight questions, each with five possible answers, only one of which is correct. Find the probability that a student who guesses the answer to every question will have:

a no correct answers

b six or more correct answers

c every question correct, given they have six or more correct answers.

21 The probability that a full forward in Australian Rules football will kick a goal from outside the 50-metre line is 0.15. If the full forward has 10 kicks at goal from outside the 50-metre line, find the probability that he will:

a kick a goal every time

b kick at least one goal

c kick more than one goal, given that he kicks at least one goal.

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9B The graph, expectation and variance of a binomial distribution

We looked at the properties of discrete probability distributions in Chapter 8. We now consider these properties for the binomial distribution.

The graph of a binomial probability distribution

As discussed in Chapter 8, a probability distribution may be represented as a rule, a table or a graph. We now investigate the shape of the graph of a binomial probability distribution for different values of the parameters n and p.

A method for plotting a binomial distribution with a CAS calculator can be found in the calculator appendices in the Interactive Textbook.

Example 5

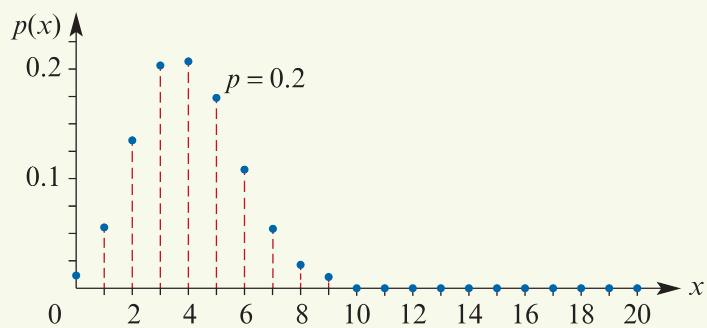
Construct and compare the graph of the binomial probability distribution for 20 trials ($n = 20$) with probability of success:

a $p = 0.2$ **b** $p = 0.5$ **c** $p = 0.8$

Solution

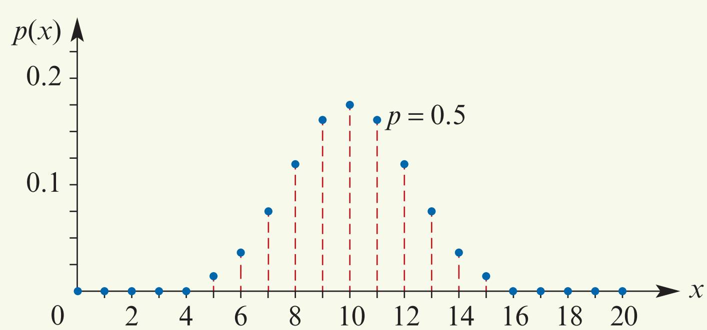
a For $p = 0.2$, the graph is positively skewed.

Mostly from 1 to 8 successes will be observed in 20 trials.



b For $p = 0.5$, the graph is symmetrical (as the probability of success is the same as the probability of failure).

Mostly from 6 to 14 successes will be observed in 20 trials.



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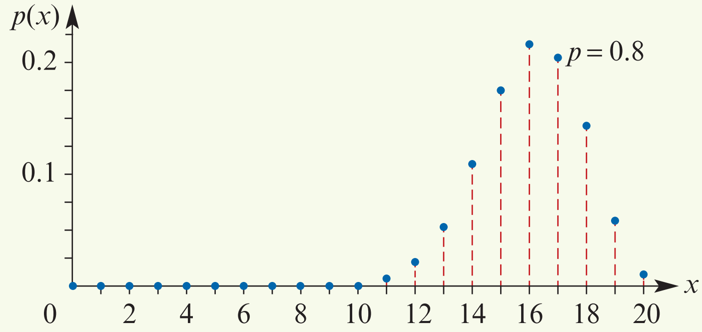
c For $p = 0.8$, the graph is negatively skewed.

Mostly from 12 to 19 successes will be observed in 20 trials.



Expectation and variance for a Bernoulli random variable

The table on the right shows the probability distribution for a Bernoulli random variable.

y	0	1
$\mathrm{P}(Y = y)$	$1 - p$	p

Thus $\quad \mathrm{E}(Y) = 0 \times (1 - p) + 1 \times p = p$

and $\quad \mathrm{E}(Y^2) = 0^2 \times (1 - p) + 1^2 \times p = p$

so $\quad \mathrm{Var}(Y) = p - p^2 = p(1 - p)$

Hence, if Y is a Bernoulli random variable with probability of success p, then

$$\mathrm{E}(Y) = p$$

$$\mathrm{Var}(Y) = p(1 - p)$$

Expectation and variance for a binomial random variable

How many heads would you expect to obtain, on average, if a fair coin was tossed 10 times?

While the exact number of heads in the 10 tosses would vary, and could theoretically take values from 0 to 10, it seems reasonable that the long-run average number of heads would be 5. It turns out that this is correct. That is, for a binomial random variable X with $n = 10$ and $p = 0.5$,

$$\mathrm{E}(X) = \sum_x x \cdot \mathrm{P}(X = x) = 5$$

In general, the expected value of a binomial random variable is equal to the number of trials multiplied by the probability of success. The variance can also be calculated from the parameters n and p.

If X is the number of successes in n trials, each with probability of success p, then the expected value and the variance of X are given by

$$\mathrm{E}(X) = np$$

$$\mathrm{Var}(X) = np(1 - p)$$

Note: These formulas are consistent with those for a Bernoulli random variable, which is a special case of a binomial random variable where $n = 1$.

While it is not necessary in this course to be familiar with the derivations of these formulas, they are included for completeness in the final section of this chapter.

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Example 6

An examination consists of 30 multiple-choice questions, each question having three possible answers. A student guesses the answer to every question. Let X be the number of correct answers.

a How many will she expect to get correct? That is, find $E(X) = \mu$.

b Find $Var(X)$.

Solution

The number of correct answers, X, is a binomial random variable with parameters $n = 30$ and $p = \frac{1}{3}$.

a The student has an expected result of $\mu = np = 10$ correct answers. (This is not enough to pass if the pass mark is 50%.)

b $Var(X) = np(1 - p)$

$$= 30 \times \frac{1}{3} \times \frac{2}{3} = \frac{20}{3}$$

Example 7

The probability of contracting influenza this winter is known to be 0.2. Of the 100 employees at a certain business, how many would the owner expect to get influenza? Find the standard deviation of the number who will get influenza and calculate $\mu \pm 2\sigma$. Interpret the interval $[\mu - 2\sigma, \mu + 2\sigma]$ for this example.

Solution

The number of employees who get influenza is a binomial random variable, X, with parameters $n = 100$ and $p = 0.2$.

The owner will expect $\mu = np = 20$ of the employees to contract influenza.

The variance is

$$\begin{aligned}\sigma^2 &= np(1 - p) \\ &= 100 \times 0.2 \times 0.8 \\ &= 16\end{aligned}$$

Hence the standard deviation is

$$\sigma = \sqrt{16} = 4$$

Thus

$$\begin{aligned}\mu \pm 2\sigma &= 20 \pm (2 \times 4) \\ &= 20 \pm 8\end{aligned}$$

The owner of the business knows there is a probability of about 0.95 that from 12 to 28 of the employees will contract influenza this winter.

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Summary 9B

If X is the number of successes in n trials, each with probability of success p, then the expected value and the variance of X are given by

- $\mathrm{E}(X) = np$
- $\mathrm{Var}(X) = np(1 - p)$

Exercise 9B

Example 5

1 Plot the graph of the probability distribution

$$\mathrm{P}(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 8$ and $p = 0.25$.

2 Plot the graph of the probability distribution

$$\mathrm{P}(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 12$ and $p = 0.35$.

3 **a** Plot the graph of the probability distribution

$$\mathrm{P}(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 10$ and $p = 0.2$.

b On the same axes, plot the graph of

$$\mathrm{P}(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 10$ and $p = 0.8$, using a different plotting symbol.

c Compare the two distributions.

d Comment on the effect of the value of p on the shape of the distribution.

Example 6

4 Find the mean and variance of the binomial random variables with parameters:

a $n = 25, \ p = 0.2$

b $n = 10, \ p = 0.6$

c $n = 500, \ p = \frac{1}{3}$

d $n = 40, \ p = 20\%$

5 A fair die is rolled six times.

a Find the expected value for the number of sixes obtained.

b Find the probability that more than the expected number of sixes is obtained.

6 The survival rate for a certain disease is 75%. Of the next 50 people who contract the disease, how many would you expect would survive?

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7 A binomial random variable X has mean 12 and variance 9. Find the parameters n and p, and hence find $P(X = 7)$.

8 A binomial random variable X has mean 30 and variance 21. Find the parameters n and p, and hence find $P(X = 20)$.

Example 7

9 A fair coin is tossed 20 times. Find the mean and standard deviation of the number of heads obtained and calculate $\mu \pm 2\sigma$. Interpret the interval $[\mu - 2\sigma, \mu + 2\sigma]$ for this example.

10 Records show that 60% of the students in a certain state attend government schools. If a group of 200 students are to be selected at random, find the mean and standard deviation of the number of students in the group who attend government schools, and calculate $\mu \pm 2\sigma$. Interpret the interval $[\mu - 2\sigma, \mu + 2\sigma]$ for this sample.

9C Finding the sample size

While we can never be absolutely certain about the outcome of a random experiment, sometimes we are interested in knowing what size sample would be required to observe a certain outcome. For example, how many times do you need to roll a die to be reasonably sure of observing a six, or how many lotto tickets must you buy to be reasonably sure that you will win a prize?

Example 8

The probability of winning a prize in a game of chance is 0.48.

a What is the least number of games that must be played to ensure that the probability of winning at least once is more than 0.95?

b What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.95?

Solution

Since the probability of winning each game is the same each time the game is played, this is an example of a binomial distribution, with the probability of success $p = 0.48$.

a The required answer is the smallest value of n such that $P(X \geq 1) > 0.95$.

$$\begin{aligned} & P(X \geq 1) > 0.95 \\ \Leftrightarrow \quad & 1 - P(X = 0) > 0.95 \\ \Leftrightarrow \quad & P(X = 0) < 0.05 \\ \Leftrightarrow \quad & 0.52^n < 0.05 \quad \text{since } P(X = 0) = 0.52^n \end{aligned}$$

This can be solved by taking logarithms of both sides:

$$\ln(0.52^n) < \ln(0.05)$$

$$n \ln(0.52) < \ln(0.05)$$

$$\therefore \quad n > \frac{\ln(0.05)}{\ln(0.52)} \approx 4.58$$

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Thus the game must be played at least five times to ensure that the probability of winning at least once is more than 0.95.

b The required answer is the smallest value of n such that $P(X \geq 2) > 0.95$, or equivalently, such that

$$P(X < 2) < 0.05$$

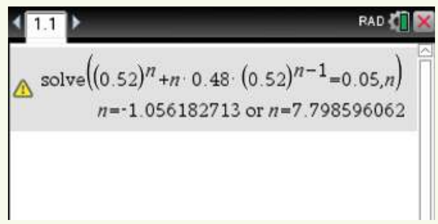
We have

$$\begin{aligned} P(X < 2) &= P(X = 0) + P(X = 1) \\ &= \binom{n}{0} 0.48^0 0.52^n + \binom{n}{1} 0.48^1 0.52^{n-1} \\ &= 0.52^n + 0.48n(0.52)^{n-1} \end{aligned}$$

So the answer is the smallest value of n such that

$$0.52^n + 0.48n(0.52)^{n-1} < 0.05$$

This equation cannot be solved algebraically; but a CAS calculator can be used to find the solution $n > 7.7985\ldots$.
Thus the game must be played at least eight times to ensure that the probability of winning at least twice is more than 0.95.



The following calculator inserts give a solution to part **b** of Example 8. Similar techniques can be used for part **a**. For further explanation, refer to the calculator appendices in the Interactive Textbook.

Using the TI-Nspire

To find the smallest value of n such that $P(X \geq 2) > 0.95$, where $p = 0.48$:

- Define the binomial CDF as shown.
  The last two parameters are the lower and upper bounds (inclusive) of the X value.

- Insert a **Lists & Spreadsheet** page. Press ctrl T to show the table of values.
- Scroll through the table to find where the probability is greater than 0.95. Hence $n = 8$.

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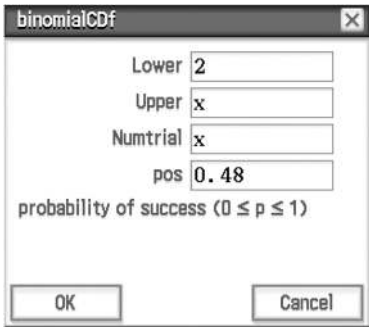
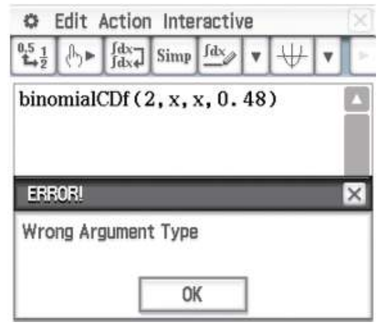
Using the Casio ClassPad

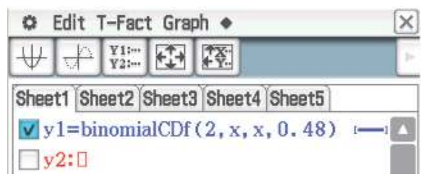
To find the smallest value of n such that $\mathrm{P}(X \geq 2) > 0.95$, where $p = 0.48$:

- In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Discrete** > **binomialCDf**.
- Enter bounds for the number of successes and the parameters as shown below.



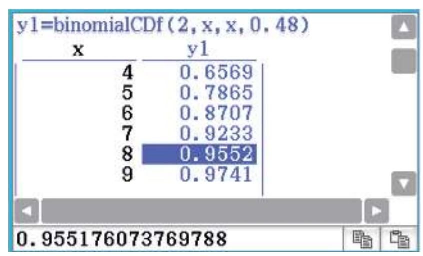


- Highlight and copy the expression.
- Go to the main menu Menu and select the **Graph & Table** application.
- Paste the expression in $y1$.



- Tap on the table icon.
- Scroll down the table of values until the probability first exceeds 0.95: the answer is $n = 8$.

Note: To view larger values of x in the table, tap the icon and set **End** at a larger value.



Skill-sheet

Exercise 9C

Example 8

1 The probability of a target shooter hitting the bullseye on any one shot is 0.2.

a If the shooter takes five shots at the target, find the probability of:

i missing the bullseye every time

ii hitting the bullseye at least once.

b What is the smallest number of shots the shooter should make to ensure a probability of more than 0.95 of hitting the bullseye at least once?

c What is the smallest number of shots the shooter should make to ensure a probability of more than 0.95 of hitting the bullseye at least twice?

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2 The probability of winning a prize with a lucky ticket on a wheel of fortune is 0.1.

a If a person buys 10 lucky tickets, find the probability of:

i winning twice

ii winning at least once.

b What is the smallest number of tickets that should be bought to ensure a probability of more than 0.7 of winning at least once?

3 Rex is shooting at a target. His probability of hitting the target is 0.6. What is the minimum number of shots needed for the probability of Rex hitting the target exactly five times to be more than 25%?

4 Janet is selecting chocolates at random out of a box. She knows that 20% of the chocolates have hard centres. What is the minimum number of chocolates she needs to select to ensure that the probability of choosing exactly three hard centres is more than 10%?

5 The probability of winning a prize in a game of chance is 0.35. What is the fewest number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

6 Geoff has determined that his probability of hitting '4' off any ball when playing cricket is 0.07. What is the fewest number of balls he must face to ensure that the probability of hitting more than one '4' is more than 0.8?

7 Monique is practising goaling for netball. She knows from past experience that her chance of making any one shot is about 70%. Her coach has asked her to keep practising until she scores 50 goals. How many shots would she need to attempt to ensure that the probability of scoring at least 50 goals is more than 0.99?

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Review

Chapter summary

Assign-ment

Nrich

- A **Bernoulli sequence** is a sequence of trials with the following properties:
  - Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
  - The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
  - The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).
- A **Bernoulli random variable** describes the outcome from a Bernoulli trial; it has a probability distribution of the form $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$.
- If X is the number of successes in n Bernoulli trials, each with probability of success p, then X is called a **binomial random variable** and is said to have a **binomial probability distribution** with parameters n and p. The probability of observing x successes in the n trials is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

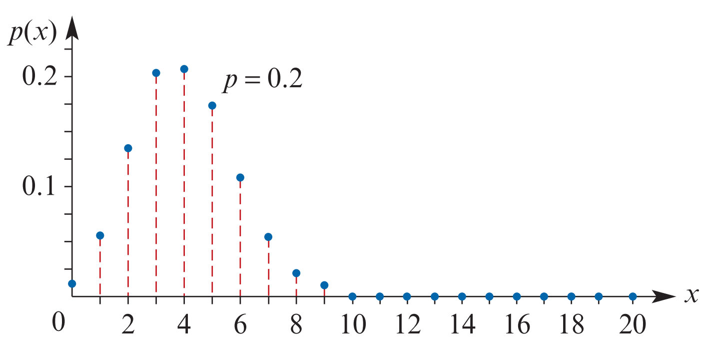
where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

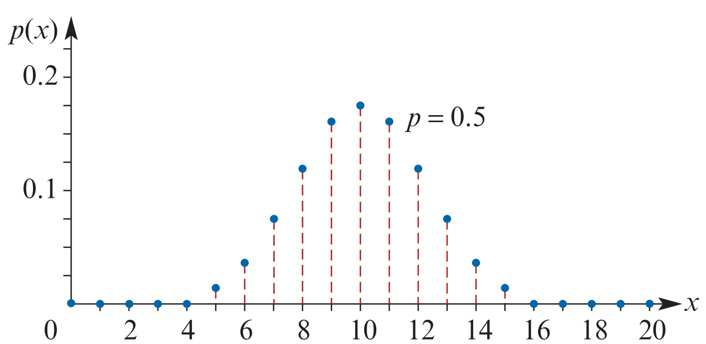
- If X has a binomial probability distribution with parameters n and p, then

$$\mathrm{E}(X) = np$$

$$\mathrm{Var}(X) = np(1 - p)$$

- The shape of the graph of a binomial probability function depends on the values of n and p.





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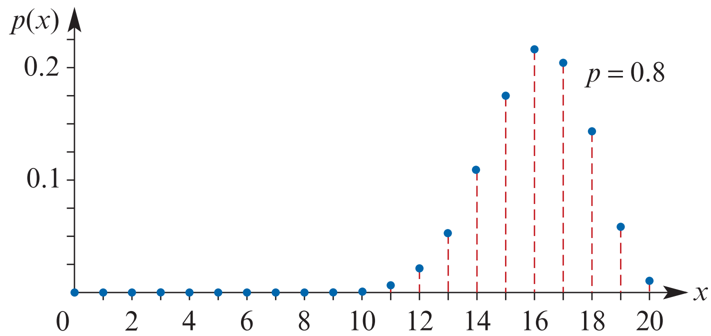

Short-answer questions

1 If X is a binomial random variable with parameters $n = 4$ and $p = \frac{1}{3}$, find:

a $\Pr(X = 0)$ **b** $\Pr(X = 1)$ **c** $\Pr(X \leq 1)$ **d** $\Pr(X \geq 1)$

2 A salesperson knows that 60% of the people who enter a particular shop will make a purchase. What is the probability that of the next three people who enter the shop exactly two will make a purchase?

3 If 10% of patients fail to improve on a certain medication, find the probability that of five patients selected at random one or more will fail to show improvement.

4 A machine has a probability of 0.1 of manufacturing a defective part.

a What is the expected number of defective parts in a random sample of 20 parts manufactured by the machine?

b What is the standard deviation of the number of defective parts?

5 An experiment consists of four independent trials. Each trial results in either a success or a failure. The probability of success in a trial is p. Find the probability of each of the following in terms of p:

a no successes **b** one success **c** at least one success

d four successes **e** at least two successes.

6 A coin is tossed 10 times. The probability of three heads is $m \times \left(\frac{1}{2}\right)^{10}$. State the value of m.

7 An experiment consists of five independent trials. Each trial results in either a success or a failure. The probability of success in a trial is p. Find, in terms of p, the probability of exactly one success given at least one success.

8 A die is rolled five times. What is the probability of obtaining an even number on the uppermost face on exactly three of the rolls?

9 In a particular city, the probability of rain on any day in June is $\frac{1}{5}$. What is the probability of it raining on three of five days?

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Extended-response questions

1 In a test to detect learning disabilities, a child is asked 10 questions, each of which has possible answers labelled A, B and C. Children with a disability of type 1 almost always answer A or B on every question, while children with a disability of type 2 almost always answer C on every question. Children without either disability have an equal chance of answering A, B or C for each question.

a What is the probability that the answers given by a child without either disability will be all As and Bs, thereby indicating a type 1 disability?

b A child is further tested for type 2 disability if he or she answers C five or more times. What is the probability that a child without either disability will test positive for type 2 disability?

2 An inspector takes a random sample of 10 items from a very large batch. If none of the items is defective, he accepts the batch; otherwise, he rejects the batch. What is the probability that a batch is accepted if the fraction of defective items is 0, 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1? Plot these probabilities against the corresponding fraction defective. Is the inspection method a good one or not?

3 It has been found in the past that 4% of the CDs produced in a certain factory are defective. A sample of 10 CDs is drawn randomly from each hour's production and the number of defective CDs is noted.

a What percentage of these hourly samples would contain at least two defective CDs?

b Find the mean and standard deviation of the number of defective CDs in a sample, and calculate $\mu \pm 2\sigma$.

c A particular sample is found to contain three defective CDs. Would this cause you to have doubts about the production process?

4 A pizza company claims that they deliver 90% of orders within 30 minutes. In a particular 2-hour period, the supervisor notes that there are 67 orders, and of these 12 orders are delivered late. If the company claim is correct, and 90% of orders are delivered on time, what is the probability that at least 12 orders are delivered late?

5 **a** A sample of six objects is to be drawn from a large population in which 20% of the objects are defective. Find the probability that the sample contains:

i three defectives **ii** fewer than three defectives.

b Another large population contains a proportion p of defective items.

i Write down an expression in terms of p for P, the probability that a sample of six items contains exactly two defectives.

ii By differentiating to find $\frac{dP}{dp}$, show that P is greatest when $p = \frac{1}{3}$.

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6 Groups of six people are chosen at random and the number, x, of people in each group who normally wear glasses is recorded. The table gives the results from 200 groups.

Number wearing glasses, x	0	1	2	3	4	5	6
Number of occurrences	17	53	65	45	18	2	0

a Calculate, from the above data, the mean value of x.

b Assuming that the situation can be modelled by a binomial distribution having the same mean as the one calculated above, state the appropriate values for the binomial parameters n and p.

c Calculate the theoretical frequencies corresponding to those in the table.

7 A sampling inspection scheme is devised as follows. A sample of size 10 is drawn at random from a large batch of articles and all 10 articles are tested. If the sample contains fewer than two faulty articles, the batch is accepted; if the sample contains three or more faulty articles, the batch is rejected; but if the sample contains exactly two faulty articles, a second sample of size 10 is taken and tested. If this second sample contains no faulty articles, the batch is accepted; but if it contains any faulty articles, the batch is rejected. Previous experience has shown that 5% of the articles in a batch are faulty.

a Find the probability that the batch is accepted after the first sample is taken.

b Find the probability that the batch is rejected.

c Find the expected number of articles to be tested.

8 Assume that dates of birth in a large population are distributed such that the probability of a randomly chosen person's birthday being in any particular month is $\frac{1}{12}$.

a Find the probability that of six people chosen at random exactly two will have a birthday in January.

b Find the probability that of eight people at least one will have a birthday in January.

c N people are chosen at random. Find the least value of N such that the probability that at least one will have a birthday in January exceeds 0.9.

9 Suppose that, in flight, aeroplane engines fail with probability q, independently of each other, and that a plane will complete the flight successfully if at least half of its engines are still working. For what values of q is a two-engine plane to be preferred to a four-engine one?

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10 Continuous random variables and their probability distributions

In this chapter

Syllabus references

Topic: General continuous random variables

Subtopics: 4.2.1 – 4.2.4

In this chapter we extend our knowledge of probability to include continuous random variables, which can take any value in an interval of the real number line. Examples include the time taken to complete a puzzle and the height of an adult. When considering the heights of adults, the range of values could be from 56 cm to 251 cm, and in principle the measurement could be any value in this interval.

We also introduce the concept of the probability density function to describe the distribution of a continuous random variable. We shall see that probabilities associated with a continuous random variable are described by areas under the probability density function, and thus integration is an important skill required to determine these probabilities.

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10A Continuous random variables

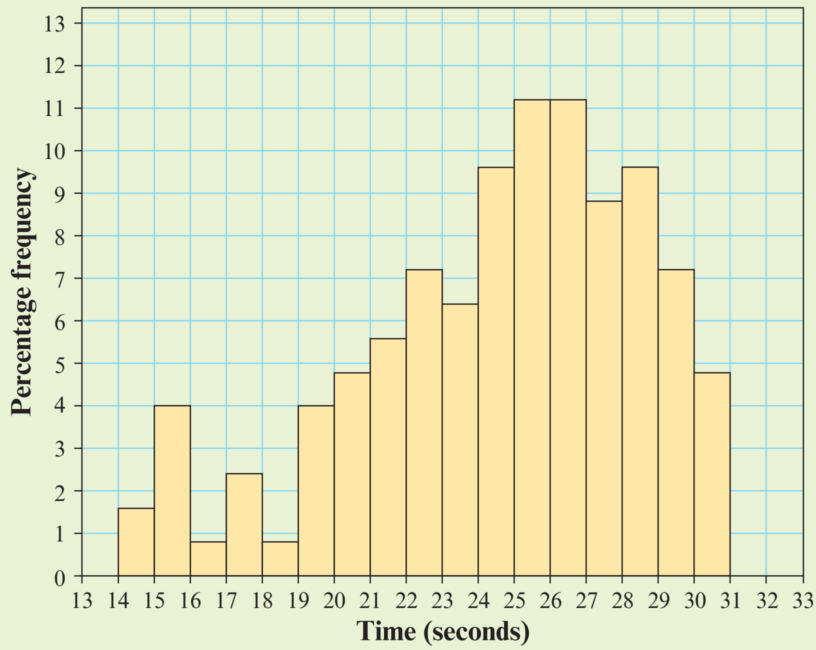
A **continuous random variable** is one that can take any value in an interval of the real number line. For example, if X is the random variable which takes its values as 'distance in metres that a parachutist lands from a marker', then X is a continuous random variable, and here the values which X may take are the non-negative real numbers.

Using data to approximate a continuous random variable

Relative frequencies obtained from data can be used to approximate the probabilities associated with a continuous random variable.

Example 1

Let T represent the time (in seconds) that it takes a student to complete a particular puzzle. The following percentage frequency histogram was obtained by recording the times taken to complete the puzzle by 500 students, with each recorded time rounded down to a whole number of seconds.



Use the histogram to estimate:

a $\text{P}(19 \le T < 22)$

b $\text{P}(T \ge 28)$

Solution

a $\text{P}(19 \le T < 22) \approx 4\% + 4.8\% + 5.6\%$
$= 14.4\%$
$= 0.144$

b $\text{P}(T \ge 28) \approx 9.6\% + 7.2\% + 4.8\%$
$= 21.6\%$
$= 0.216$

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An example of a continuous random variable

A continuous random variable has no limit as to the accuracy with which it can be measured. For example, let W be the random variable with values 'a person's weight in kilograms' and let W_i be the random variable with values 'a person's weight in kilograms measured to the ith decimal place'.

Then $W_0 = 83$ implies $82.5 \leq W < 83.5$

$W_1 = 83.3$ implies $83.25 \leq W < 83.35$

$W_2 = 83.28$ implies $83.275 \leq W < 83.285$

$W_3 = 83.281$ implies $83.2805 \leq W < 83.2815$

and so on. Thus, the random variable W cannot take an exact value, since it is always rounded to the limits imposed by the method of measurement used. Hence, the probability of W being exactly equal to a particular value is zero, and this is true for all continuous random variables.

That is,

$$\text{P}(W = w) = 0 \quad \text{for all } w$$

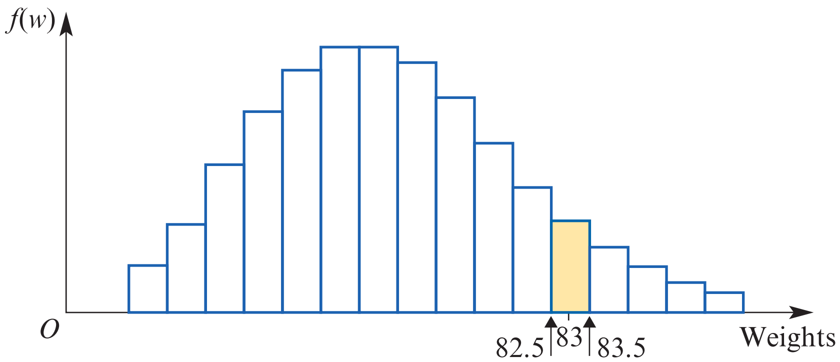
In practice, considering W_i taking a particular value is equivalent to W taking a value in an appropriate interval.

Thus, from above:

$$\text{P}(W_0 = 83) = \text{P}(82.5 \leq W < 83.5)$$

To determine the value of this probability, you could begin by measuring the weight of a large number of randomly chosen people, and determine the proportion of the people in the group who have weights in this interval.

Suppose after doing this a histogram of weights was obtained as shown.



From this histogram:

$$\text{P}(W_0 = 83) = \text{P}(82.5 \leq W < 83.5)$$

$$= \frac{\text{shaded area from 82.5 to 83.5}}{\text{total area}}$$

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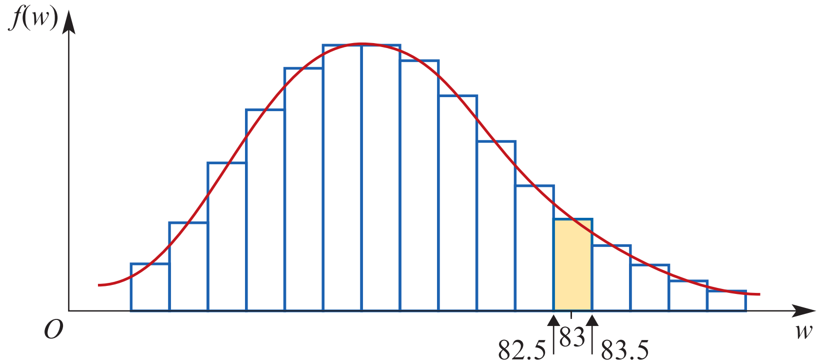
If the histogram is scaled so that the total area under the blocks is 1, then

$$\begin{aligned} P(W_0 = 83) &= P(82.5 \leq W < 83.5) \\ &= \text{area under block from 82.5 to 83.5} \end{aligned}$$

Now suppose that the sample size gets larger and that the class interval width gets smaller. If theoretically this process is continued so that the intervals are arbitrarily small, then the histogram can be modelled by a smooth curve, as shown in the following diagram.



The curve obtained here is of great importance for a continuous random variable.

The function f whose graph models the histogram as the number of intervals is increased is called the **probability density function**. The probability density function f is used to describe the probability distribution of a continuous random variable X.

Now, the probability of interest is no longer represented by the area under the histogram, but by the area under the curve. That is,

$$\begin{aligned} P(W_0 = 83) &= P(82.5 \leq W < 83.5) \\ &= \text{area under the graph of the function with rule } f(w) \text{ from 82.5 to 83.5} \\ &= \int_{82.5}^{83.5} f(w)\, dw \end{aligned}$$

Probability density functions

In general, a **probability density function** f is a function with domain some interval (e.g. domain $[c, d]$ or $\mathbb{R}$) such that:

1 $f(x) \geq 0$ for all x in the interval, and

2 the area under the graph of the function is equal to 1.

If the domain of f is $[c, d]$, then the second condition corresponds to $\int_c^d f(x)\, dx = 1$.

In many cases, however, the domain of f will be an 'unbounded' interval such as $[1, \infty)$ or $\mathbb{R}$. Therefore, some new notation is necessary.

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- If the probability density function f has domain $[1, \infty)$, then $\int_1^{\infty} f(x)\,dx = 1$. This integral is computed as $\lim_{k\to\infty} \int_1^k f(x)\,dx$.
- If the probability density function f has domain $\mathbb{R}$, then $\int_{-\infty}^{\infty} f(x)\,dx = 1$. This integral is computed as $\lim_{k\to\infty} \int_{-k}^k f(x)\,dx$.

Note: Definite integrals which have one or both limits infinite are called **improper integrals**. There are possible complications with such integrals which we avoid in this course; you will only need the methods of evaluation illustrated in Examples 2 and 4.

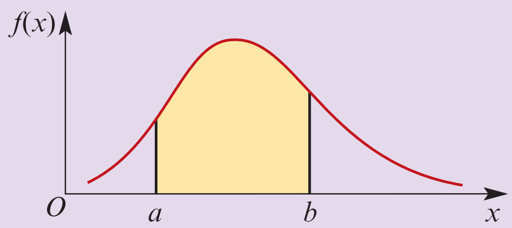
The probability density function of a random variable

Now consider a continuous random variable X with range $[c, d]$. (Alternatively, the range of X may be an unbounded interval such as $(-\infty, d]$, $[c, \infty)$ or $\mathbb{R}$.) Let f be a probability density function with domain $[c, d]$. Then:

We say that f is the **probability density function of X** if

$$P(a < X < b) = \int_a^b f(x)\,dx$$

for all $a < b$ in the range of X.



Notes:

- The values of a probability density function f are not probabilities, and $f(x)$ may take values greater than 1.
- The probability of any specific value of X is 0. That is, $P(X = a) = 0$. It follows that all of the following expressions have the same numerical value:
  - $P(a < X < b)$
  - $P(a \le X < b)$
  - $P(a < X \le b)$
  - $P(a \le X \le b)$
- If f has domain $[c, d]$ and $a \in [c, d]$, then $P(X < a) = P(X \le a) = \int_c^a f(x)\,dx$.

The natural extension of a probability density function

Any probability density function f with domain $[c, d]$ (or any other interval) may be extended to a function f^* with domain $\mathbb{R}$ by defining

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in [c, d] \\ 0 & \text{if } x \notin [c, d] \end{cases}$$

This leads to the following:

A probability density function f (or its natural extension) must satisfy the following two properties:

1 $f(x) \ge 0$ for all x **2** $\int_{-\infty}^{\infty} f(x)\,dx = 1$

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Example 2

Consider the function f with the rule:

$$f(x) = \begin{cases} 1.5(1 - x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

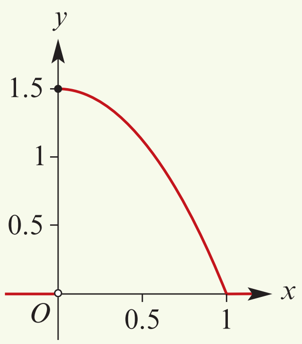
a Sketch the graph of f.

b Show that f is a probability density function.

c Find $\Pr(X > 0.5)$, where the random variable X has probability density function f.

Solution

a For $0 \le x \le 1$, the graph of $y = f(x)$ is part of a parabola with intercepts at $(0, 1.5)$ and $(1, 0)$.



b From the graph, we can see that $f(x) \ge 0$ for all x, and so the first condition holds.

The second condition to check is that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 1.5(1 - x^2)\,dx$ since $f(x) = 0$ elsewhere

$$= 1.5\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 1.5\left(1 - \frac{1}{3}\right)$$

$$= 1$$

Thus the second condition holds, and hence f is a probability density function.

c $\Pr(X > 0.5) = \int_{0.5}^1 1.5(1 - x^2)\,dx$

$$= 1.5\left[x - \frac{x^3}{3}\right]_{0.5}^1$$

$$= 1.5\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= 0.3125$$

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Example 3

Suppose that the random variable X has the probability density function with rule:

$$f(x) = \begin{cases} cx & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x > 2 \text{ or } x < 0 \end{cases}$$

a Find the value of c that makes f a probability density function.

b Find $\mathrm{P}(X > 1.5)$.

c Find $\mathrm{P}(1 \le X \le 1.5)$.

Solution

a Since f is a probability density function, we know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\text{Now } \int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 cx\,dx \qquad \text{since } f(x) = 0 \text{ elsewhere}$$

$$= \left[\frac{cx^2}{2}\right]_0^2$$

$$= 2c$$

Therefore $2c = 1$ and so $c = 0.5$.

b

$$\mathrm{P}(X > 1.5) = \int_{1.5}^{2} 0.5x\,dx$$

$$= 0.5\left[\frac{x^2}{2}\right]_{1.5}^{2}$$

$$= 0.5\left(\frac{4}{2} - \frac{2.25}{2}\right)$$

$$= 0.4375$$

c

$$\mathrm{P}(1 \le X \le 1.5) = \int_{1}^{1.5} 0.5x\,dx$$

$$= 0.5\left[\frac{x^2}{2}\right]_{1}^{1.5}$$

$$= 0.5\left(\frac{2.25}{2} - \frac{1}{2}\right)$$

$$= 0.3125$$

Probability density functions with unbounded domain

Some intervals for which definite integrals need to be evaluated are of the form $(-\infty, a]$ or $[a, \infty)$ or $(-\infty, \infty)$. For a function f with non-negative values, such integrals are defined as follows (provided the limits exist):

- To integrate over the interval $(-\infty, a]$, find $\lim_{k \to -\infty} \int_k^a f(x)\,dx$.
- To integrate over the interval $[a, \infty)$, find $\lim_{k \to \infty} \int_a^k f(x)\,dx$.
- To integrate over the interval $(-\infty, \infty)$, find $\lim_{k \to \infty} \int_{-k}^k f(x)\,dx$.

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Example 4

Consider the exponential probability density function f with the rule:

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

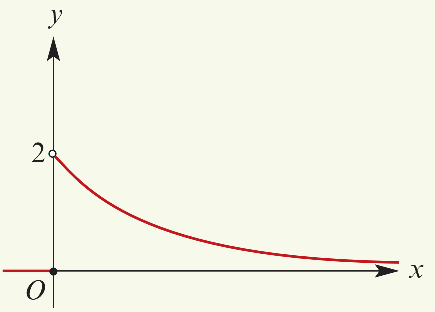
a Sketch the graph of f.

b Show that f is a probability density function.

c Find $\Pr(X > 1)$, where the random variable X has probability density function f.

Solution

a For $x > 0$, the graph of $y = f(x)$ is part of the graph of an exponential function with y-axis intercept 2. As $x \to \infty$, $y \to 0$.



b Since $f(x) \ge 0$ for all x, the first condition holds.

The second condition to check is that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 2e^{-2x}\,dx$ since $f(x) = 0$ elsewhere

$$\begin{aligned}
&= \lim_{k\to\infty} \int_0^k 2e^{-2x}\,dx \\
&= \lim_{k\to\infty} \left[\frac{2e^{-2x}}{-2}\right]_0^k \\
&= \lim_{k\to\infty} \left[-e^{-2x}\right]_0^k \\
&= \lim_{k\to\infty} \left((-e^{-2k}) - (-e^{-0})\right) \\
&= 0 + e^0 \\
&= 1
\end{aligned}$$

Thus f satisfies the two conditions for a probability density function.

c
$$\begin{aligned}
\Pr(X > 1) &= \lim_{k\to\infty} \int_1^k 2e^{-2x}\,dx \\
&= \lim_{k\to\infty} \left[-e^{-2x}\right]_1^k \\
&= \lim_{k\to\infty} \left((-e^{-2k}) - (-e^{-2})\right) \\
&= 0 + e^{-2} \\
&= \frac{1}{e^2} \\
&= 0.1353 \quad \text{correct to four decimal places}
\end{aligned}$$

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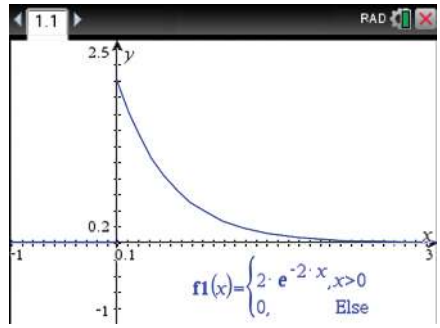
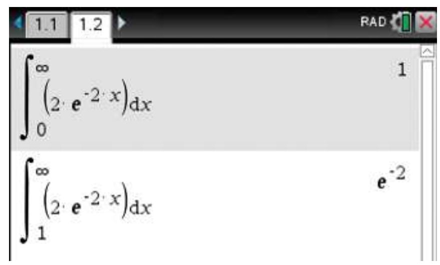
Using the TI-Nspire

This is an application of integration.

a The graph is as shown. The piecewise function template has been used in this example; access the template using (template key).



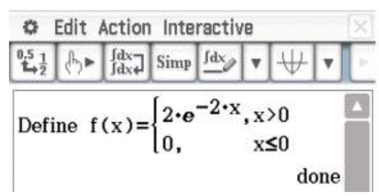
b, c The two required integrations are shown. The symbol ∞ can be found using (π▸) or (ctrl)(book).

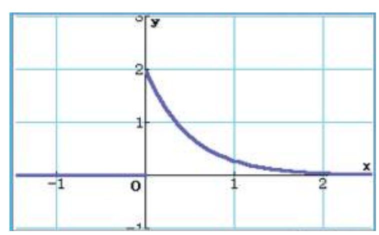


Using the Casio ClassPad

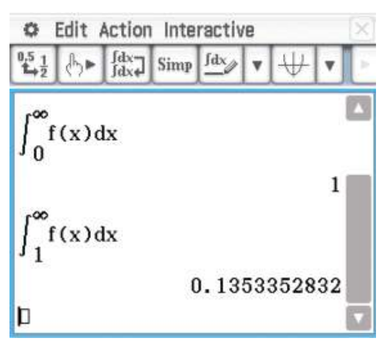
a To sketch the graph:

- Select the (Math3) keyboard and tap on the piecewise template.
- Enter the function as shown, highlight and go to **Interactive** > **Define**.
- Now select (graph icon), highlight $f(x)$ and drag into the graph screen.
- Adjust the window using (window icon).





b, c Find the definite integrals as shown.



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Conditional probability

Next is an example involving conditional probability with continuous random variables.

Example 5

The time (in seconds) that it takes a student to complete a puzzle is a random variable X with a density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \geq 5 \\ 0 & x < 5 \end{cases}$$

a Find the probability that a student takes less than 12 seconds to complete the puzzle.

b Find the probability that a student takes between 8 and 10 seconds to complete the puzzle, given that he takes less than 12 seconds.

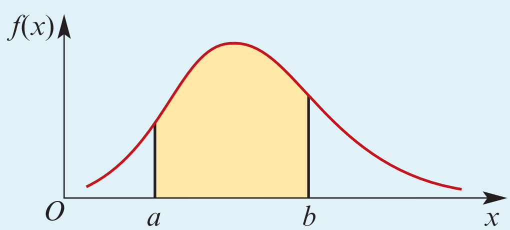
Solution

a
$$\begin{aligned} P(X < 12) &= \int_5^{12} f(x)\,dx \\ &= \int_5^{12} \frac{5}{x^2}\,dx \\ &= \left[-\frac{5}{x}\right]_5^{12} \\ &= -\frac{5}{12} + 1 \\ &= \frac{7}{12} \end{aligned}$$

b
$$\begin{aligned} &P(8 < X < 10 \mid X < 12) \\ &= \frac{P(8 < X < 10 \cap X < 12)}{P(X < 12)} \\ &= \frac{P(8 < X < 10)}{P(X < 12)} \\ &= \frac{\int_8^{10} f(x)\,dx}{\int_5^{12} f(x)\,dx} \\ &= \frac{-\frac{1}{2} + \frac{5}{8}}{\frac{7}{12}} = \frac{3}{14} \end{aligned}$$

Summary 10A

- A probability density function f (or its natural extension) must satisfy the following two properties:
  1 $f(x) \geq 0$ for all x **2** $\int_{-\infty}^{\infty} f(x)\,dx = 1$
- If X is a continuous random variable with density function f, then
  $$P(a < X < b) = \int_a^b f(x)\,dx$$
  which is the area of the shaded region.



- Definite integrals may need to be evaluated over unbounded intervals:
  - To integrate over the interval $(-\infty, a]$, find $\lim_{k \to -\infty} \int_k^a f(x)\,dx$.
  - To integrate over the interval $[a, \infty)$, find $\lim_{k \to \infty} \int_a^k f(x)\,dx$.
  - To integrate over the interval $(-\infty, \infty)$, find $\lim_{k \to \infty} \int_{-k}^k f(x)\,dx$.

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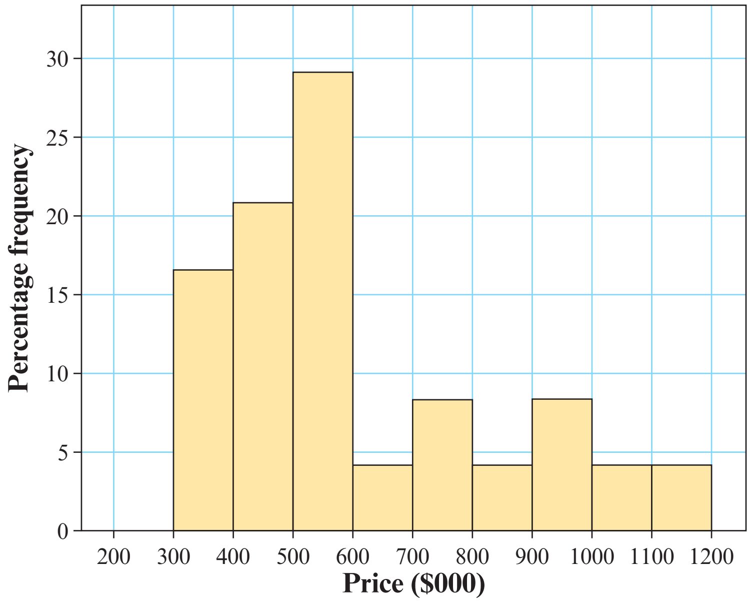
Skill-sheet

Exercise 10A

Example 1

1 The following percentage frequency histogram summarises the prices, $\$X$, of apartments in a certain town.

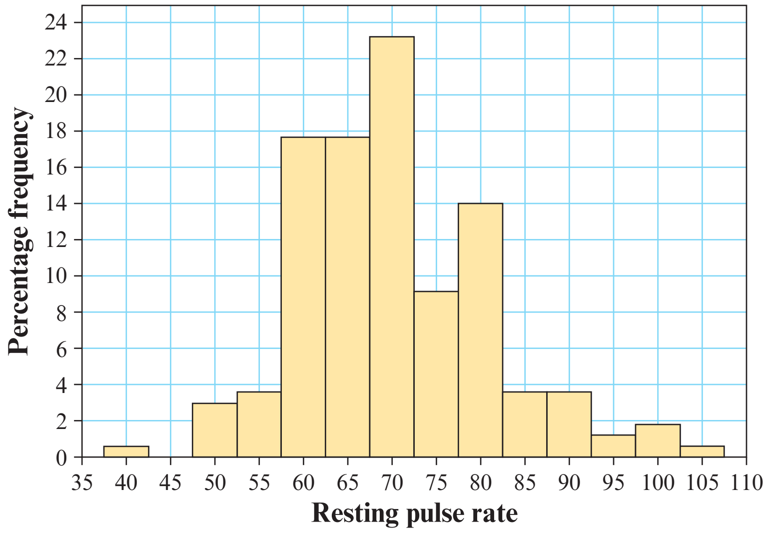


Use the histogram to estimate:

a $P(X \geq 900\,000)$

b $P(600\,000 \leq X < 1\,000\,000)$

2 Let X be the resting pulse rate (in beats per minute) of a randomly chosen person. The following percentage frequency data was obtained from a random sample of people, with each recorded pulse rate rounded to the nearest 5 beats per minute.



Use the histogram to estimate:

a $P(62.5 \leq X < 67.5)$

b $P(67.5 \leq X < 87.5)$

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Example 2

3 Show that the function f with the following rule is a probability density function:

$$f(x) = \begin{cases} \dfrac{24}{x^3} & 3 \le x \le 6 \\ 0 & x < 3 \text{ or } x > 6 \end{cases}$$

Example 3

4 Let X be a continuous random variable with the following probability density function:

$$f(x) = \begin{cases} x^2 + kx + 1 & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Determine the constant k such that f is a valid probability density function.

5 Consider the random variable X having the probability density function with the rule:

$$f(x) = \begin{cases} 12x^2(1-x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $\mathrm{P}(X < 0.5)$.

c Shade the region which represents this probability on your sketch graph.

Example 4

6 Consider the random variable Y with the probability density function:

$$f(y) = \begin{cases} ke^{-y} & y \ge 0 \\ 0 & y < 0 \end{cases}$$

a Find the constant k.

b Find $\mathrm{P}(Y \le 2)$.

Example 5

7 The quarantine period for a certain disease is between 5 and 11 days after contact. The probability of showing the first symptoms at various times during the quarantine period is described by the probability density function:

$$f(t) = \frac{1}{36}(t-5)(11-t)$$

a Sketch the graph of the function.

b Find the probability that the symptoms appear within 7 days.

c Find the probability that the symptoms appear within 7 days, given that they appear after 5.5 days.

d Find the probability that the symptoms appear within 7 days, given that they appear within 10 days.

8 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = k\sin\left(\frac{\pi(x-7)}{10}\right), \quad 7 \le x \le 17$$

a Show that $k = \dfrac{\pi}{20}$.

b Hence find the percentage of 2-year-old children whose mass is:

i greater than 16 kg

ii between 12 kg and 13 kg.

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9 A probability density function for the lifetime, T hours, of Electra light bulbs has rule

$$f(t) = ke^{\left(\frac{-t}{200}\right)}, \quad t > 0$$

a Find the value of the constant k.

b Find the probability that an Electra light bulb will last more than 1000 hours.

10 A random variable X has a probability density function given by

$$f(x) = \begin{cases} k(1+x) & -1 \le x \le 0 \\ k(1-x) & 0 < x \le 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

where $k > 0$.

a Sketch the graph of the probability density function. **b** Evaluate k.

c Find the probability that X lies between -0.5 and 0.5.

11 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $\Pr(0.25 < X < 0.75)$ and illustrate this on your graph.

12 A random variable X has a probability density function f with the rule:

$$f(x) = \begin{cases} \frac{1}{100}(10+x) & \text{if } -10 < x \le 0 \\ \frac{1}{100}(10-x) & \text{if } 0 < x \le 10 \\ 0 & \text{if } x \le -10 \text{ or } x > 10 \end{cases}$$

a Sketch the graph of f. **b** Find $\Pr(-1 \le X < 1)$.

13 The life, X hours, of a type of light bulb has a probability density function with the rule:

$$f(x) = \begin{cases} \frac{k}{x^2} & x > 1000 \\ 0 & x \le 1000 \end{cases}$$

a Evaluate k. **b** Find the probability that a bulb will last at least 2000 hours.

14 The weekly demand for petrol, X (in thousands of litres), at a particular service station is a random variable with probability density function:

$$f(x) = \begin{cases} 2\left(1 - \frac{1}{x^2}\right) & 1 \le x \le 2 \\ 0 & x < 1 \text{ or } x > 2 \end{cases}$$

a Determine the probability that more than 1.5 thousand litres are bought in one week.

b Determine the probability that the demand for petrol in one week is less than 1.8 thousand litres, given than it is more than 1.5 thousand litres.

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15 The length of time, X minutes, between the arrival of customers at an ATM is a random variable with probability density function:

$$f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Find the probability that more than 8 minutes elapses between successive customers.

b Find the probability that more than 12 minutes elapses between successive customers, given that more than 8 minutes has passed.

16 A random variable X has density function given by

$$f(x) = \begin{cases} 0.2 & -1 < x \leq 0 \\ 0.2 + 1.2x & 0 < x \leq 1 \\ 0 & x \leq -1 \text{ or } x > 1 \end{cases}$$

a Find $\Pr(X \leq 0.5)$.

b Hence find $\Pr(X > 0.5 \mid X > 0.1)$.

17 The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Sketch the graph of f.

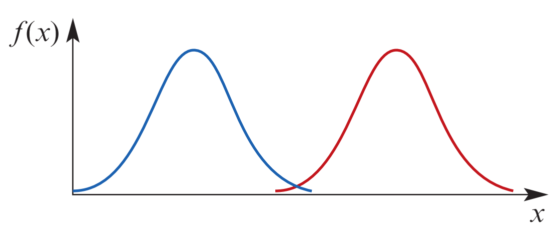
b Find:

i $\Pr(X < 0.5)$ **ii** $\Pr(X \geq 1)$ **iii** $\Pr(X \geq 1 \mid X > 0.5)$

10B Mean and median for a continuous random variable

The centre is an important summary feature of a probability distribution.

The following diagram shows two probability distributions which are identical except for their centres.



More than one measure of centre may be determined for a continuous random variable, and each gives useful information about the random variable under consideration. The most generally useful measure of centre is the mean.

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Mean

We defined the mean for a discrete random variable in Section 8D. We can also define the mean for a continuous random variable.

For a continuous random variable X with probability density function f, the **mean** or **expected value** of X is given by

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

provided the integral exists. The mean is denoted by the Greek letter μ (mu).

If $f(x) = 0$ for all $x \notin [c, d]$, then $\mathrm{E}(X) = \int_{c}^{d} x f(x)\, dx$.

This definition is consistent with the definition of the expected value for a discrete random variable. As in the case of a discrete random variable, the expected value of a continuous random variable is the long-run average value of the variable. For example, consider the daily demand for petrol at a service station. The mean of this variable tells us the average daily demand for petrol over a very long period of time.

Example 6

Find the expected value of the random variable X which has probability density function with rule:

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

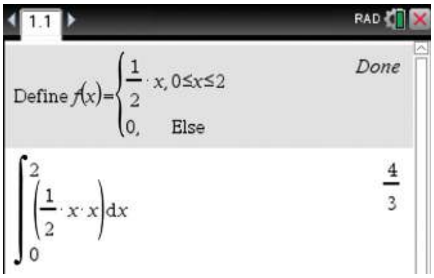
Solution

By definition,

$$\begin{aligned} \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx &= \int_{0}^{2} x \times 0.5x\, dx \qquad \text{since } f(x) = 0 \text{ elsewhere} \\ &= 0.5 \int_{0}^{2} x^2\, dx \\ &= 0.5 \left[\frac{x^3}{3} \right]_0^2 \\ &= \frac{4}{3} \end{aligned}$$

Using the TI-Nspire

Define the function f as shown; access the piecewise function template using [template key].



Notes:

- Leave the domain for the last function piece blank; it will autofill as 'Else'.
- To obtain an exact answer, enter $\frac{1}{2}x$ instead of $0.5x$.

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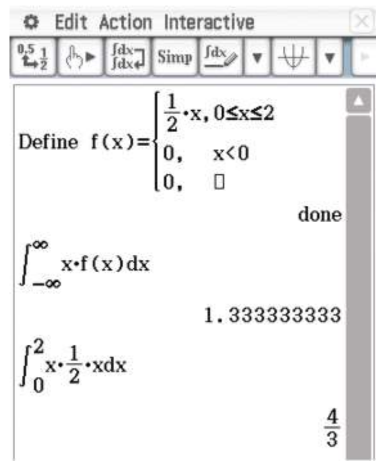
Using the Casio ClassPad

- Tap the piecewise template twice.
- Define the function f as shown.
- Find $\mathrm{E}(X)$ by evaluating the definite integral as shown.

Note: Using the defined function to find $\mathrm{E}(X)$ gives the decimal answer only.



The mean of a function of X is calculated as follows. (In this case, the function of X is denoted by $g(X)$ and is the composition of the random variable X followed by the function g.)

The expected value of $g(X)$ is given by

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

provided the integral exists.

Example 7

Let X be a random variable with probability density function f given by

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Find:

a the expected value of X^2

b the expected value of e^X.

Solution

a
$$\begin{aligned}\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\, dx \\ &= \int_0^2 x^2 \times 0.5x\, dx \\ &= 0.5 \int_0^2 x^3\, dx \\ &= 0.5 \left[\frac{x^4}{4}\right]_0^2 \\ &= 2\end{aligned}$$

b
$$\begin{aligned}\mathrm{E}(e^X) &= \int_{-\infty}^{\infty} e^x f(x)\, dx \\ &= \int_0^2 e^x f(x)\, dx \\ &= \int_0^2 e^x \times 0.5x\, dx \\ &= 4.195\end{aligned}$$

correct to three decimal places.

Where g is a linear function:

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad \text{(for } a, b \text{ constant)}$$

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Percentiles and the median

Another value of interest is the value of X which bounds a particular area under the probability density function. For example, a teacher may wish determine the mark, p, below which lie 75% of all students' marks. This is called the 75th percentile of the population, and is found by solving

$$\int_{-\infty}^{p} f(x)\, dx = 0.75$$

This can be stated more generally:

Percentiles

The value p of X which is the solution of an equation of the form

$$\int_{-\infty}^{p} f(x)\, dx = q$$

is called a **percentile** of the distribution.

For example, the 75th percentile is the value p found by taking $q = 75\% = 0.75$.

Example 8

The duration of telephone calls to the order department of a large company is a random variable, X minutes, with probability density function:

$$f(x) = \begin{cases} \frac{1}{3}e^{-\frac{x}{3}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find the value of a such that 90% of phone calls last less than a minutes.

Solution

To find the value of a, solve the equation:

$$\int_{0}^{a} \frac{1}{3}e^{-\frac{x}{3}}\, dx = 0.9$$

$$\left[-e^{-\frac{x}{3}}\right]_0^a = 0.9$$

$$1 - e^{-\frac{a}{3}} = 0.9$$

$$-\frac{a}{3} = \ln 0.1$$

$$\therefore \quad a = 3 \ln 10$$

$$= 6.908 \quad \text{(correct to three decimal places)}$$

So 90% of the calls to this company last less than 6.908 minutes.

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A percentile of special interest is the **median**, or 50th percentile. The median is the middle value of the distribution. That is, the probability of X taking a value below the median is 0.5, and the probability of X taking a value above the median is 0.5. Thus, if m is the median value of the distribution, then

$$P(X \le m) = P(X > m) = 0.5$$

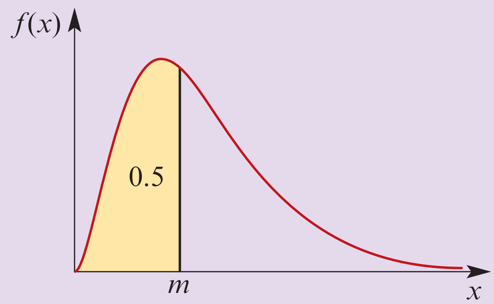
Graphically, the median is the value of the random variable which divides the area under the probability density function in half.

The median

The median is another measure of centre for a continuous probability distribution.

The median, m, of a continuous random variable X is the value of X such that

$$\int_{-\infty}^{m} f(x)\,dx = 0.5$$



Example 9

Suppose the probability density function of weekly sales of topsoil, X (in tonnes), is given by the rule:

$$f(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Find the median value of X, and interpret.

Solution

The median m is such that

$$\int_0^m 2(1-x)\,dx = 0.5$$

$$2\left[x - \frac{x^2}{2}\right]_0^m = 0.5$$

$$2m - m^2 = 0.5$$

$$m^2 - 2m + 0.5 = 0$$

$$\therefore \quad m = 0.293 \text{ or } m = 1.707$$

But since $0 \le x \le 1$, the median is $m = 0.293$ tonnes.

This means that, in the long run, 50% of weekly sales will be less than 0.293 tonnes, and 50% will be more.

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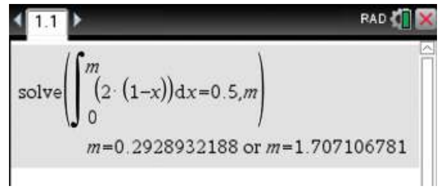
Using the TI-Nspire

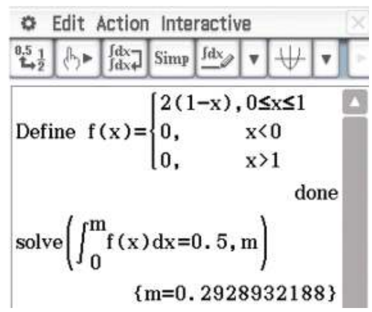
This is an application of integration.

- Solve the definite integral equal to 0.5 as shown to find m (the median value).
- Since $0 \le x \le 1$, the median is $m = 0.293$.



Using the Casio ClassPad

- Define the function f.
- Solve the definite integral equal to 0.5 as shown to find m (the median value).



Summary 10B

For a continuous random variable X with probability density function f:

- the **mean** or **expected value** of X is given by $\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$
- the expected value of $g(X)$ is given by $\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$
- the **median** of X is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0.5$

Skill-sheet

Exercise 10B

Example 6

1 Find the mean, E(X), of the continuous random variables with the following probability density functions:

a $f(x) = 2x, \ 0 < x < 1$

b $f(x) = \dfrac{1}{2\sqrt{x}}, \ 0 < x < 1$

c $f(x) = 6x(1 - x), \ 0 < x < 1$

d $f(x) = \dfrac{1}{x^2}, \ x \ge 1$

2 For each of the following, use your calculator to check that f is a probability density function and then to find the mean, E(X), of the corresponding continuous random variable:

a $f(x) = \sin x, \ 0 < x < \dfrac{\pi}{2}$

b $f(x) = \ln x, \ 1 < x < e$

c $f(x) = \dfrac{1}{\sin^2 x}, \ \dfrac{\pi}{4} < x < \dfrac{\pi}{2}$

d $f(x) = -4x \ln x, \ 0 < x < 1$

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3 A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} 2x^3 - x + 1 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find μ, the mean value of X.

b Find the probability that X takes a value less than or equal to the mean.

4 Consider the probability density function given by

$$f(x) = \frac{1}{2\pi}(1 + \cos x), \quad -\pi \le x \le \pi$$

Find the expected value of X.

5 A random variable Y has the probability density function:

$$f(y) = \begin{cases} Ay & 0 \le y \le B \\ 0 & y < 0 \text{ or } y > B \end{cases}$$

Find A and B if the mean of Y is 2.

Example 7

6 A random variable X has the probability density function given by

$$f(x) = \begin{cases} 12x^2(1 - x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find $\mathrm{E}\left(\dfrac{1}{X}\right)$.

b Find $\mathrm{E}(e^X)$.

Example 8

7 The time, X seconds, between arrivals of particles at a radiation counter has been found to have a probability density function f with the rule:

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \ge 0 \end{cases}$$

a Find $\mathrm{P}(X \le 1)$.

b Find $\mathrm{P}(1 \le X \le 2)$.

c Find the median, m, of X.

8 The random variable X has a probability density function given by

$$f(x) = \begin{cases} k & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find the value of k.

b Find the median, m, of X.

9 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 5(1 - x)^4 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Find the median, m, of X correct to four decimal places.

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10 Suppose that the time (in minutes) between telephone calls received at a pizza restaurant has the probability density function:

$$f(x) = \begin{cases} \frac{1}{4}e^{-\frac{x}{4}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the median time between calls.

Example 9 **11** A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x < 0 \text{ or } x \geq 2 \end{cases}$$

a Find μ, the expected value of X.

b Find m, the median value of X.

12 Let the probability density function of X be given by

$$f(x) = \begin{cases} 30x^4(1 - x) & 0 < x < 1 \\ 0 & x \leq 0 \text{ or } x \geq 1 \end{cases}$$

a Find the expected value, μ, of X.

b Find the median value, m, of X and hence show the mean is less than the median.

13 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = \frac{\pi}{20}\sin\left(\frac{\pi(x - 7)}{10}\right), \quad 7 \leq x \leq 17$$

Find the median value, m, of X.

14 A random variable X has density function given by

$$f(x) = \begin{cases} 0.2 & -1 \leq x \leq 0 \\ 0.2 + 1.2x & 0 < x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

a Find μ, the expected value of X.

b Find m, the median value of X.

15 The exponential probability distribution describes the distribution of the time between random events, such as phone calls. The general form of the exponential distribution with parameter λ is

$$f(x) = \begin{cases} \frac{1}{\lambda}e^{-\frac{x}{\lambda}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Differentiate $(kx + 1)e^{-kx}$ and hence find an antiderivative of kxe^{-kx}.

b Show that the mean of an exponential random variable is λ.

c On the same axes, sketch the graphs of the distribution for $\lambda = \frac{1}{2}$, $\lambda = 1$ and $\lambda = 2$.

d Describe the effect of varying the value of λ on the graph of the distribution.

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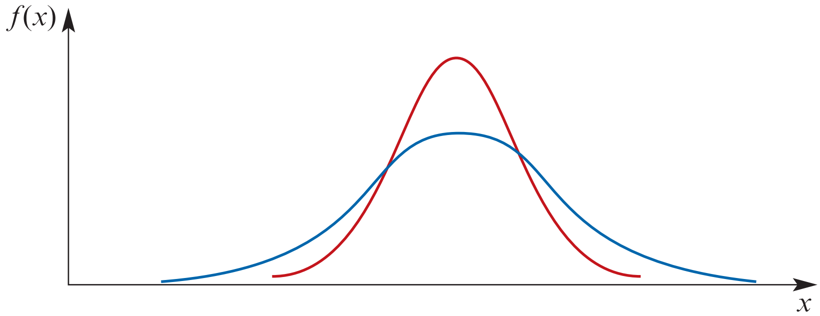
10C Measures of spread

Another important summary feature of a distribution is variation or spread. The following diagram shows two distributions that are identical except for their spreads.



As in the case of centre, there is more than one measure of spread. The most commonly used is the variance, together with its companion measure, the standard deviation. Others that you may be familiar with are the range and the interquartile range.

Variance and standard deviation

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as:

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2] \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx\end{aligned}$$

As for discrete random variables, the variance is usually denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

Variance may be considered as the long-run average value of the square of the distance from X to μ. This means that the variance is not in the same units of measurement as the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

The **standard deviation** of X is defined as:

$$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

The standard deviation is usually denoted by σ.

As in the case of discrete random variables, an alternative (computational) formula for variance is generally used.

To calculate variance, use

$$\mathrm{Var(X)} = \mathrm{E}(X^2) - \mu^2$$

Note: This alternative is not mentioned in the SCSA syllabus.

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Proof The computational form of the expression for variance is derived as follows:

$$\begin{aligned}\text{Var}(X) &= \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx \\ &= \int_{-\infty}^{\infty}(x^2 - 2\mu x + \mu^2) f(x)\,dx \\ &= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \int_{-\infty}^{\infty} 2\mu x f(x)\,dx + \int_{-\infty}^{\infty} \mu^2 f(x)\,dx \\ &= \text{E}(X^2) - 2\mu \int_{-\infty}^{\infty} x f(x)\,dx + \mu^2 \int_{-\infty}^{\infty} f(x)\,dx\end{aligned}$$

Since $\int_{-\infty}^{\infty} x f(x)\,dx = \mu$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we obtain

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - 2\mu^2 + \mu^2 \\ &= \text{E}(X^2) - \mu^2\end{aligned}$$

Example 10

Find the variance and standard deviation of the random variable X which has the probability density function f with rule:

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Solution

Use the computational formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$.

First evaluate $\text{E}(X^2)$:

$$\begin{aligned}\text{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\ &= \int_0^2 x^2 \times 0.5x\,dx \\ &= 0.5\int_0^2 x^3\,dx \\ &= 0.5\left[\frac{x^4}{4}\right]_0^2 \\ &= 0.5 \times 4 \\ &= 2\end{aligned}$$

Since $\text{E}(X) = \dfrac{4}{3}$ from Example 6, we now have

$$\text{Var}(X) = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$$

and $\quad \text{sd}(X) = \sqrt{\dfrac{2}{9}} = \dfrac{\sqrt{2}}{3} = 0.471$ (correct to three decimal places)

It helps to make the standard deviation more meaningful to give it an interpretation which relates to the probability distribution. As already stated for discrete random variables, it is also the case for many continuous random variables that about 95% of the distribution lies within two standard deviations either side of the mean.

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In general, for many continuous random variables X,

$$P(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Example 11

The life of a certain brand of battery, X hours, is a continuous random variable with mean 50 and variance 16. Find an (approximate) interval for the time period for which 95% of the batteries would be expected to last.

Solution

$P(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$

Since $\mu = 50$ and $\sigma = \sqrt{16} = 4$, we expect 95% of the batteries to last between 42 hours and 58 hours.

Interquartile range

The **interquartile range** is the range of the middle 50% of the distribution; it is the difference between the 75th percentile (also known as Q3) and the 25th percentile (also known as Q1).

Example 12

Determine the interquartile range of the random variable X which has the probability density function:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Solution

To find the 25th percentile a, solve:

$$\int_0^a 2x\,dx = 0.25$$
$$[x^2]_0^a = 0.25$$
$$a^2 = 0.25$$
$$\therefore \quad a = \sqrt{0.25} = 0.5$$

To find the 75th percentile b, solve:

$$\int_0^b 2x\,dx = 0.75$$
$$[x^2]_0^b = 0.75$$
$$b^2 = 0.75$$
$$\therefore \quad b = \sqrt{0.75} \approx 0.866$$

Thus the interquartile range is $0.866 - 0.5 = 0.366$, correct to three decimal places.

Note that the negative solutions to these equations were not appropriate, as $0 \le x \le 1$.

Summary 10C

- To calculate the **variance** of a continuous random variable X, use

  $$\text{Var}(X) = \text{E}(X^2) - \mu^2$$

- The **standard deviation** of X is defined by $\sigma = \sqrt{\text{Var}(X)}$.

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- The **interquartile range** of X is

  $$\text{IQR} = b - a$$

  where a and b are such that

  $$\int_{-\infty}^{a} f(x)\,dx = 0.25 \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx = 0.75$$

  and where f is the probability density function of X.

Exercise 10C

Example 10

1 A random variable X has probability density function:

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & x \le 0 \text{ or } x \ge 1 \end{cases}$$

Find the variance of X, and hence find the standard deviation of X.

Example 11

2 The life of a certain brand of light bulb, X hours, is a continuous random variable with mean 400 and variance 64. Find an (approximate) interval for the time period for which 95% of the light bulbs would be expected to last.

Example 12

3 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find a such that $P(X \le a) = 0.25$.

b Find b such that $P(X \le b) = 0.75$.

c Find the interquartile range of X.

4 A random variable X has the probability density function given by

$$f(x) = \begin{cases} 0.5e^{x} & x \le 0 \\ 0.5e^{-x} & x > 0 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find the interquartile range of X, giving your answer correct to three decimal places.

5 A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \dfrac{k}{x} & 1 \le x \le 9 \\ 0 & x < 1 \text{ or } x > 9 \end{cases}$$

a Find the value of k.

b Find the mean and variance of X, giving your answer correct to three decimal places.

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6 A continuous random variable X has density function f given by

$$f(x) = \begin{cases} 0 & x < 0 \\ 2 - 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

a Find the interquartile range of X.
b Find the mean and variance of X.

7 A random variable X has probability density function f with the rule:

$$f(x) = \begin{cases} 0 & x < 0 \\ 2xe^{-x^2} & x \ge 0 \end{cases}$$

Find the interquartile range of X.

8 A random variable X has a probability density function given by

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{2} & 0 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

a Find the interquartile range of X.
b Find the mean and variance of X.

9 The queuing time, X minutes, of a traveller at the ticket office of a large railway station has probability density function f defined by

$$f(x) = \begin{cases} kx(100 - x^2) & 0 \le x \le 10 \\ 0 & x > 10 \text{ or } x < 0 \end{cases}$$

a Find the value of k.
b Find the mean of the distribution.
c Find the standard deviation of the distribution, correct to two decimal places.

10 A probability density function is given by

$$f(x) = \begin{cases} k(a^2 - x^2) & -a \le x \le a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

a Find k in terms of a.
b Find the value of a which gives a standard deviation of 2.

11 A continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} k(3 - x) & 0 \le x \le 3 \\ k(x - 3) & 3 < x \le 6 \\ 0 & x > 6 \text{ or } x < 0 \end{cases}$$

where k is a constant.

a Sketch the graph of f.
b Hence, or otherwise, find the value of k.
c Verify that the mean of X is 3.
d Find $\text{Var}(X)$.

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10D Properties of mean and variance

The expected value of a function of X is not necessarily equal to that function of the expected value of X. That is, in general,

$$\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$$

An exception is the case where the function g is linear: the mean of a linear function of X is equal to the linear function of the mean of X.

The mean and variance of $aX + b$

For any continuous random variable X,

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

Proof The validity of this statement can be readily demonstrated:

$$\begin{aligned} \mathrm{E}(aX + b) &= \int_{-\infty}^{\infty} (ax + b) f(x)\, dx \\ &= \int_{-\infty}^{\infty} ax f(x)\, dx + \int_{-\infty}^{\infty} b f(x)\, dx \\ &= a \int_{-\infty}^{\infty} x f(x)\, dx + b \int_{-\infty}^{\infty} f(x)\, dx \\ &= a\mathrm{E}(X) + b \qquad \left(\text{since } \int_{-\infty}^{\infty} f(x)\, dx = 1\right) \end{aligned}$$

We can also obtain a formula for the variance of a linear function of X.

For any continuous random variable X,

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$$

Proof Consider the variance of a linear function of X:

$$\mathrm{Var}(aX + b) = \mathrm{E}[(aX + b)^2] - [\mathrm{E}(aX + b)]^2$$

Now $\quad [\mathrm{E}(aX + b)]^2 = [a\mathrm{E}(X) + b]^2 = (a\mu + b)^2 = a^2\mu^2 + 2ab\mu + b^2$

and
$$\begin{aligned} \mathrm{E}[(aX + b)^2] &= \mathrm{E}(a^2X^2 + 2abX + b^2) \\ &= a^2\mathrm{E}(X^2) + 2ab\mu + b^2 \end{aligned}$$

Thus
$$\begin{aligned} \mathrm{Var}(aX + b) &= a^2\mathrm{E}(X^2) + 2ab\mu + b^2 - a^2\mu^2 - 2ab\mu - b^2 \\ &= a^2\mathrm{E}(X^2) - a^2\mu^2 \\ &= a^2\mathrm{Var}(X) \end{aligned}$$

Although initially the absence of b in the variance may seem surprising, on reflection it makes sense that adding a constant merely translates the probability density function, and has no effect on its spread.

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Example 13

Suppose that X is a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 2$.

a Find $E(2X + 1)$. **b** Find $\text{Var}(1 - 3X)$.

Solution

a $E(2X + 1) = 2E(X) + 1$
$= 2 \times 10 + 1$
$= 21$

b $\text{Var}(1 - 3X) = (-3)^2\text{Var}(X)$
$= 9 \times 2$
$= 18$

Summary 10D

Linear function of a continuous random variable:

$$E(aX + b) = aE(X) + b$$
$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

Exercise 10D

1 The amount of flour used each day in a bakery is a continuous random variable X with a mean of 4 tonnes. The cost of the flour is $C = 300X + 100$. Find $E(C)$.

2 The mean weight of the oranges grown at a certain orchard is 100 g, with a standard deviation of 5 g. The oranges are sold in bags of 10. If the weight of the empty bag is 10 g, find the mean and standard deviation of the weight of a bag of oranges.

3 For certain glass ornaments, the proportion of impurities per ornament, X, is a random variable with a density function given by

$$f(x) = \begin{cases} \dfrac{3x^2}{2} + x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

The value of each ornament (in dollars) is $V = 100 - 1.5X$.

a Find $E(X)$ and $\text{Var}(X)$.

b Hence find the mean and standard deviation of V.

Example 13

4 Let X be a random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{3x^2}{2} & \text{if } -1 \le x \le 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$$

Find:

a $E(3X)$ and $\text{Var}(3X)$

b $E(3 - X)$ and $\text{Var}(3 - X)$

c $E(3X + 1)$ and $\text{Var}(3X + 1)$

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5 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} x^2\left(2x + \frac{3}{2}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\mathrm{E}(X)$.

b Hence find $\mathrm{E}(V)$, where $V = 2X + 3$.

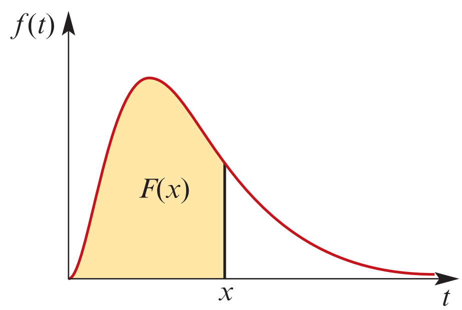
10E Cumulative distribution functions

Another function of importance in describing a continuous random variable is the **cumulative distribution function** or **CDF**. For a continuous random variable X, with probability density function f defined on the interval $[c, d]$, the cumulative distribution function F is given by

$$\begin{aligned} F(x) &= \mathrm{P}(X \leq x) \\ &= \int_c^x f(t)\, dt \end{aligned}$$

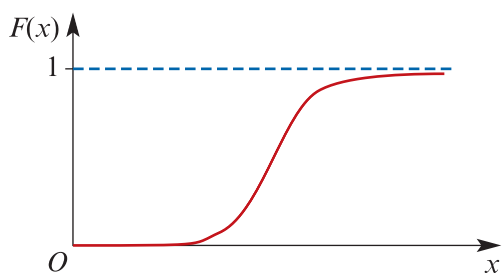
where t is the variable of integration. The cumulative distribution function at a particular value x gives the probability that the random variable X takes a value less than or equal to x.

The diagram on the right shows the relationship between the probability density function f and the cumulative distribution function F.



The function F describes the area under the graph of the probability density function between the lower bound of the domain of f and x. (In the diagram, the lower bound is 0.)

For every continuous random variable X, the cumulative distribution function F is continuous.



Using the general version of the fundamental theorem of calculus, it can be shown that the derivative of the cumulative distribution function is the density function. More precisely, we have $F'(x) = f(x)$, for each value of x at which f is continuous.

There are three important properties of a cumulative distribution function.

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For a continuous random variable X with range $[c, d]$:

1. The probability that X takes a value less than or equal to c is 0. That is, $F(c) = 0$.
2. The probability that X takes a value less than or equal to d is 1. That is, $F(d) = 1$.
3. If x_1 and x_2 are values of X with $x_1 \le x_2$, then $\mathrm{P}(X \le x_1) \le \mathrm{P}(X \le x_2)$. That is,

$$x_1 \le x_2 \quad \text{implies} \quad F(x_1) \le F(x_2)$$

The function F is a **non-decreasing** function.

For a probability density function f defined on $\mathbb{R}$, the cumulative distribution is given by

$$\begin{aligned} F(x) &= \mathrm{P}(X \le x) \\ &= \int_{-\infty}^{x} f(t)\,dt \end{aligned}$$

In this case, we have $F(x) \to 0$ as $x \to -\infty$, and $F(x) \to 1$ as $x \to \infty$.

The importance of the cumulative distribution function is that probabilities for various intervals can be computed directly from $F(x)$.

Example 14

The time, X seconds, that it takes a student to complete a puzzle is a random variable with density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \ge 5 \\ 0 & x < 5 \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Use the cumulative distribution function to find:

i $\mathrm{P}(X \le 7)$ **ii** $\mathrm{P}(X \ge 6)$ **iii** $\mathrm{P}(10 \le X \le 20)$

Solution

a $$\begin{aligned} F(x) = \int_5^x f(t)\,dt &= \int_5^x \frac{5}{t^2}\,dt \\ &= \left[\frac{-5}{t}\right]_5^x \\ &= \frac{-5}{x} + 1 \end{aligned}$$

Thus $F(x) = 1 - \dfrac{5}{x}$ for $x \ge 5$.

b **i** $$\begin{aligned} \mathrm{P}(X \le 7) &= F(7) \\ &= 1 - \frac{5}{7} = \frac{2}{7} \end{aligned}$$

ii $$\begin{aligned} \mathrm{P}(X \ge 6) &= 1 - \mathrm{P}(X < 6) \\ &= 1 - F(6) \\ &= 1 - \left(1 - \frac{5}{6}\right) = \frac{5}{6} \end{aligned}$$

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iii $P(10 \le X \le 20) = P(X \le 20) - P(X < 10)$

$$= F(20) - F(10)$$

$$= \left(1 - \frac{5}{20}\right) - \left(1 - \frac{5}{10}\right) = \frac{1}{4}$$

Summary 10E

- The **cumulative distribution function** of a continuous random variable X is defined by

$$F(x) = P(X \le x)$$

- For a probability density function f defined on $\mathbb{R}$, we have

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

- For each value of x at which f is continuous, we have

$$F'(x) = f(x)$$

Exercise 10E

Example 14

1 The probability density function for a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{5} & \text{if } 0 < x < 5 \\ 0 & \text{if } x \le 0 \text{ or } x \ge 5 \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Hence find $P(X \le 3)$.

2 A continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \le x < 1 \\ \frac{x^3}{5} & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Solve the equation $F(x) = 0.5$ for x to find the median value of X.

3 A random variable X has the cumulative distribution function with rule:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^2} & x \ge 0 \end{cases}$$

a Sketch the graph of $y = F(x)$.

b Find $P(X \ge 2)$.

c Find $P(X \ge 2 \mid X < 3)$.

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4 The continuous random variable X has cumulative distribution function F given by

$$F(x) = \begin{cases} 0 & x < 0 \\ kx^2 & 0 \le x \le 6 \\ 1 & x > 6 \end{cases}$$

a Determine the value of the constant k.

b Calculate $\mathrm{P}\left(\frac{1}{2} \le X \le 1\right)$.

5 The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - \dfrac{10}{x} & x > 10 \\ 0 & x \le 10 \end{cases}$$

Use the cumulative distribution function to determine:

a $\mathrm{P}(X < 30)$

b the median of X

c the interquartile range for X.

6 The maximum daily temperature, X°C, at a ski resort during winter has the cumulative distribution function $F(x) = \dfrac{1}{1 + e^{-0.5x}}$ for $x \in \mathbb{R}$.

a With the help of your calculator, sketch the graph of F.

b Given that the probability density function f of X is continuous everywhere, determine the rule for $f(x)$ and sketch the graph of f.

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Review

Chapter summary

Assign-ment

Nrich

- A **continuous random variable** is one that can take any value in an interval of the real number line.
- A continuous random variable can be described by a **probability density function** f. There are many different probability density functions with different shapes and properties. However, they all have the following two fundamental properties:
  1. For any value of x, the value of $f(x)$ is non-negative. That is,
     $$f(x) \geq 0 \text{ for all } x$$
  2. The total area enclosed by the graph of f and the x-axis is equal to 1. That is,
     $$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

  The probability of X taking a value in the interval (a, b) is found by determining the area under the probability density curve between a and b. That is,
  $$\mathrm{P}(a < X < b) = \int_a^b f(x)\, dx$$
- The **mean** or **expected value** of a continuous random variable X with probability density function f is given by
  $$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$
  provided the integral exists.
- If $g(X)$ is a function of X, then the expected value of $g(X)$ is given by
  $$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$
  provided the integral exists. In general, $\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$.
- The **median** of a continuous random variable X is the value m such that
  $$\int_{-\infty}^{m} f(x)\, dx = 0.5$$
- The **variance** of a continuous random variable X with probability density function f is defined by
  $$\begin{aligned}\sigma^2 = \mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2] \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx\end{aligned}$$
  provided the integral exists. To calculate the variance, use
  $$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$$
- The **standard deviation** of X is defined by
  $$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$
- Linear function of a continuous random variable:
  $$\begin{aligned}\mathrm{E}(aX+b) &= a\mathrm{E}(X) + b \\ \mathrm{Var}(aX+b) &= a^2\mathrm{Var}(X)\end{aligned}$$
- In general, for many continuous random variables X,
  $$\mathrm{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$$

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- The **interquartile range** of X is IQR $= b - a$, where a and b are such that

$$\int_{-\infty}^{a} f(x)\,dx = 0.25 \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx = 0.75$$

and where f is the probability density function of X.
- The **cumulative distribution function** of a continuous random variable X is defined by

$$F(x) = \mathrm{P}(X \le x)$$

- For a probability density function f defined on $\mathbb{R}$, we have $F(x) = \int_{-\infty}^{x} f(t)\,dt$.
- For each value of x at which f is continuous, we have $F'(x) = f(x)$.

Short-answer questions

1 The probability density function of X is given by

$$f(x) = \begin{cases} kx & \text{if } 1 \le x \le \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

a Find k. **b** Find $\mathrm{P}(1 < X < 1.1)$. **c** Find $\mathrm{P}(1 < X < 1.2)$.

2 If the probability density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & x > 1 \text{ or } x < 0 \end{cases}$$

and $\mathrm{E}(X) = \dfrac{2}{3}$, find a and b.

3 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{\sin x}{2} & 0 \le x \le \pi \\ 0 & x > \pi \text{ or } x < 0 \end{cases}$$

Find the median of X.

4 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{4} & 1 \le x < 5 \\ 0 & x < 1 \text{ or } x \ge 5 \end{cases}$$

a Find $\mathrm{P}(1 < X < 3)$. **b** Find $\mathrm{P}(X > 2 \mid 1 < X < 3)$. **c** Find $\mathrm{P}(X > 4 \mid X > 2)$.

5 Consider the random variable X having the probability density function given by

$$f(x) = \begin{cases} 12x^2(1 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $\mathrm{P}(X < 0.5)$ and illustrate this probability on your sketch graph.

c Find $F(x)$, the cumulative distribution function of X.

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Review

6 The probability density function of a random variable X is

$$f(x) = \begin{cases} kx^2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Determine k.

b Find the probability that X is less than $\frac{2}{3}$.

c Find the probability that X is less than $\frac{1}{3}$, given that X is less than $\frac{2}{3}$.

7 Let X be a continuous random variable with probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\mathrm{P}(X < 0.2)$. **b** Find $\mathrm{P}(X < 0.2 \mid X < 0.3)$.

8 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the median value, m, of X.

9 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \frac{x+2}{16} & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\mathrm{E}(X)$. **b** Find a such that $\mathrm{P}(X \le a) = \frac{5}{32}$.

10 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} c(1-x^2) & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find c. **b** Find $\mathrm{E}(X)$.

11 Show that

$$f(x) = \begin{cases} n(1-x)^{n-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability function, where the constant n is a natural number.

12 The probability density function of X is given by

$$f(x) = \begin{cases} \frac{1}{x} & 1 \le x \le e \\ 0 & x > e \text{ or } x < 1 \end{cases}$$

a Find the median value of X. **b** Find the interquartile range of X.

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13 The amount of fluid, X mL, in a can of soft drink is a continuous random variable with mean 330 and standard deviation 5. Find an (approximate) interval for the amount of soft drink contained in 95% of the cans.

14 The weight, X g, of cereal in a packet is a continuous random variable with mean 250 and variance 4. Find an (approximate) interval for the weight of cereal contained in 95% of the packets.

Extended-response questions

1 The distribution of X, the life of a certain electronic component in hours, is described by the following probability density function:

$$f(x) = \begin{cases} \dfrac{a}{100}\left(1 - \dfrac{x}{100}\right) & 100 < x < 1000 \\ 0 & x \le 100 \text{ and } x \ge 1000 \end{cases}$$

a What is the value of a?

b Find the expected value of the life of the components.

c Find the median value of the life of the components.

2 The cumulative distribution function of a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x - x^2 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

a Find $\mathrm{P}(X > 0.5)$. **b** Find a such that $\mathrm{P}(X < a) = 0.8$. **c** Find $\mathrm{E}(X)$ and $\mathrm{E}(\sqrt{X})$.

3 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{\pi}{20}\cos\left(\dfrac{\pi(x-6)}{10}\right) & 1 \le x \le 11 \\ 0 & x < 1 \text{ or } x > 11 \end{cases}$$

a Find the median and the interquartile range of X.

b Find the mean and the variance of X.

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Review

4 A hardware shop sells a certain size nail either in a small packet at \$1 per packet, or loose at \$4 per kilogram. On any shopping day, the number, X, of packets sold is a binomial random variable with number of trials $n = 8$ and probability of success $p = 0.6$, and the weight, Y kg, of nails sold loose is a continuous random variable with probability density function f given by

$$f(y) = \begin{cases} \dfrac{2(y-1)}{25} & 1 \le y \le 6 \\ 0 & y < 1 \text{ or } y > 6 \end{cases}$$

a Find the probability that the weight of nails sold loose on any shopping day will be between 4 kg and 5 kg.

b Calculate the expected money received on any shopping day from the sale of this size nail in the shop.

5 The continuous random variable X has the probability density function f, where

$$f(x) = \begin{cases} \dfrac{x-2}{2} & 2 \le x \le 4 \\ 0 & x < 2 \text{ or } x > 4 \end{cases}$$

By first expanding $(X-c)^2$, or otherwise, find two values of c such that $\mathrm{E}[(X-c)^2] = \dfrac{2}{3}$.

6 The continuous random variable X has probability density function f, where

$$f(x) = \begin{cases} \dfrac{k}{12(x+1)^3} & 0 \le x \le 4 \\ 0 & x < 0 \text{ or } x > 4 \end{cases}$$

a Find k.

b Evaluate $\mathrm{E}(X+1)$. Hence, find the mean of X.

c Use your calculator to verify your answer to part **b**.

d Find the value of $c > 0$ for which $\mathrm{P}(X \le c) = c$.

7 The yield of a variety of corn has probability density function:

$$f(x) = \begin{cases} kx & 0 \le x < 2 \\ k(4-x) & 2 \le x \le 4 \\ 0 & x < 0 \text{ or } x > 4 \end{cases}$$

a Find k.

b Find the expected value, μ, and the variance of the yield of corn.

c Find the probability $\mathrm{P}(\mu - 1 < X < \mu + 1)$.

d Find the value of a such that $\mathrm{P}(X > a) = 0.6$, giving your answer correct to one decimal place.

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8 Continuous uniform distributions

a A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \dfrac{1}{7} & \text{if } 1 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases}$$

i Sketch the graph of f.

ii Find $F(x)$, the cumulative distribution function of X.

iii Find $\mathrm{E}(X)$, the expected value of X.

iv Find $\mathrm{Var}(X)$, the variance of X.

b In general, a continuous random variable X is said to have a **uniform distribution** if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where a and b are real constants with $a < b$. Find:

i the cumulative distribution function of X

ii $\mathrm{E}(X)$

iii $\mathrm{Var}(X)$

c Suppose that the amount of time that a person must wait for a bus is uniformly distributed between 0 and 15 minutes, inclusive. Let X be the waiting time in minutes. Find:

i the cumulative distribution function of X

ii $\mathrm{E}(X)$

iii $\mathrm{Var}(X)$

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11 The normal distribution

In this chapter

Syllabus references

Topic: Normal distributions

Subtopics: 4.2.5 – 4.2.7

The most useful continuous distribution, and one that occurs frequently, is the normal distribution. The probability density functions of normal random variables are symmetric, single-peaked, bell-shaped curves.

Data sets occurring in nature will often have such a bell-shaped distribution, as measurements on many random variables are closely approximated by a normal probability distribution.

Variables such as height, weight, IQ and the volume of milk in a milk carton are all examples of normally distributed random variables.

As well as helping us to understand better the behaviour of many real-world variables, the normal distribution also underpins the development of statistical estimation, which is the topic of Chapter 12.

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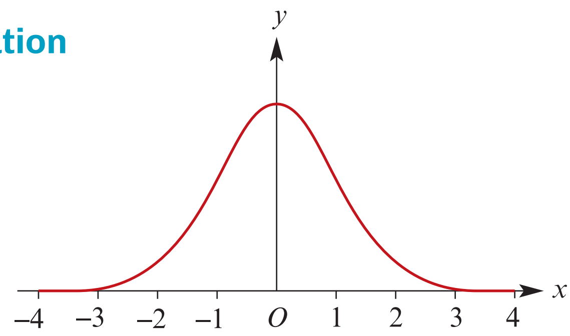
11A The normal distribution

The standard normal distribution

The simplest form of the normal distribution is a random variable with probability density function f given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

The domain of f is $\mathbb{R}$.



Because it is the simplest form of the normal distribution, it is given a special name: the **standard normal distribution**. The graph of the standard normal distribution is as shown.

The graph of the standard normal probability density function f is symmetric about $x = 0$, since $f(-x) = f(x)$. That is, the function f is even.

The line $y = 0$ is an asymptote: as $x \to \pm\infty$, $y \to 0$. Almost all of the area under the probability density function lies between $x = -3$ and $x = 3$.

The mean and standard deviation of the standard normal distribution

It can be seen from the graph that the mean and median of this distribution are the same, and are equal to 0. While the probability density function for the standard normal distribution cannot be integrated exactly, the value of the mean can be verified by observing the symmetry of the two integrals formed below. One is just the negative of the other.

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}x^2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} x e^{-\frac{1}{2}x^2}\, dx + \int_{-\infty}^{0} x e^{-\frac{1}{2}x^2}\, dx \right) \end{aligned}$$

Thus the mean, $\mathrm{E}(X)$, of the standard normal distribution is 0.

What can be said about the standard deviation of this distribution? It can be shown that

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2}\, dx = 1$$

Therefore

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 1 - 0 = 1 \quad \text{and} \quad \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} = 1$$

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Standard normal distribution

A random variable with the standard normal distribution has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

The standard normal distribution has mean $\mu = 0$ and standard deviation $\sigma = 1$.

Henceforth, we will denote the random variable of the standard normal distribution by Z.

The general normal distribution

The normal distribution does not apply just to the special circumstances where the mean is 0 and the standard deviation is 1.

The rule for the general normal distribution

If X is a **normally distributed random variable** with mean μ and standard deviation σ, then the probability density function of X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

and

$$\mathrm{P}(X \leq a) = \mathrm{P}\left(Z \leq \frac{a-\mu}{\sigma}\right)$$

where Z is the random variable of the standard normal distribution.

The general form of the normal density function involves two parameters, μ and σ, which are the mean (μ) and the standard deviation (σ) of that particular distribution.

When a random variable has a distribution described by a normal density function, the random variable is said to have a **normal distribution**.

As with all probability density functions, the normal density function has the fundamental properties that:

- probability corresponds to an area under the curve
- the total area under the curve is 1.

However, it has some additional special properties.

The graph of a normal density function is symmetric and bell-shaped and:

- its **centre** is determined by the **mean** of the distribution
- its **width** is determined by the **standard deviation** of the distribution.

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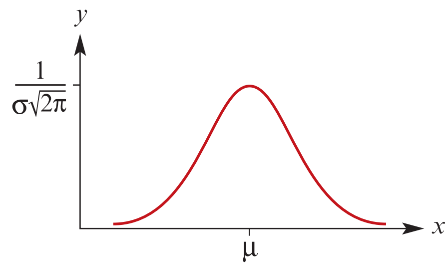
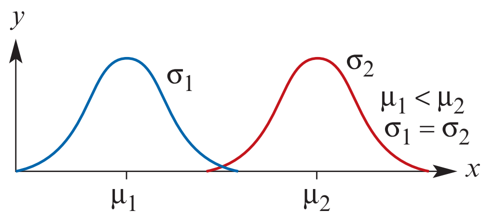
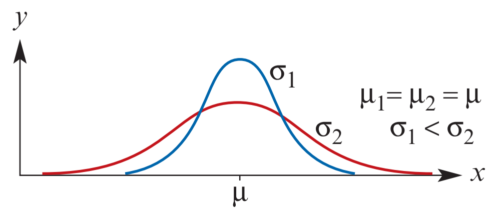
The graph of $y = f(x)$ is shown on the right.

The graph is symmetric about the line $x = \mu$, and has a maximum value of $\dfrac{1}{\sigma\sqrt{2\pi}}$, which occurs when $x = \mu$.



Thus the *location* of the curve is determined by the value of μ, and the *steepness* of the curve by the value of σ.





Irrespective of the values of the mean and standard deviation of a particular normal density function, the area under the curve within a given number of standard deviations from the mean is always the same.

Example 1

On the same set of axes, sketch the graphs of the probability density functions of the standard normal distribution and the normal distribution with:

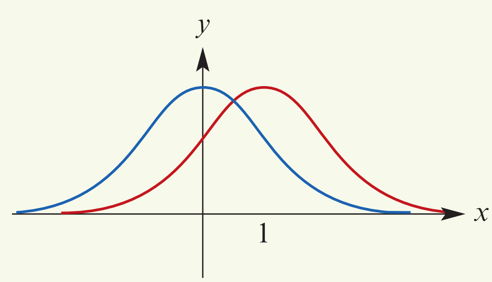
a mean 1 and standard deviation 1

b mean 1 and standard deviation 2.

(A calculator can be used to help.)

Solution

a The graph has been translated 1 unit in the positive direction of the x-axis.



The rules of the two density functions are $y = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and $y = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}$.

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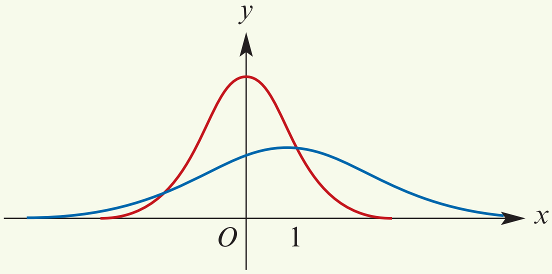
b The graph has been dilated parallel to the x-axis by factor 2 and parallel to the y-axis by factor $\frac{1}{2}$, and then translated 1 unit in the positive direction of the x-axis.



The rules of the two density functions are $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and $y = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{2}\right)^2}$.

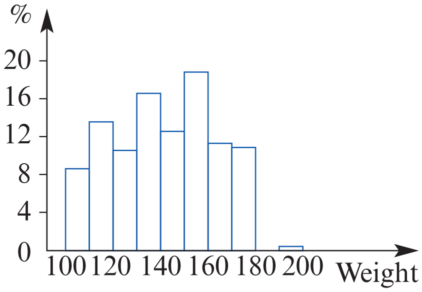
Exercise 11A

Example 1

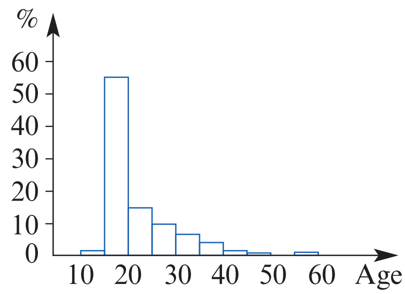
1 Both the random variables X_1 and X_2 are normally distributed, with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively. If $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$, sketch both distributions on the same diagram.

2 Which of the following data distributions are approximately normally distributed?

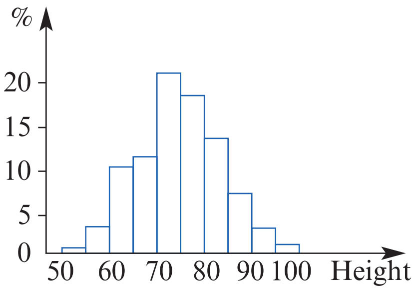
a



b



c



3 Consider the normal probability density function:

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}, \quad x \in \mathbb{R}$$

a Use your calculator to find $\int_{-\infty}^{\infty} f(x)\,dx$.

b **i** Express $E(X)$ as an integral.

ii Use your calculator to evaluate the integral found in part **i**.

c **i** Write down an expression for $E(X^2)$. **ii** What is the value of $E(X^2)$?

iii What is the value of σ?

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4 Consider the normal probability density function:

$$f(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2}, \quad x \in \mathbb{R}$$

a Use your calculator to find $\int_{-\infty}^{\infty} f(x)\, dx$.

b **i** Express $\mathrm{E}(X)$ as an integral.

ii Use your calculator to evaluate the integral found in part **i**.

c **i** Write down an expression for $\mathrm{E}(X^2)$. **ii** What is the value of $\mathrm{E}(X^2)$?

iii What is the value of σ?

5 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{10}\right)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

6 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+3)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

7 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{\sqrt{18\pi}} e^{-\frac{1}{2}\left(\frac{x}{3}\right)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

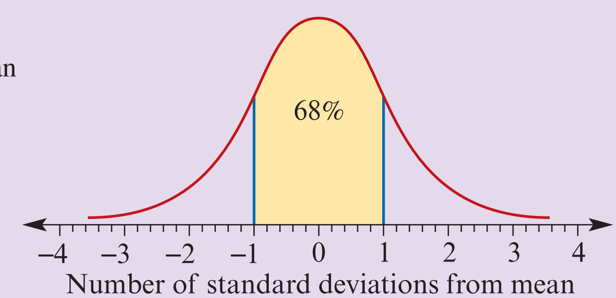
11B Standardisation and the 68–95–99.7% rule

For a set of data values that are normally distributed, approximately 68% of the values will lie within one standard deviation of the mean, approximately 95% of the values will lie within two standard deviations of the mean, and almost all (99.7%) within three standard deviations. This gives rise to what is known as the **68–95–99.7% rule**.

The 68–95–99.7% rule

For a normally distributed random variable, approximately:

- 68% of the values lie within one standard deviation of the mean



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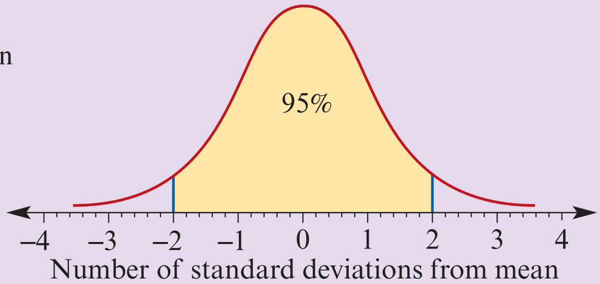
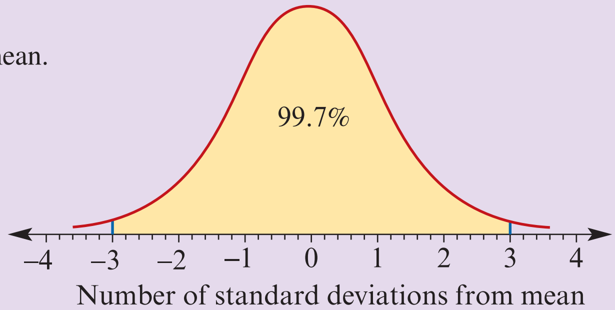
- 95% of the values lie within two standard deviations of the mean



- 99.7% of the values lie within three standard deviations of the mean.



If we know that a random variable is approximately normally distributed, and we know its mean and standard deviation, then we can use the 68–95–99.7% rule to quickly make some important statements about the way in which the data values are distributed.

Example 2

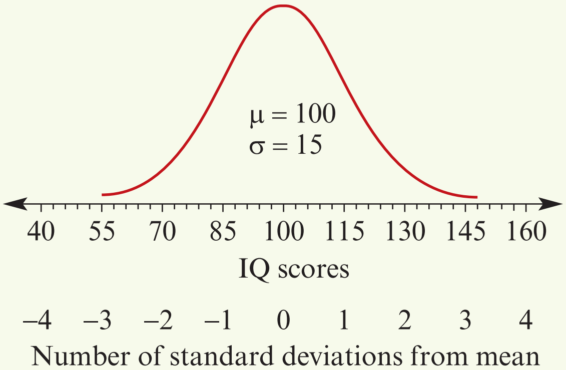
Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

Approximately what percentage of the distribution lies within one, two or three standard deviations of the mean?

Solution

Since the scores are normally distributed with $\mu = 100$ and $\sigma = 15$, the 68–95–99.7% rule means that approximately:

- 68% of the scores will lie between 85 and 115
- 95% of the scores will lie between 70 and 130
- 99.7% of the scores will lie between 55 and 145.



Note: In this example, we are using a continuous distribution to model a discrete situation.

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Statements can also be made about the percentage of scores that lie in the tails of the distribution, by using the symmetry of the distribution and noting that the total area under the curve is 100%.

Example 3

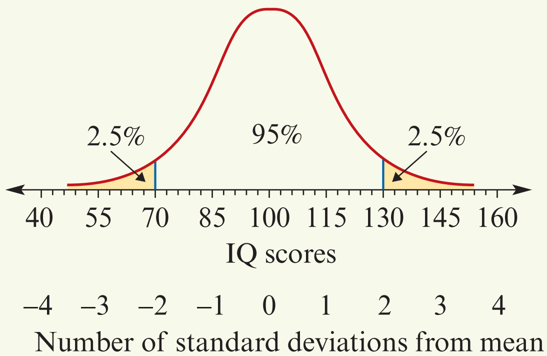
From Example 2, we know that 95% of the scores in the IQ distribution lie between 70 and 130 (that is, within two standard deviations of the mean). What percentage of the scores are *more* than two standard deviations above or below the mean (in this instance, less than 70 or greater than 130)?

Solution

If we focus our attention on the tails of the distribution, we see that 5% of the IQ scores lie outside this region.

Using the symmetry of the distribution, we can say that 2.5% of the scores are below 70, and 2.5% are above 130.

That is, if you obtained a score greater than 130 on this test, you would be in the top 2.5% of the group.



Standardised values

Clearly, the standard deviation is a natural measuring stick for normally distributed data. For example, a person who obtained a score of 112 on an IQ test with a mean of $\mu = 100$ and a standard deviation of $\sigma = 15$ is less than one standard deviation from the mean. Their score is typical of the group as a whole, as it lies well within the middle 68% of scores. In contrast, a person who scored 133 has done exceptionally well; their score is more than two standard deviations from the mean and this puts them in the top 2.5%.

Because of the additional insight provided, it is usual to convert normally distributed data to a new set of units which shows the number of standard deviations each data value lies from the mean of the distribution. These new values are called **standardised values** or ***z*-values**. To standardise a data value x, we first subtract the mean μ of the normal random variable from the value and then divide the result by the standard deviation σ. That is,

$$\text{standardised value} = \frac{\text{data value} - \text{mean of the normal curve}}{\text{standard deviation of the normal curve}}$$

or symbolically,

$$z = \frac{x - \mu}{\sigma}$$

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Standardised values can be positive or negative:

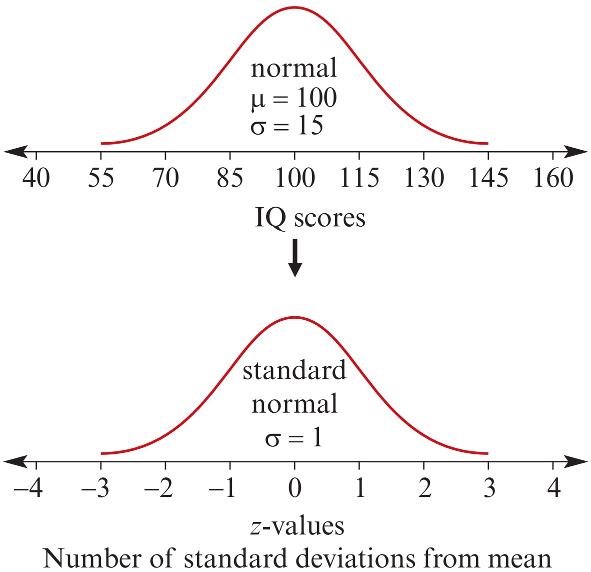
- A **positive z-value** indicates that the data value it represents lies **above** the mean.
- A **negative z-value** indicates that the data value it represents lies **below** the mean.

For example, an IQ score of 90 lies *below* the mean and has a standardised value of

$$z = \frac{x - \mu}{\sigma} = \frac{90 - 100}{15} = \frac{-10}{15} \approx -0.67$$

There are as many different normal curves as there are values of μ and σ. But if the measurement scale is changed to 'standard deviations from the mean' or z-values, all normal curves reduce to the same normal curve with mean $\mu = 0$ and standard deviation $\sigma = 1$.

The figures on the right show how standardising IQ scores transforms a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$ into the standard normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.



Exercise 11B

Example 2

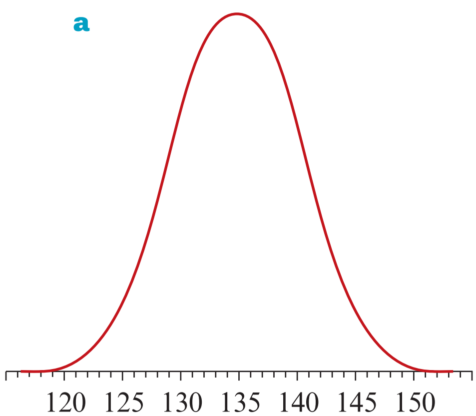
1 The scores obtained on an IQ test can be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. What percentage of scores lie:

a above 115 **b** below 85

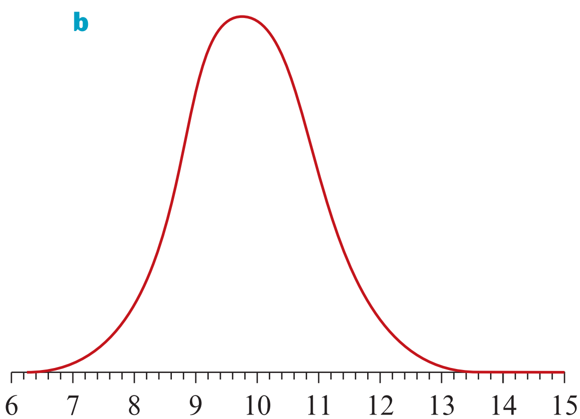
c above 130 **d** below 70?

2 State the values of the mean and standard deviation of the normal distributions shown:

a



b



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Example 3

3 The heights of young women are normally distributed with mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. What percentage of the women would you expect to have heights:

a between 152 cm and 168 cm

b greater than 168 cm

c less than 136 cm?

4 Fill in the blanks in the following paragraph.

The age at marriage of males in the US in the 1980s was approximately normally distributed with a mean of $\mu = 27.3$ years and a standard deviation of $\sigma = 3.1$ years. From this data, we can conclude that in the 1980s about 95% of males married between the ages of ☐ and ☐.

5 Fill in the blanks in the following statement of the 68–95–99.7% rule.

For any normal distribution, about:

- 68% of the values lie within ☐ standard deviation of the mean
- ☐% of the values lie within two standard deviations of the mean
- ☐% of the values lie within ☐ standard deviations of the mean.

6 If you are told that in Australian adults, nostril width is approximately normally distributed with a mean of $\mu = 2.3$ cm and a standard deviation of $\sigma = 0.3$ cm, find the percentage of people with nostril widths less than 1.7 cm.

7 The distribution of IQ scores for the inmates of a certain prison is approximately normal with mean $\mu = 85$ and standard deviation $\sigma = 15$.

a What percentage of the prison population have an IQ of 100 or higher?

b If someone with an IQ of 70 or less can be classified as having special needs, what percentage of the prison population could be classified as having special needs?

8 The distribution of the heights of navy officers was found to be normal with a mean of $\mu = 175$ cm and a standard deviation of $\sigma = 5$ cm. Determine:

a the percentage of navy officers with heights between 170 cm and 180 cm

b the percentage of navy officers with heights greater than 180 cm

c the approximate percentage of navy officers with heights greater than 185 cm.

9 The distribution of blood pressures (systolic) among women of similar ages is normal with a mean of 120 (mm of mercury) and a standard deviation of 10 (mm of mercury). Determine the percentage of women with a systolic blood pressure:

a between 100 and 140

b greater than 130

c greater than 120

d between 90 and 150.

10 The heights of women are normally distributed with mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. What is the standardised value for the height of a woman who is:

a 160 cm tall

b 150 cm tall

c 172 cm tall?

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11 The length of pregnancy for a human is approximately normally distributed with a mean of $\mu = 270$ days and a standard deviation of $\sigma = 10$ days. How many standard deviations away from the mean is a pregnancy of length:

a 256 days **b** 281 days **c** 305 days?

12 Michael scores 85 on the mathematics section of a scholastic aptitude test, the results of which are known to be normally distributed with a mean of 78 and a standard deviation of 5. Cheryl sits for a different mathematics ability test and scores 27. The scores from this test are normally distributed with a mean of 18 and a standard deviation of 6. Assuming that both tests measure the same kind of ability, who has the better score?

13 The following table gives a student's results in Biology and History. For each subject, the table gives the student's mark (x) and also the mean (μ) and standard deviation (σ) for the class.

	Mark (x)	Mean (μ)	Standard deviation (σ)	Standardised mark (z)
Biology	77	68.5	4.9	
History	79	75.3	4.1	

Complete the table by calculating the student's standardised mark for each subject, and use this to determine in which subject the student did best *relative* to her peers.

14 Three students took different tests in French, English and Mathematics:

Student	Subject	Mark (x)	Mean (μ)	Standard deviation (σ)	Standardised mark (z)
Mary	French	19	15	4	
	English	42	35	8	
	Mathematics	20	20	5	
Steve	French	21	23	4	
	English	39	42	3	
	Mathematics	23	18	4	
Sue	French	15	15	5	
	English	42	35	10	
	Mathematics	19	20	5	

a Determine the standardised mark for each student on each test.

b Who is the best student in:

i French **ii** English **iii** Mathematics?

c Who is the best student overall? Give reasons for your answer.

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11C Determining normal probabilities

A CAS calculator can be used to determine areas under normal curves, allowing us to find probabilities for ranges of values other than one, two or three standard deviations from the mean. The following example is for the standard normal distribution, but the same procedures can be used for any normal distribution by entering the appropriate values for μ and σ.

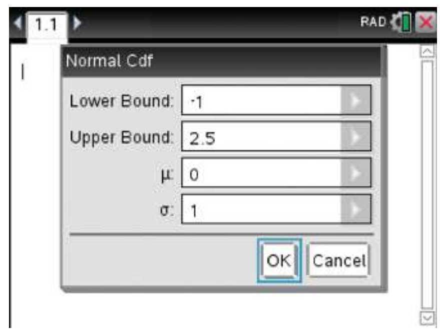
Example 4

Suppose that Z is a standard normal random variable (that is, it has mean $\mu = 0$ and standard deviation $\sigma = 1$). Find:

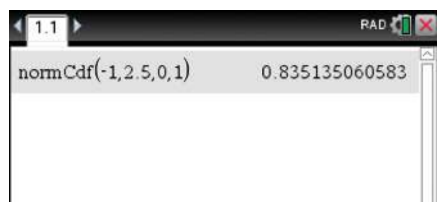
a $P(-1 < Z < 2.5)$ **b** $P(Z > 1)$

Using the TI-Nspire

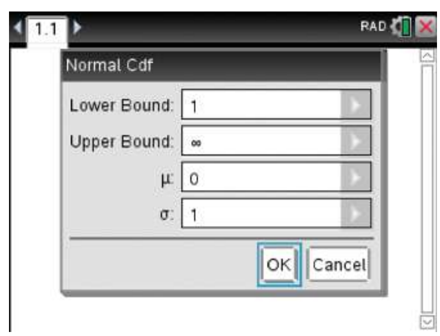
a Use (menu) > **Probability** > **Distributions** > **Normal Cdf** and complete as shown. (Use (tab) or ▼ to move between cells.)



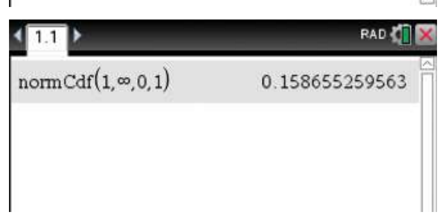
The answer is:
$P(-1 < Z < 2.5) = 0.8351$



b Use (menu) > **Probability** > **Distributions** > **Normal Cdf** and complete as shown. (The symbol ∞ can be found using (π▸) or (ctrl)(📖).)



The answer is:
$P(Z > 1) = 0.1587$



Note: You can enter the commands and parameters directly if preferred. The commands are not case sensitive.

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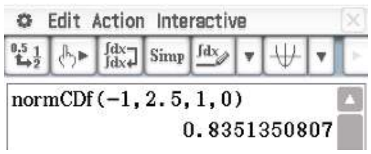
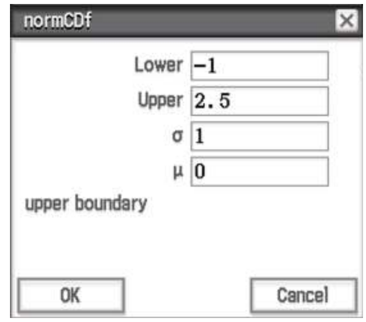
Using the Casio ClassPad

Method 1

a ■ In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Continuous** > **normCDf**.

■ Enter the lower and upper bounds and tap `OK`.





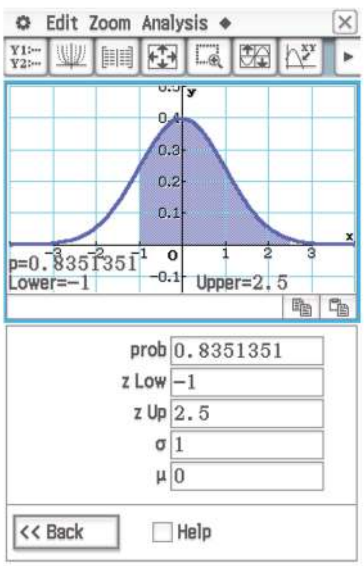
Method 2

■ In **Statistics** [Statistics icon], go to **Calc** > **Distribution** and select **Normal CD**. Tap `Next`.

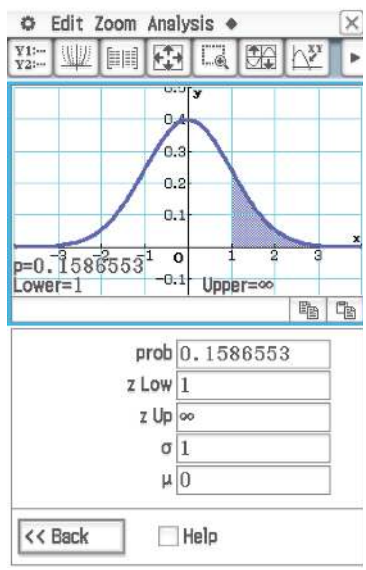
■ Enter values for the lower and upper bounds. Tap `Next` to view the answer.

■ Select [graph icon] to view the graph with the answer.

a



b



The calculator can also be used to determine **percentiles** of any normal distribution.

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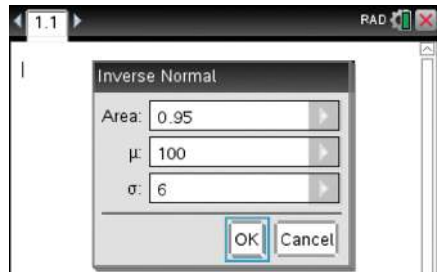
Example 5

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.

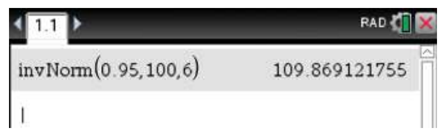
Find k such that $P(X \le k) = 0.95$.

Using the TI-Nspire

Use (menu) > **Probability** > **Distributions** > **Inverse Normal** and complete as shown.



The value of k is 109.869.



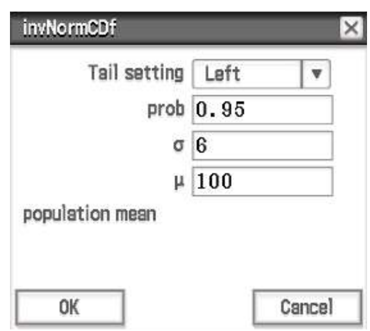
Note: You can enter the command and parameters directly if preferred. The command is not case sensitive.

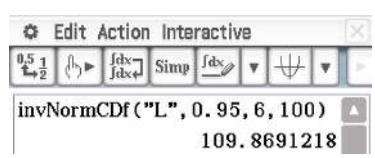
Using the Casio ClassPad

Method 1

- In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Inverse** > **InvNormCDf**.
- Set the 'Tail setting' as 'Left'.
- Enter the probability, 0.95, to the left of the required value k.
- Enter the standard deviation σ and the mean μ.
- Tap OK.

Note: The tail setting is 'Left' to indicate that we seek the value k such that 95% of the area lies to the left of k for this normal distribution.





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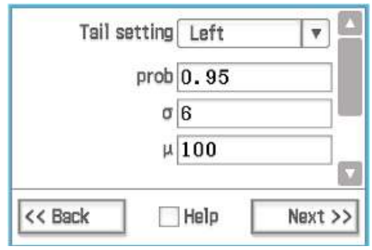
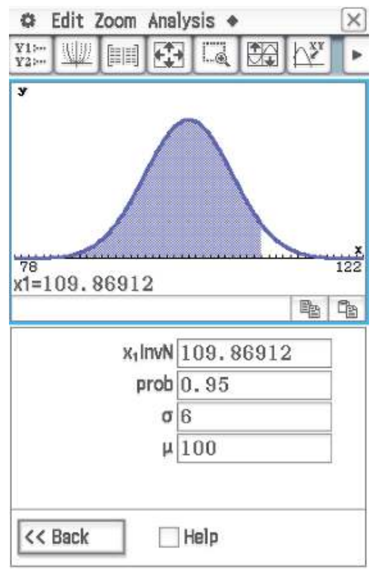
Method 2

- In the **Statistics** application, go to **Calc** > **Inv. Distribution**.
- Select **Inverse Normal CD** and tap Next.
- Set the 'Tail setting' as 'Left'.
- Enter the probability, 0.95, to the left.
- Enter the standard deviation σ and the mean μ.



- Tap Next to view the answer.
- Select to view the graph with the answer.



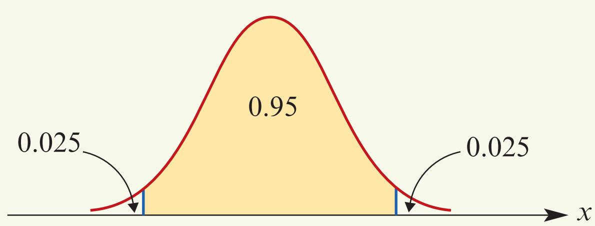
Example 6

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.

Find c_1 and c_2 such that $\mathrm{P}(c_1 < X < c_2) = 0.95$.

Solution

Examining the normal curve, we see that there are (infinitely) many intervals which enclose an area of 0.95. By convention, we choose the interval which leaves equal areas in each tail.



To find c_1 using the inverse-normal facility of your calculator, enter 0.025 as the area. To find c_2, enter 0.975.

This will give the answer $c_1 = 88.240$ and $c_2 = 111.760$.

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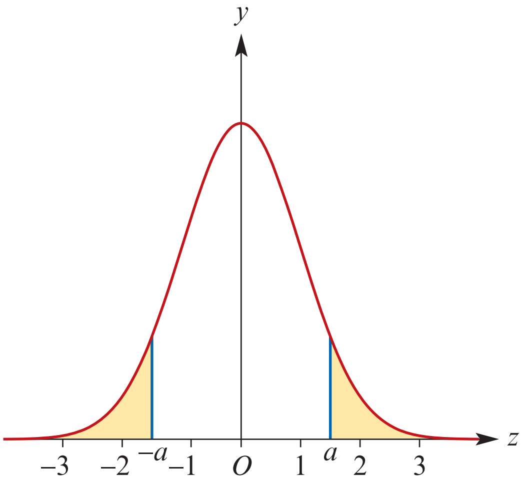
Symmetry properties

Probabilities associated with a normal distribution can often be determined by using its symmetry properties.

Here we work with the standard normal distribution, as it is easiest to use the symmetry properties in this situation:

- $P(Z > a) = 1 - P(Z \le a)$
- $P(Z < -a) = P(Z > a)$
- $P(-a < Z < a) = 1 - 2P(Z \ge a)$
  $= 1 - 2P(Z \le -a)$



Exercise 11C

Example 4

1 Suppose Z is a standard normal random variable (that is, it has mean $\mu = 0$ and standard deviation $\sigma = 1$). Find the following probabilities, drawing an appropriate diagram in each case:

a $P(Z < 2)$ **b** $P(Z < 2.5)$ **c** $P(Z \le 2.5)$ **d** $P(Z < 2.53)$
e $P(Z \ge 2)$ **f** $P(Z > 1.5)$ **g** $P(Z \ge 0.34)$ **h** $P(Z > 1.01)$

2 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $P(Z > -2)$ **b** $P(Z > -0.5)$ **c** $P(Z > -2.5)$ **d** $P(Z \ge -1.283)$
e $P(Z < -2)$ **f** $P(Z < -2.33)$ **g** $P(Z \le -1.8)$ **h** $P(Z \le -0.95)$

3 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $P(-1 < Z < 1)$ **b** $P(-2 < Z < 2)$ **c** $P(-3 < Z < 3)$

How do these results compare with the 68–95–99.7% rule discussed in Section 14B?

4 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $P(2 < Z < 3)$ **b** $P(-1.5 < Z < 2.5)$
c $P(-2 < Z < -1.5)$ **d** $P(-1.4 < Z < -0.8)$

Example 5

5 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $P(Z \le c) = 0.9$.

6 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $P(Z \le c) = 0.75$.

7 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $P(Z \le c) = 0.975$.

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8 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $P(Z \geq c) = 0.95$.

9 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $P(Z \geq c) = 0.8$.

10 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $P(Z \leq c) = 0.10$.

11 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $P(Z \leq c) = 0.025$.

12 Let X be a normal random variable with mean $\mu = 100$ and standard deviation $\sigma = 6$. Find:

a $P(X < 110)$ **b** $P(X < 105)$ **c** $P(X > 110)$ **d** $P(105 < X < 110)$

13 Let X be a normal random variable with mean $\mu = 40$ and standard deviation $\sigma = 5$. Find:

a $P(X < 48)$ **b** $P(X < 36)$ **c** $P(X > 32)$ **d** $P(32 < X < 36)$

14 Let X be a normal random variable with mean $\mu = 6$ and standard deviation $\sigma = 2$.

a Find c such that $P(X < c) = 0.95$.

b Find k such that $P(X < k) = 0.90$.

15 Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 3$.

a Find c such that $P(X < c) = 0.50$.

b Find k such that $P(X < k) = 0.975$.

16 The 68–95–99.7% rule tells us approximately the percentage of a normal distribution which lies within one, two or three standard deviations of the mean. If Z is the standard normal random variable find, correct to two decimal places:

a a such that $P(-a < Z < a) = 0.68$

b b such that $P(-b < Z < b) = 0.95$

c c such that $P(-c < Z < c) = 0.997$

17 Given that X is a normally distributed random variable with a mean of 22 and a standard deviation of 7, find:

a $P(X < 26)$

b $P(25 < X < 27)$

c $P(X < 26 \mid 25 < X < 27)$

d c such that $P(X < c) = 0.95$

e k such that $P(X > k) = 0.9$

f c_1 and c_2 such that $P(c_1 < X < c_2) = 0.95$

18 Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 0.5$. Find:

a $P(X < 11)$

b $P(X < 11 \mid X < 13)$

c c such that $P(X < c) = 0.95$

d k such that $P(X < k) = 0.2$

e c_1 and c_2 such that $P(c_1 < X < c_2) = 0.95$

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11D Solving problems using the normal distribution

The normal distribution can be used to solve many practical problems.

Example 7

The time taken to complete a logical reasoning task is normally distributed with a mean of 55 seconds and a standard deviation of 8 seconds.

a Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task.

b Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task, if it is known that this person took less than 60 seconds to complete the task.

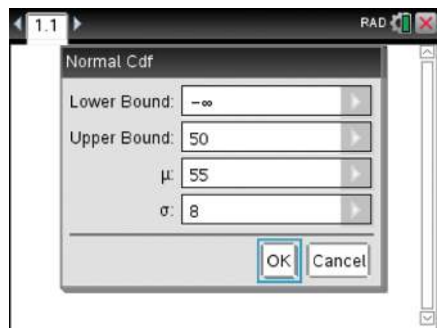
Using the TI-Nspire

a Method 1

Use menu > **Probability** > **Distributions** > **Normal Cdf** and complete as shown.

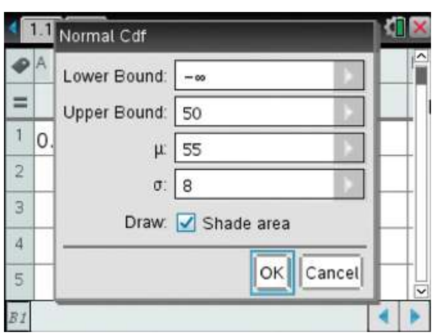
The answer is:

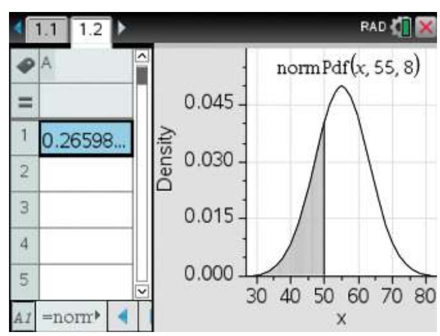
$P(X < 50) = 0.2660$



Method 2

You can also solve this problem in a **Lists & Spreadsheet** page and plot the graph. Use menu > **Statistics** > **Distributions** > **Normal Cdf** and complete as shown below.





b $P(X < 50 | X < 60) = \dfrac{P(X < 50 \cap X < 60)}{P(X < 60)}$

$$= \frac{P(X < 50)}{P(X < 60)} = \frac{0.2660}{0.7340} = 0.3624$$

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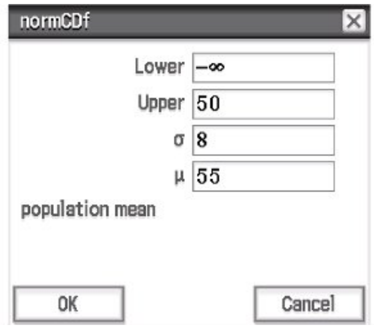
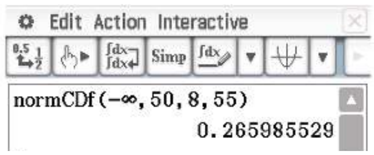
Using the Casio ClassPad

a Method 1

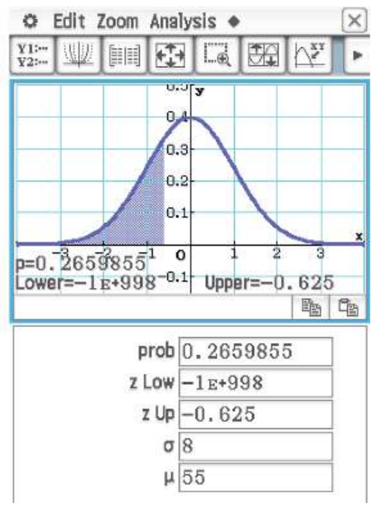
- In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Continuous** > **normCDf**.
- Enter values for the lower and upper bounds, the standard deviation and the mean. Tap OK.





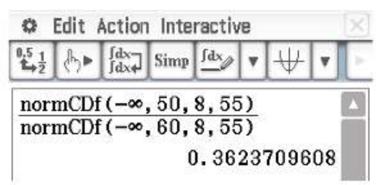
Method 2

- In Statistics, go to **Calc** > **Distribution** and select **Normal CD**. Tap Next.
- Enter the lower and upper bounds, the standard deviation and the mean.
- Tap Next to view the answer.
- Select the graph icon to view the graph with the answer.



b

- In Main $\sqrt{\alpha}$, select the fraction template.
- Enter as shown and tap EXE.



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When the mean and standard deviation of a normal distribution are unknown, it is sometimes necessary to transform to the standard normal distribution. This is demonstrated in the following example.

Example 8

Limits of acceptability imposed on the lengths of a certain batch of metal rods are 1.925 cm and 2.075 cm. It is observed that, on average, 5% are rejected as undersized and 5% are rejected as oversized.

Assuming that the lengths are normally distributed, find the mean and standard deviation of the distribution.

Solution

It is given that $\text{P}(X > 2.075) = 0.05$ and $\text{P}(X < 1.925) = 0.05$.

Symmetry tells us that the mean is equal to

$$\mu = \frac{2.075 + 1.925}{2} = 2$$

Transforming to the standard normal gives

$$\text{P}\left(Z > \frac{2.075 - \mu}{\sigma}\right) = 0.05 \quad \text{and} \quad \text{P}\left(Z < \frac{1.925 - \mu}{\sigma}\right) = 0.05$$

The first equality can be rewritten as

$$\text{P}\left(Z < \frac{2.075 - \mu}{\sigma}\right) = 0.95$$

Use the inverse-normal facility of your calculator to obtain

$$\frac{2.075 - \mu}{\sigma} = 1.6448\ldots \qquad \text{and} \qquad \frac{1.925 - \mu}{\sigma} = -1.6448\ldots$$

These equations confirm that $\mu = 2$.

Substitute $\mu = 2$ into the first equation and solve for σ:

$$\frac{2.075 - 2}{\sigma} = 1.6448\ldots$$

$$\therefore \qquad \sigma = 0.045\,596\ldots$$

Thus $\sigma = 0.0456$, correct to four decimal places.

Skill-sheet

Exercise 11D

Example 7

1 Suppose that IQ scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

a What is the probability that a person chosen at random has an IQ:

i greater than 110

ii less than 75

iii greater than 130, given that they have an IQ greater than 110?

b To be allowed to join an elite club, a potential member must have an IQ in the top 5% of the population. What IQ score would be necessary to join this club?

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2 The heights of women are normally distributed with a mean of $\mu = 160$ cm and a standard deviation of $\sigma = 8$ cm.

a What is the probability that a woman chosen at random would be:

i taller than 155 cm

ii shorter than 170 cm

iii taller than 170 cm, given that her height is between 168 cm and 174 cm?

b What height would put a woman among the tallest 10% of the population?

c What height would put a woman among the shortest 20% of the population?

3 The results of a mathematics exam are normally distributed with mean $\mu = 50$ and standard deviation $\sigma = 7$.

a What is the probability that a student chosen at random has an exam mark:

i greater than 60

ii less than 75

iii greater than 60, given that they passed? (Assume a pass mark of 50.)

b The top 15% of the class are to be awarded a distinction. What mark would be required to gain a distinction in this exam?

4 The lengths of a species of fish are normally distributed with a mean length of 40 cm and a standard deviation of 4 cm. Find the percentage of these fish having lengths:

a greater than 45 cm

b between 35.5 cm and 45.5 cm.

Example 8

5 The weights of cats are normally distributed. It is known that 10% of cats weigh more than 1.8 kg, and 15% of cats weigh less than 1.35 kg. Find the mean and the standard deviation of this distribution.

6 The marks of a large number of students in a statistics examination are normally distributed with a mean of 48 marks and a standard deviation of 15 marks.

a If the pass mark is 53, find the percentage of students who passed the examination.

b If 8% of students gained an A on the examination by scoring a mark of at least c, find the value of c.

7 The height of a certain population of adult males is normally distributed with mean 176 cm and standard deviation 7 cm.

a Find the probability that the height of a randomly selected male will exceed 190 cm.

b If two males are selected at random, find the probability that both of their heights will exceed 190 cm.

c Suppose 10 males are selected at random. Find the probability that at least two will have heights that exceed 190 cm.

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8 **a** Machine A is packaging bags of mints with a mean weight of 300 grams. The bags are considered underweight if they weigh less than 295 grams. It is observed that, on average, 5% of bags are rejected as underweight. Assuming that the weights of the bags are normally distributed, find the standard deviation of the distribution.

b In the same factory, machine B is packaging bags of liquorice. The bags from this machine are considered underweight if they weigh less than 340 grams. It is observed that, on average, 2% of bags from machine B are rejected as underweight. Assuming that the weights are normally distributed with a standard deviation of 5 grams, find the mean of the distribution.

9 The volume of soft drink in a 1-litre bottle is normally distributed. The soft drink company needs to calibrate its filling machine. They don't want to put too much soft drink into each bottle, as it adds to their expense. However, they know they will be fined if more than 2% of bottles are more than 2 millilitres under volume. The standard deviation of the volume dispensed by the filling machine is 2.5 millilitres. What should they choose as the target volume (i.e. the mean of the distribution)? Give your answer to the nearest millilitre.

10 The weights of pumpkins sold to a greengrocer are normally distributed with a mean of 1.2 kg and a standard deviation of 0.4 kg. The pumpkins are sold in three sizes:

Small: under 0.8 kg Medium: from 0.8 kg to 1.8 kg Large: over 1.8 kg

a Find the proportions of pumpkins in each of the three sizes.

b The prices of the pumpkins are \$2.80 for a small, \$3.50 for a medium, and \$5.00 for a large. Find the expected cost for 100 pumpkins chosen at random from the greengrocer's supply.

11 Potatoes are delivered to a chip factory in semitrailer loads. A sample of 1 kg of the potatoes is chosen from each load and tested for starch content. From past experience it is known that the starch content is normally distributed with a standard deviation of 2.1.

a For a semitrailer load of potatoes with a mean starch content of 22.0:

i What is the probability that the test reading is 19.5 or less?

ii What reading will be exceeded with a probability of 0.98?

b If the starch content is greater than 22.0, the potatoes cannot be used for chips, and so the semitrailer load is rejected. What is the probability that a load with a mean starch content of 18.0 will be rejected?

12 The amount of a certain chemical in a type A cell is normally distributed with a mean of 10 and a standard deviation of 1. The amount in a type B cell is normally distributed with a mean of 14 and a standard deviation of 2. To determine whether a cell is type A or type B, the amount of chemical in the cell is measured. The cell is classified as type A if the amount is less than a specified value c, and as type B otherwise.

a If $c = 12$, calculate the probability that a type A cell will be misclassified, and the probability that a type B cell will be misclassified.

b Find the value of c for which the two probabilities of misclassification are equal.

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Review

Chapter summary

Assign-ment

Nrich

- A special continuous random variable X, called a **normal random variable**, has a probability density function given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ and σ are the mean and standard deviation of X.
- In the special case that $\mu = 0$ and $\sigma = 1$, this probability density function defines the **standard normal distribution**. A random variable with this distribution is usually denoted by Z.
- The graph of a normal density function is a symmetric, bell-shaped curve; its centre is determined by the mean, μ, and its width by the standard deviation, σ.
- The **68–95–99.7% rule** states that, for any normal distribution:
  - approximately 68% of the values lie within one standard deviation of the mean
  - approximately 95% of the values lie within two standard deviations of the mean
  - approximately 99.7% of the values lie within three standard deviations of the mean.
- If X is a normally distributed random variable with mean μ and standard deviation σ, then to **standardise** a value x of X we subtract the mean and divide by the standard deviation:

$$z = \frac{x - \mu}{\sigma}$$

The standardised value z indicates the number of standard deviations that the value x lies above or below the mean.
- A calculator can be used to evaluate the cumulative distribution function of a normal random variable – that is, to find the area under the normal curve up to a specified value.
- The inverse-normal facility of a calculator can be used to find the value of a normal random variable corresponding to a specified area under the normal curve.

Short-answer questions

1 Given that $\mathrm{P}(Z \le a) = p$ for the standard normal random variable Z, find in terms of p:

a $\mathrm{P}(Z > a)$ **b** $\mathrm{P}(Z < -a)$ **c** $\mathrm{P}(-a \le Z \le a)$

2 Let X be a normal random variable with mean 4 and standard deviation 1. Let Z be the standard normal random variable.

a If $\mathrm{P}(X < 3) = \mathrm{P}(Z < a)$, then $a =$ ☐.

b If $\mathrm{P}(X > 5) = \mathrm{P}(Z > b)$, then $b =$ ☐.

c $\mathrm{P}(X > 4) =$ ☐

3 A normal random variable X has mean 8 and standard deviation 3. Give the rule for a transformation that maps the graph of the density function of X to the graph of the density function for the standard normal distribution.

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4 Let X be a normal random variable with mean μ and standard deviation σ. If $\mu < a < b$ with $P(X < b) = p$ and $P(X < a) = q$, find:

a $P(X < a \mid X < b)$ **b** $P(X < 2\mu - a)$ **c** $P(X > b \mid X > a)$

5 Let X be a normal random variable with mean 4 and standard deviation 2. Write each of the following probabilities in terms of Z:

a $P(X < 5)$ **b** $P(X < 3)$ **c** $P(X > 5)$
d $P(3 < X < 5)$ **e** $P(3 < X < 6)$

In Questions 6 to 8, you will use the following:

$$P(Z < 1) = 0.84$$
$$P(Z < 2) = 0.98$$
$$P(Z < 0.5) = 0.69$$

6 A machine produces metal rods with mean diameter 2.5 mm and standard deviation 0.05 mm. Let X be the random variable of the normal distribution. Find:

a $P(X < 2.55)$ **b** $P(X < 2.5)$
c $P(X < 2.45)$ **d** $P(2.45 < X < 2.55)$

7 Nuts are packed in tins such that the mean weight of the tins is 500 g and the standard deviation is 5 g. The weights are normally distributed with random variable W. Find:

a $P(W > 505)$ **b** $P(500 < W < 505)$
c $P(W > 505 \mid W > 500)$ **d** $P(W > 510)$

8 A random variable X has a normal distribution with mean 6 and standard deviation 1. Find:

a $P(X < 6.5)$ **b** $P(6 < X < 6.5)$
c $P(6.5 < X < 7)$ **d** $P(5 < X < 7)$

9 Suppose that three tests were given in your mathematics course. The class means and standard deviations, together with your scores, are listed in the table.

	μ	σ	Your score
Test A	50	11	62
Test B	47	17	64
Test C	63	8	73

On which test did you do best and on which did you do worst?

10 Let X be a normally distributed random variable with mean 10 and variance 4, and let Z be a random variable with the standard normal distribution.

a Find $P(X > 10)$.

b Find b such that $P(X > 13) = P(Z < b)$.

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Review

Extended-response questions

1 A test devised to measure mathematical aptitude gives scores that are normally distributed with a mean of 50 and a standard deviation of 10. If we wish to categorise the results so that the highest 10% of scores are designated as high aptitude, the next 20% as moderate aptitude, the middle 40% as average, the next 20% as little aptitude and the lowest 10% as no aptitude, then what ranges of scores will be covered by each of these five categories?

2 If X is normally distributed with $\mu = 10$ and $\sigma = 2$, find the value of k such that

$$P(\mu - k \leq X \leq \mu + k) = 0.95$$

3 Records kept by a manufacturer of car tyres suggest that the distribution of the mileage from their tyres is normal, with mean 60 000 km and standard deviation 5000 km.

a What proportion of the company's tyres last:

i less that 55 000 km

ii more than 50 000 km but less than 74 000 km

iii more than 72 000 km, given that they have already lasted more than 60 000 km?

b The company's advertising manager wishes to claim that '90% of our tyres last longer than c km'. What should c be?

c What is the probability that a customer buys five tyres at the same time and finds that they all last longer than 72 000 km?

4 The owner of a new van complained to the dealer that he was using, on average, 18 litres of petrol to drive 100 km. The dealer pointed out that the 15 litres per 100 km referred to in an advertisement was 'just a guide and actual consumption will vary'. Suppose that the distribution of fuel consumption for this make of van is normal, with a mean of 15 litres per 100 km and a standard deviation of 0.75 litres per 100 km.

a How probable is it that such a van uses at least 18 litres per 100 km?

b What does your answer to part **a** suggest about the manufacturer's claim?

c Find c_1 and c_2 such that the van's fuel consumption is more than c_1 but less than c_2 with a probability of 0.95.

5 Suppose that L, the useful life (in hours) of a fluorescent tube designed for indoor gardening, is normally distributed with a mean of 600 and a standard deviation of 4. The fluorescent tubes are sold in boxes of 10. Find the probability that at least three of the tubes in a randomly selected box last longer than 605 hours.

6 The amount of anaesthetic required to cause surgical anaesthesia in patients is normally distributed, with a mean of 50 mg and a standard deviation of 10 mg. The lethal dose is also normally distributed, with a mean of 110 mg and a standard deviation of 20 mg. If a dosage that brings 90% of patients to surgical anaesthesia were used, what percentage of patients would be killed by this dose?

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7 In a given manufacturing process, components are rejected if they have a particular dimension greater than 60.4 mm or less than 59.7 mm. It is found that 3% are rejected as being too large and 5% are rejected as being too small. Assume that the dimension is normally distributed.

a Find the mean and standard deviation of the distribution of the dimension, correct to one decimal place.

b Use the result of part **a** to find the percentage of rejects if the limits for acceptance are changed to 60.3 mm and 59.6 mm.

8 The hardness of a metal may be determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point. Suppose that the hardness of a particular alloy is normally distributed with mean 70 and standard deviation 3.

a If a specimen is acceptable only if its hardness is between 65 and 75, what is the probability that a randomly chosen specimen has an acceptable hardness?

b If the acceptable range of hardness was $(70 - c, 70 + c)$, for what value of c would 95% of all specimens have acceptable hardness?

c If the acceptable range is the same as in part **a**, and the hardness of each of 10 randomly selected specimens is independently determined, what is the expected number of acceptable specimens among the 10?

d What is the probability that at most eight out of 10 randomly selected specimens have a hardness less than 73.84?

e The profit on an acceptable specimen is \$20, while unacceptable specimens result in a loss of \$5. If $\$P$ is the profit on a randomly selected specimen, find the mean and variance of P.

9 The weekly error (in seconds) of a brand of watch is known to be normally distributed. Only those watches with an error of less than 5 seconds are acceptable.

a Find the mean and standard deviation of the distribution of error if 3% of watches are rejected for losing time and 3% are rejected for gaining time.

b Determine the probability that fewer than two watches are rejected in a batch of 10 such watches.

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10 A brand of detergent is sold in bottles of two sizes: standard and large. For each size, the content (in litres) of a randomly chosen bottle is normally distributed with mean and standard deviation as given in the table:

	Mean	Standard deviation
Standard bottle	0.760	0.008
Large bottle	1.010	0.009

a Find the probability that a randomly chosen standard bottle contains less than 0.75 litres.

b Find the probability that a box of 10 randomly chosen standard bottles contains at least three bottles whose contents are each less than 0.75 litres.

c Using the results

$$\mathrm{E}(aX - bY) = a\mathrm{E}(X) - b\mathrm{E}(Y)$$
$$\mathrm{Var}(aX - bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

find the probability that there is more detergent in four randomly chosen standard bottles than in three randomly chosen large bottles. (Assume that $aX - bY$ is normally distributed.)

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12 Sampling and estimation

In this chapter

Syllabus references

Topics: Random sampling; Sample proportions; Confidence intervals for proportions

Subtopics: 4.3.1 – 4.3.10

There is more to a complete statistical investigation than data analysis. First, we should concern ourselves with the methods used to collect the data. In practice, the purpose of selecting a sample and analysing the information collected from the sample is to make some sort of conclusion, or inference, about the population from which the sample was drawn. Therefore we want the sample we select to be representative of this population.

For example, consider the following questions:

- What proportion of Year 12 students intend to take a gap year?
- What proportion of people aged 18–25 regularly attend church?
- What proportion of secondary students take public transport to school?

While we can answer each of these questions for a sample of people from the group, we really want to know something about the whole group. How can we generalise information gained from a sample to the population, and how confident can we be in that generalisation?

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12A Populations and samples

The set of all eligible members of a group which we intend to study is called a **population**. For example, if we are interested in the IQ scores of the Year 12 students at ABC Secondary College, then this group of students could be considered a population; we could collect and analyse all the IQ scores for these students. However, if we are interested in the IQ scores of all Year 12 students across Australia, then this becomes the population.

Often, dealing with an entire population is not practical:

- The population may be too large – for example, all Year 12 students in Australia.
- The population may be hard to access – for example, all blue whales in the Pacific Ocean.
- The data collection process may be destructive – for example, testing every battery to see how long it lasts would mean that there were no batteries left to sell.

Nevertheless, we often wish to make statements about a property of a population when data about the entire population is unavailable.

The solution is to select a subset of the population – called a **sample** – in the hope that what we find out about the sample is also true about the population it comes from. Dealing with a sample is generally quicker and cheaper than dealing with the whole population, and a well-chosen sample will give much useful information about this population. How to select the sample then becomes a very important issue.

Random samples

Suppose we are interested in investigating the effect of sustained computer use on the eyesight of a group of university students. To do this we go into a lecture theatre containing the students and select all the students sitting in the front two rows as our sample. This sample may be quite inappropriate, as students who already have problems with their eyesight are more likely to be sitting at the front, and so the sample may not be typical of the population. To make valid conclusions about the population from the sample, we would like the sample to have a similar nature to the population.

While there are many sophisticated methods of selecting samples, the general principle of sample selection is that the method of choosing the sample should not favour or disfavour any subgroup of the population. Since it is not always obvious if the method of selection will favour a subgroup or not, we try to choose the sample so that every member of the population has an equal chance of being in the sample. In this way, all subgroups have a chance of being represented. The way we do this is to choose the sample at random.

> A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.

To choose a sample from the group of university students, we could put the name of every student in a hat and then draw out, one at a time, the names of the students who will be in the sample.

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Choosing the sample in an appropriate manner is critical in order to obtain usable results.

Example 1

A researcher wishes to evaluate how well the local library is catering to the needs of a town's residents. To do this she hands out a questionnaire to each person entering the library over the course of a week. Will this method result in a random sample?

Solution

Since the members of the sample are already using the library, they are possibly satisfied with the service available. Additional valuable information might well be obtained by finding out the opinion of those who do not use the library.

A better sample would be obtained by selecting at random from the town's entire population, so the sample contains both people who use the library and people who do not.

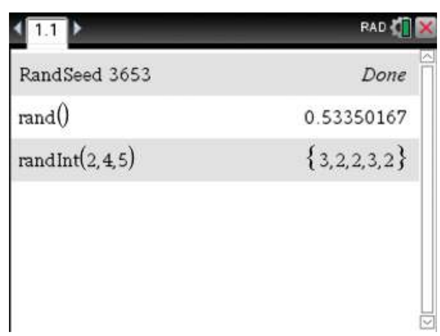
Thus, we have a very important consideration when sampling if we wish to generalise from the results of the sample.

In order to make valid conclusions about a population from a sample, we would like the sample chosen to be representative of the population as a whole. This means that all the different subgroups present in the population appear in the sample in similar proportions as they do in the population.

One very useful method for drawing random samples is to generate random numbers using a calculator or a computer.

Using the TI-Nspire

- In a **Calculator** page, go to Menu > **Probability** > **Random** > **Seed** and enter the last 4 digits of your phone number. This ensures that your random-number starting point differs from the calculator default.
- For a random number between 0 and 1, use Menu > **Probability** > **Random** > **Number**.
- For a random integer, use Menu > **Probability** > **Random** > **Integer**. To obtain five random integers between 2 and 4 inclusive, use the command randInt(2, 4, 5) as shown.



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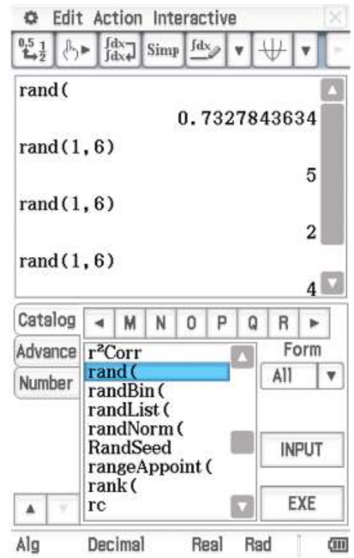
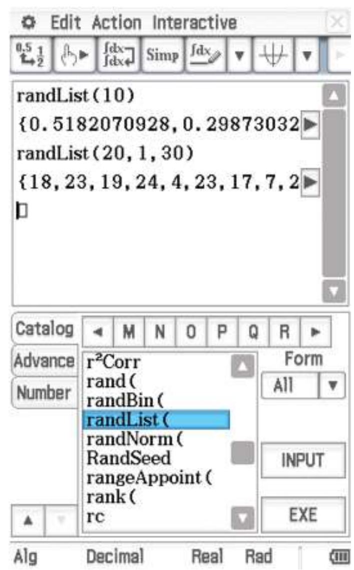
Using the Casio ClassPad

- In $\sqrt{\alpha}$ (Main), press the Keyboard button.
- Find and then select Catalog by first tapping ▼ at the bottom of the left sidebar.
- Scroll across the alphabet to the letter R.



- To generate a random number between 0 and 1:
  - In Catalog, select **rand(**.
  - Tap EXE.
- To generate three random integers between 1 and 6 inclusive:
  - In Catalog, select **rand(**.
  - Type: 1, 6)
  - Tap EXE three times.



- To generate a list of 10 random numbers between 0 and 1:
  - In Catalog, select **randList(**.
  - Type: 10)
  - Tap EXE.
  - Tap ► to view all the numbers.
- To generate a list of 20 random integers between 1 and 30 inclusive:
  - In Catalog, select **randList(**.
  - Type: 20, 1, 30)
  - Tap EXE.
  - Tap ► to view all the integers.

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Example 2

Use a random number generator to select a group of six students from the following class:

Denise	Sharyn	Miller	Tom	Steven
Matt	Mark	William	David	Jane
Teresa	Peter	Anne	Sally	Georgia
Sue	Nick	Darren	Janelle	Jaimie

Solution

First assign a number to each member of the class:

Denise (1)	Sharyn (5)	Miller (9)	Tom (13)	Steven (17)
Matt (2)	Mark (6)	William (10)	David (14)	Jane (18)
Teresa (3)	Peter (7)	Anne (11)	Sally (15)	Georgia (19)
Sue (4)	Nick (8)	Darren (12)	Janelle (16)	Jaimie (20)

Generating six random integers from 1 to 20 gives on this occasion: 4, 19, 9, 2, 13, 14.
The sample chosen is thus:

Sue, Georgia, Miller, Matt, Tom, David

Note: In this example, we want a list of six random integers without repeats. We do not add a randomly generated integer to our list if it is already in the list.

The sample proportion as a random variable

Suppose that our population of interest is the class of students from Example 2, and suppose further that we are particularly interested in the proportion of female students in the class. This is called the **population proportion** and is generally denoted by p. The population proportion p is constant for a particular population.

$$\text{Population proportion } p = \frac{\text{number in population with attribute}}{\text{population size}}$$

Since there are 10 males and 10 females, the proportion of female students in the class is

$$p = \frac{10}{20} = \frac{1}{2}$$

Now consider the proportion of female students in the sample chosen:

Sue, Georgia, Miller, Matt, Tom, David

The proportion of females in the sample may be calculated by dividing the number of females in the sample by the sample size. In this case, the proportion of female students in the sample is $\frac{2}{6} = \frac{1}{3}$. This value is called the **sample proportion** and is denoted by $\hat{p}$. (We say 'p hat'.)

$$\text{Sample proportion } \hat{p} = \frac{\text{number in sample with attribute}}{\text{sample size}}$$

Note that different symbols are used for the sample proportion and the population proportion, so that we don't confuse them.

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In this particular case, $\hat{p} = \frac{1}{3}$, which is not the same as the population proportion $p = \frac{1}{2}$. This does not mean there is a problem. In fact, each time a sample is selected the number of females in the sample will vary. Sometimes the sample proportion $\hat{p}$ will be $\frac{1}{2}$, and sometimes it will not.

- The population proportion p is a **population parameter**; its value is constant.
- The sample proportion $\hat{p}$ is a **sample statistic**; its value is not constant, but varies from sample to sample.

Example 3

Use a random number generator to select another group of six students from the same class, and determine the proportion of females in the sample.

Denise (1)	Sharyn (5)	Miller (9)	Tom (13)	Steven (17)
Matt (2)	Mark (6)	William (10)	David (14)	Jane (18)
Teresa (3)	Peter (7)	Anne (11)	Sally (15)	Georgia (19)
Sue (4)	Nick (8)	Darren (12)	Janelle (16)	Jaimie (20)

Solution

Generating another six random integers from 1 to 20 gives 19, 3, 11, 9, 15, 1.

The sample chosen is thus: Georgia, Teresa, Anne, Miller, Sally, Denise

For this sample, we have

$$\hat{p} = \frac{5}{6}$$

Since $\hat{p}$ varies according to the contents of the random samples, we can consider the sample proportions $\hat{p}$ as being the values of a random variable, which we will denote by $\hat{P}$. We investigate this idea further in the next section.

Summary 12A

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute, and is constant.
- The **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing the attribute, and varies from sample to sample.
- The sample proportions $\hat{p}$ are the values of a random variable $\hat{P}$.

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Exercise 12A

Example 1

1 In order to determine the sort of film in which to invest his money, a producer waits outside a theatre and asks people as they leave whether they prefer comedy, drama, horror or science fiction. Do you think this is an appropriate way of selecting a random sample of movie goers? Explain your answer.

2 A market researcher wishes to find out how people spend their leisure time. She positions herself in a shopping mall and asks shoppers as they pass to fill out a short questionnaire.

a Do you think this sample will be representative of the general population? Explain.

b How would you suggest that the sample could be chosen?

3 To investigate people's attitudes to control of gun ownership, a television station conducts a phone-in poll, where people are asked to telephone one number if they are in favour of tighter gun control, and another if they are against. Is this an appropriate method of choosing a random sample? Give reasons for your answer.

4 A researcher wishes to select five guinea pigs at random from a large cage containing 20 guinea pigs. In order to select her sample, she reaches into the cage and (gently) pulls out five guinea pigs.

a Do you think this sample will be representative of the general population? Explain.

b How would you suggest the sample could be chosen?

5 In order to estimate how much money young people spend on takeaway food, a questionnaire is sent to several schools randomly chosen from a list of all schools in the state, to be given to a random selection of students in the school. Is this an appropriate method of choosing a random sample? Give reasons for your answer.

Example 2

6 Use a random number generator to select a random sample of size 3 from the following list of people:

- Karen
- Kate
- Trinh
- Paul
- Lina
- Alexander
- Juliet
- Craig
- Conrad
- Fairlie
- Kylie
- Edward
- Kelly
- Rani
- Maree
- Janet
- Fleur
- Connie
- Aden
- Wolfgang
- Zoe
- Cara
- Noel
- Judy
- Andrew

7 In a survey to obtain adults' views on unemployment, people were stopped by interviewers as they came out of:

a a travel agency **b** a supermarket **c** an employment-services centre.

What is wrong with each of the methods of sampling listed here? Describe a better method of choosing the sample.

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8 A marine biologist wishes to estimate the total number of crabs on a rock platform which is 10 metres square. It would be impossible to count them all individually, so she places a 1-metre-square frame at five random locations on the rock platform, and counts the number of crabs in the frame. To estimate the total number, she will multiply the average number in the frame by the total area of the rock platform.

a Explain how a random number generator could be used to select the five locations for the frame.

b Will this give a good estimate of the crab population?

9 In order to survey the attitude of parents to the current uniform requirements, the principal of a school selected 100 students at random from the school roll, and then interviewed their parents. Do you think this group of parents would form a simple random sample?

10 A television station carried out a poll to find out if the public felt that mining should be allowed in a particular area. People were asked to ring one number to register a 'yes' vote and another to register a 'no' vote. The results showed that 77% of people were in favour of mining proceeding. Comment on the results.

11 A market-research company decided to collect information concerning the way people use their leisure time by phoning a randomly chosen group of 1000 people at home between 7 p.m. and 10 p.m. on weeknights. The final report was based on the responses of only the 550 people of those sampled who could be found at home. Comment on the validity of this report.

12 In a certain school, 35% of the students travel on the school bus. A group of 100 students were selected in a random sample, and 42 of them travel on the school bus. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

13 Of a random sample of 100 homes, 22 were found to have central heating.

a What proportion of these homes have central heating?

b Is this the value of the population proportion p or the sample proportion $\hat{p}$?

Example 3

14 Use a random number generator to select another group of six students from the class listed below, and determine the proportion of females in the sample:

- Denise (1)
- Matt (2)
- Teresa (3)
- Sue (4)
- Sharyn (5)
- Mark (6)
- Peter (7)
- Nick (8)
- Miller (9)
- William (10)
- Anne (11)
- Darren (12)
- Tom (13)
- David (14)
- Sally (15)
- Janelle (16)
- Steven (17)
- Jane (18)
- Georgia (19)
- Jaimie (20)

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12B The exact distribution of the sample proportion

We have seen that the sample proportion varies from sample to sample. We can use our knowledge of probability to further develop our understanding of the sample proportion.

Sampling from a small population

Suppose we have a bag containing six blue balls and four red balls, and from the bag we take a sample of size 4. We are interested in the proportion of blue balls in the sample. We know that the population proportion is equal to $\frac{6}{10} = \frac{3}{5}$. That is,

$$p = 0.6$$

The probabilities associated with the possible values of the sample proportion $\hat{p}$ can be calculated either by direct consideration of the sample outcomes or by using our knowledge of selections. Recall that

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

is the number of different ways to select x objects from n objects.

Example 4

A bag contains six blue balls and four red balls. If we take a random sample of size 4, what is the probability that there is one blue ball in the sample $\left(\hat{p} = \frac{1}{4}\right)$?

Solution

Method 1

Consider selecting the sample by taking one ball from the bag at a time (without replacement). The favourable outcomes are RRRB, RRBR, RBRR and BRRR, with

$$\begin{aligned} &\Pr(\{\text{RRRB}, \text{RRBR}, \text{RBRR}, \text{BRRR}\}) \\ &= \left(\frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} \times \frac{6}{7}\right) + \left(\frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \times \frac{2}{7}\right) + \left(\frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} \times \frac{2}{7}\right) + \left(\frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7}\right) \\ &= \frac{4}{35} \end{aligned}$$

Method 2

In total, there are $\binom{10}{4} = 210$ ways to select 4 balls from 10 balls.

There are $\binom{4}{3} = 4$ ways of choosing 3 red balls from 4 red balls, and there are $\binom{6}{1} = 6$ ways of choosing one blue ball from 6 blue balls.

Thus the probability of obtaining 3 red balls and one blue ball is equal to

$$\frac{\binom{4}{3} \times \binom{6}{1}}{\binom{10}{4}} = \frac{24}{210} = \frac{4}{35}$$

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The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of size 4 from the bag.

Number of blue balls in the sample	0	1	2	3	4
Proportion of blue balls in the sample, $\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Probability	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

We can see from the table that we can consider the sample proportion as a random variable, $\hat{P}$, and we can write:

- $\mathrm{P}(\hat{P} = 0) = \frac{1}{210}$
- $\mathrm{P}\left(\hat{P} = \frac{1}{4}\right) = \frac{24}{210}$
- $\mathrm{P}\left(\hat{P} = \frac{1}{2}\right) = \frac{90}{210}$
- $\mathrm{P}\left(\hat{P} = \frac{3}{4}\right) = \frac{80}{210}$
- $\mathrm{P}(\hat{P} = 1) = \frac{15}{210}$

The possible values of $\hat{p}$ and their associated probabilities together form a probability distribution for the random variable $\hat{P}$, which can summarised as follows:

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\mathrm{P}(\hat{P} = \hat{p})$	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

The distribution of a statistic which is calculated from a sample (such as the sample proportion) has a special name – it is called a **sampling distribution**.

Example 5

A bag contains six blue balls and four red balls. Use the sampling distribution in the previous table to determine the probability that the proportion of blue balls in a sample of size 4 is more than $\frac{1}{4}$.

Solution

$$\begin{aligned}
\mathrm{P}\left(\hat{P} > \frac{1}{4}\right) &= \mathrm{P}\left(\hat{P} = \frac{1}{2}\right) + \mathrm{P}\left(\hat{P} = \frac{3}{4}\right) + \mathrm{P}(\hat{P} = 1) \\
&= \frac{90}{210} + \frac{80}{210} + \frac{15}{210} \\
&= \frac{185}{210} \\
&= \frac{37}{42}
\end{aligned}$$

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Sampling from a large population

Generally, when we select a sample, it is from a population which is too large or too difficult to enumerate or even count – populations such as all the people in Australia, or all the cows in Texas, or all the people who will ever have asthma. When the population is so large, we assume that the probability of observing the attribute we are interested in remains constant with each selection, irrespective of prior selections for the sample.

Suppose we know that 70% of all 17-year-olds in Australia attend school. That is,

$$p = 0.7$$

We will assume that this probability remains constant for all selections for the sample.

Now consider selecting a random sample of size 4 from the population of all 17-year-olds in Australia. This time we can use our knowledge of binomial distributions to calculate the associated probability for each possible value of the sample proportion $\hat{p}$, using the probability function

$$\mathrm{P}(X = x) = \binom{4}{x} 0.7^x 0.3^{4-x} \qquad x = 0, 1, 2, 3, 4$$

The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of four 17-year-olds.

Number at school in the sample	0	1	2	3	4
Proportion at school in the sample, $\hat{p}$	0	0.25	0.5	0.75	1
Probability	0.0081	0.0756	0.2646	0.4116	0.2401

Once again, we can summarise the sampling distribution of the sample proportion as follows:

$\hat{p}$	0	0.25	0.5	0.75	1
$\mathrm{P}(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

The population that the sample of size $n = 4$ is being taken from is such that each item selected has a probability $p = 0.7$ of success. Thus we can define the random variable

$$\hat{P} = \frac{X}{4}$$

where X is a binomial random variable with parameters $n = 4$ and $p = 0.7$. To emphasise this we can write:

x	0	1	2	3	4
$\hat{p} = \frac{x}{4}$	0	0.25	0.5	0.75	1
$\mathrm{P}(\hat{P} = \hat{p}) = \mathrm{P}(X = x)$	0.0081	0.0756	0.2646	0.4116	0.2401

Note: The probabilities for the sample proportions, $\hat{p}$, correspond to the probabilities for the numbers of successes, x.

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Example 6

Use the sampling distribution in the previous table to determine the probability that, in a random sample of four Australian 17-year-olds, the proportion attending school is less than 50%.

Solution

$$\begin{aligned} \mathrm{P}(\hat{P} < 0.5) &= \mathrm{P}(\hat{P} = 0) + \mathrm{P}(\hat{P} = 0.25) \\ &= 0.0081 + 0.0756 \\ &= 0.0837 \end{aligned}$$

The mean and standard deviation of the sample proportion

Since the sample proportion $\hat{P}$ is a random variable with a probability distribution, we can determine values for the mean and standard deviation, as illustrated in the following example.

Example 7

Use the probability distribution to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6.

$\hat{p}$	0	0.25	0.5	0.75	1
$\mathrm{P}(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

Solution

By definition, the mean of $\hat{P}$ is given by

$$\begin{aligned} \mathrm{E}(\hat{P}) &= \sum \hat{p} \cdot \mathrm{P}(\hat{P} = \hat{p}) \\ &= 0 \times 0.0081 + 0.25 \times 0.0756 + 0.5 \times 0.2646 + 0.75 \times 0.4116 + 1 \times 0.2401 \\ &= 0.7 \end{aligned}$$

Similarly, by definition,

$$\mathrm{sd}(\hat{P}) = \sqrt{\mathrm{E}(\hat{P}^2) - [\mathrm{E}(\hat{P})]^2}$$

We have

$$\begin{aligned} \mathrm{E}(\hat{P}^2) &= 0^2 \times 0.0081 + 0.25^2 \times 0.0756 + 0.5^2 \times 0.2646 + 0.75^2 \times 0.4116 + 1^2 \times 0.2401 \\ &= 0.5425 \end{aligned}$$

Thus

$$\mathrm{sd}(\hat{P}) = \sqrt{0.5425 - 0.7^2} = 0.2291$$

We can see from Example 7 that the mean of the sampling distribution in this case is actually the same as the value of the population proportion (0.7). Is this always true? Can we determine the mean and standard deviation of the sample proportion without needing to find the probability distribution?

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If we are selecting a random sample of size n from a large population, then we can assume that the sample proportion is of the form

$$\hat{P} = \frac{X}{n}$$

where X is a binomial random variable with parameters n and p. From Chapter 9, the mean and variance of X are given by

$$\mathrm{E}(X) = np \qquad \text{and} \qquad \mathrm{Var}(X) = np(1-p)$$

Thus we can determine

$$\begin{aligned} \mathrm{E}(\hat{P}) &= \mathrm{E}\left(\frac{X}{n}\right) \\ &= \frac{1}{n}\mathrm{E}(X) && \text{since } \mathrm{E}(aX+b) = a\mathrm{E}(X)+b \\ &= \frac{1}{n} \times np \\ &= p \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(\hat{P}) &= \mathrm{Var}\left(\frac{X}{n}\right) \\ &= \left(\frac{1}{n}\right)^2 \mathrm{Var}(X) && \text{since } \mathrm{Var}(aX+b) = a^2\mathrm{Var}(X) \\ &= \frac{1}{n^2} \times np(1-p) \\ &= \frac{p(1-p)}{n} \end{aligned}$$

If we are selecting a random sample of size n from a large population, then the mean and standard deviation of the sample proportion $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

(The standard deviation of a sample statistic is called the **standard error**.)

Example 8

Use these rules to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6. Are they the same as those found in Example 7?

Solution

$$\mathrm{E}(\hat{P}) = p = 0.7$$

$$\mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.7(1-0.7)}{4}} = 0.2291$$

These are the same as those obtained in Example 7.

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Example 9

Suppose that 70% of 17-year-olds in Australia attend school. If a random sample of size 20 is chosen from this population, find:

a the probability that the sample proportion is equal to the population proportion (0.7)

b the probability that the sample proportion lies within one standard deviation of the population proportion

c the probability that the sample proportion lies within two standard deviations of the population proportion.

Solution

a If the sample proportion is $\hat{p} = 0.7$ and the sample size is 20, then the number of school students in the sample is $0.7 \times 20 = 14$. Thus

$$\begin{aligned} \Pr(\hat{P} = 0.7) &= \Pr(X = 14) \\ &= \binom{20}{14} 0.7^{14} 0.3^{6} = 0.1916 \end{aligned}$$

b We have

$$\begin{aligned} \operatorname{sd}(\hat{P}) &= \sqrt{\frac{p(1-p)}{n}} \\ &= \sqrt{\frac{0.7(1-0.7)}{20}} = 0.1025 \end{aligned}$$

Since $0.7 - 0.1025 = 0.5975$ and $0.7 + 0.1025 = 0.8025$, we find

$$\begin{aligned} \Pr(0.5975 \le \hat{P} \le 0.8025) &= \Pr(11.95 \le X \le 16.05) \\ &= \Pr(12 \le X \le 16) \qquad \text{since } X \text{ takes integer values} \\ &= 0.7795 \end{aligned}$$

c Since $0.7 - 2 \times 0.1025 = 0.495$ and $0.7 + 2 \times 0.1025 = 0.905$, we find

$$\begin{aligned} \Pr(0.495 \le \hat{P} \le 0.905) &= \Pr(9.9 \le X \le 18.1) \\ &= \Pr(10 \le X \le 18) \\ &= 0.9752 \end{aligned}$$

Summary 12B

- The distribution of a statistic which is calculated from a sample is called a **sampling distribution**.
- The **sample proportion** $\hat{P} = \dfrac{X}{n}$ is a random variable, where X is the number of favourable outcomes in a sample of size n.
- The distribution of $\hat{P}$ is known as the **sampling distribution** of the sample proportion.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using our knowledge of selections.

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- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined by assuming that X is a binomial random variable with parameters n and p. In this case, the mean and standard deviation of $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

Skill-sheet

Exercise 12B

Example 4
Example 5

1 Consider a bag containing five blue and five red balls.

a What is p, the proportion of blue balls in the bag?

b If samples of size 3 are taken from the bag, without replacement, then a sample could contain 0, 1, 2 or 3 blue balls. What are the possible values of the sample proportion $\hat{p}$ of blue balls associated with each of these samples?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of blue balls when samples of size 3 are taken from the bag, without replacement.

d Use the sampling distribution from part **c** to determine the probability that the proportion of blue balls in the sample is more than 0.5. That is, find $\mathrm{P}(\hat{P} > 0.5)$.

2 A company employs a sales team of 20 people, consisting of 12 men and 8 women.

a What is p, the proportion of men in the sales team?

b Five salespeople are to be selected at random to attend an important conference. What are the possible values of the sample proportion $\hat{p}$ of men in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of men when samples of size 5 are selected from the sales team.

d Use the sampling distribution from part **c** to determine the probability that the proportion of men in the sample is more than 0.7.

e Find $\mathrm{P}(0 < \hat{P} < 0.7)$ and hence find $\mathrm{P}(\hat{P} < 0.7 \mid \hat{P} > 0)$.

3 A pond contains eight gold and eight black fish.

a What is p, the proportion of gold fish in the pond?

b Three fish are to be selected at random. What are the possible values of the sample proportion $\hat{p}$ of gold fish in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of gold fish when samples of size 3 are selected from the pond.

d Use the sampling distribution from part **c** to determine the probability that the proportion of gold fish in the sample is more than 0.25.

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4 A random sample of three items is selected from a batch of 10 items which contains four defectives.

a What is p, the proportion of defectives in the batch?

b What are the possible values of the sample proportion $\hat{p}$ of defectives in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of defectives in the sample.

d Use the sampling distribution from part **c** to determine the probability that the proportion of defectives in the sample is more than 0.5.

e Find $\Pr(0 < \hat{P} < 0.5)$ and hence find $\Pr(\hat{P} < 0.5 \mid \hat{P} > 0)$.

Example 6

5 Suppose that a fair coin is tossed four times, and the number of heads observed.

a What is p, the probability that a head is observed when a fair coin is tossed?

b What are the possible values of the sample proportion $\hat{p}$ of heads in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of heads in the sample.

d Use the sampling distribution from part **c** to determine the probability that the proportion of heads in the sample is more than 0.7.

6 Suppose that the probability of a male child is 0.5, and that a family has five children.

a What are the possible values of the sample proportion $\hat{p}$ of male children in the family?

b Construct a probability distribution table which summarises the sampling distribution of the sample proportion of male children in the family.

c Use the sampling distribution from part **b** to determine the probability that the proportion of male children in the family is less than 0.4.

d Find $\Pr(\hat{P} > 0 \mid \hat{P} < 0.8)$.

7 Suppose that, in a certain country, the probability that a person is left-handed is 0.2.

a If four people are selected at random from that country, what are the possible values of the sample proportion $\hat{p}$ of left-handed people in the sample?

b Construct a probability distribution table which summarises the sampling distribution of the sample proportion of left-handed people in the sample.

c Find $\Pr(\hat{P} > 0.5 \mid \hat{P} > 0)$.

Example 7

8 Use the sampling distribution from Question **5** to determine the mean and standard deviation of the sample proportion $\hat{P}$ of heads observed when a fair coin is tossed four times.

9 Use the sampling distribution from Question **6** to determine the mean and standard deviation of the sample proportion $\hat{P}$ of male children in a family of five children.

10 Use the sampling distribution from Question **7** to determine the mean and standard deviation of the sample proportion $\hat{P}$ of left-handed people when a sample of four people are selected.

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Example 8 **11** Suppose that the probability of rain on any day is 0.3. Find the mean and standard deviation of the sample proportion of rainy days which might be observed in the month of June.

12 In a certain country, it is known that 40% of people speak more than one language. If a sample of 100 people is selected, find the mean and standard deviation of the sample proportion of people who speak more than one language.

13 An examination consists of 100 multiple-choice questions, each with five possible answers. Find the mean and standard deviation of the sample proportion of correct answers that will be achieved if a student guesses every answer.

Example 9 **14** Suppose that 65% of people in Australia support an AFL team. If a random sample of size 20 is chosen from this population, find:

a the probability that the sample proportion is equal to the population proportion

b the probability that the sample proportion lies within one standard deviation of the population proportion

c the probability that the sample proportion lies within two standard deviations of the population proportion.

12C Approximating the distribution of the sample proportion

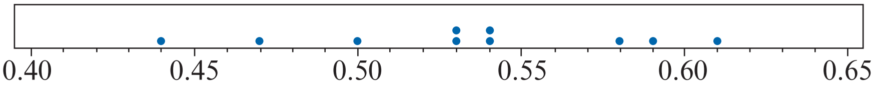
In the previous section, we used our knowledge of probability to determine the exact distribution of the sample proportion. Working out the exact probabilities associated with a sample proportion is really only practical when the sample size is quite small (say less than 10). In practice, we are rarely working with such small samples. But we can overcome this problem by approximating the distribution of the sample proportion.

Suppose, for example, we know that 55% of people in Australia have blue eyes ($p = 0.55$) and that we are interested in the values of the sample proportion $\hat{p}$ which might be observed when samples of size 100 are drawn at random from the population.

If we select one sample of 100 people and find that 50 people have blue eyes, then the value of the sample proportion is $\hat{p} = \dfrac{50}{100} = 0.5$.

If a second sample of 100 people is selected and this time 58 people have blue eyes, then the value of the sample proportion for this second sample is $\hat{p} = \dfrac{58}{100} = 0.58$.

Continuing in this way, after selecting 10 samples, the values of $\hat{p}$ that are observed might look like those in the following dotplot:



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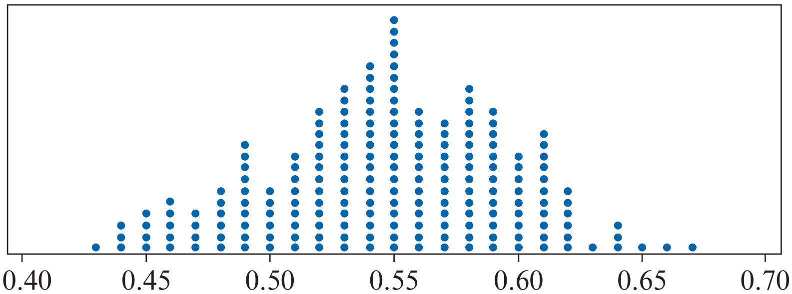
It is clear that the proportion of people with blue eyes in the sample, $\hat{p}$, is varying from sample to sample: from as low as 0.44 to as high as 0.61 for these particular 10 samples.

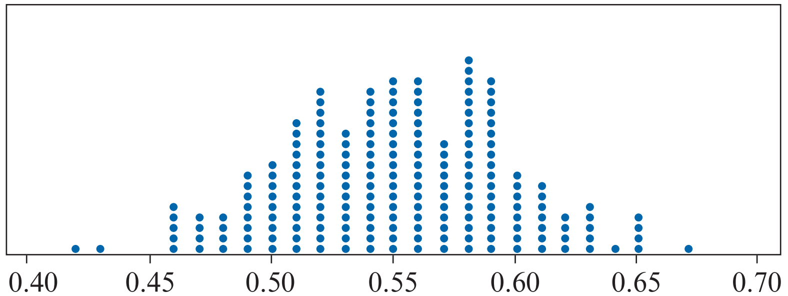
What does the distribution of the sample proportions look like if we continue with this sampling process?

The following dotplot summarises the values of $\hat{p}$ observed when 200 samples (each of size 100) were selected from a population in which the probability of having blue eyes is 0.55. We can see from the dotplot that the distribution is reasonably symmetric, centred at 0.55, and has values ranging from 0.43 to 0.67.



What does the distribution look like when another 200 samples (each of size 100) are selected at random from the same population?

The following dotplot shows the distribution obtained when this experiment was repeated. Again, the distribution is reasonably symmetric, centred at 0.55, and has values ranging from 0.42 to 0.67.



It seems reasonable to infer from these examples that, while there will be variation in the details of the distribution each time we take a collection of samples, the distribution of the values of $\hat{p}$ observed tends to conform to a predictable shape, centre and spread.

When the sample size is large enough, the distribution of a binomial random variable is well approximated by the normal distribution. The rule of thumb for the normal approximation to the binomial distribution to apply is that both np and $n(1 - p)$ should be greater than 5.

The dotplots confirm the reasonableness of the normality assumption with regard to the sample proportion $\hat{P}$, which can be considered to be a linear function of a binomial random variable.

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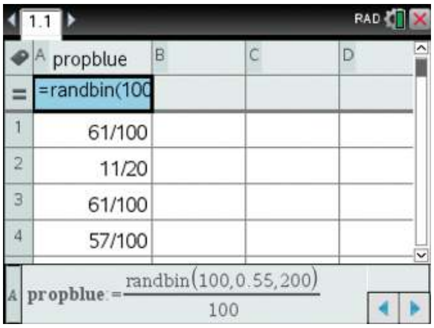
Repeated sampling can be investigated using a calculator.

Example 10

Assume that 55% of people in Australia have blue eyes. Use your calculator to illustrate a possible distribution of sample proportions $\hat{p}$ that may be obtained when 200 different samples (each of size 100) are selected from the population.

Using the TI-Nspire

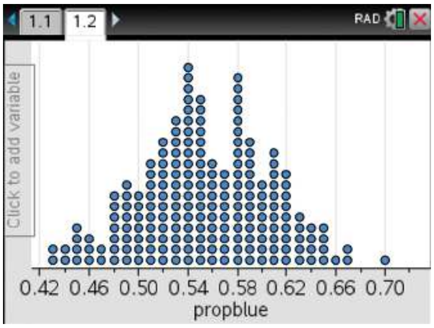
- To generate the sample proportions:
  - Start from a **Lists & Spreadsheet** page.
  - Name the list 'propblue' in Column A.
  - In the formula cell of Column A, enter the formula using (Menu) > **Data** > **Random** > **Binomial** and complete as: = randbin(100, 0.55, 200)/100



Note: The syntax is: randbin(*sample size*, *population proportion*, *number of samples*) To calculate as a proportion, divide by the sample size.

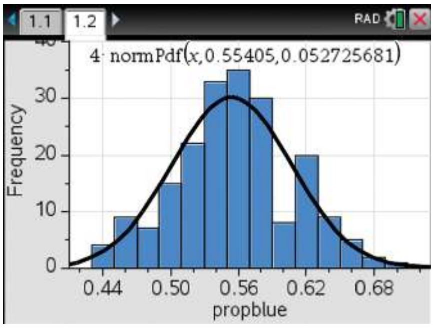
- To display the distribution of sample proportions:
  - Insert a **Data & Statistics** page ((ctrl)(I) or (ctrl)(doc ▾)).
  - Click on 'Click to add variable' on the x-axis and select 'propblue'. A dotplot is displayed.

Note: You can recalculate the random sample proportions by using (ctrl)(R) while in the **Lists & Spreadsheet** page.



- To fit a normal curve to the distribution:
  - (Menu) > **Plot Type** > **Histogram**
  - (Menu) > **Analyze** > **Show Normal PDF**

Note: The calculated Normal PDF, based on the data set, is superimposed on the plot, showing the mean and standard deviation of the sample proportion.



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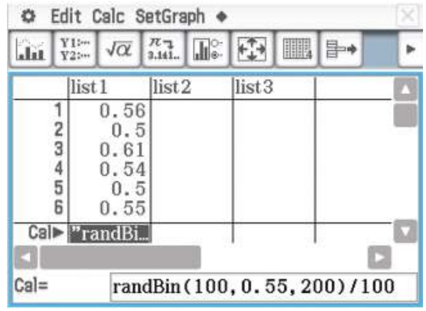
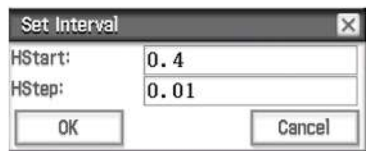
Using the Casio ClassPad

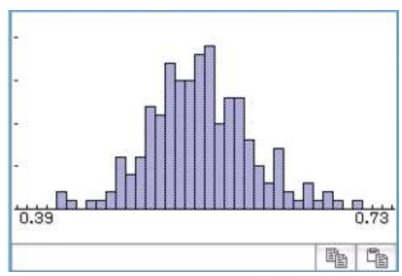
- To generate the sample proportions:
  - Open the **Statistics** application.
  - Tap the 'Calculation' cell at the bottom of list1.
  - Type: randBin(100, 0.55, 200)/100
  - Tap `Set`.



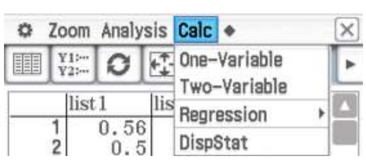
Note: The syntax is: randBin(*sample size*, *population proportion*, *number of samples*)
To calculate as a proportion, divide by the sample size.

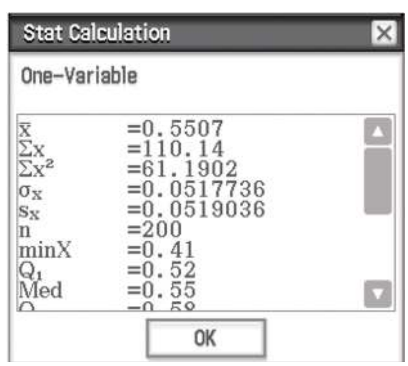
- To display the distribution of sample proportions:
  - Tap on the **Set StatGraphs** icon, select the type 'Histogram' and tap `Set`.
  - Tap on the graph icon in the toolbar.
  - In the **Set Interval** window, enter the values shown below and tap `OK`.





- To obtain statistics from the distribution, select **Calc** > **One-Variable**. Tap `OK`.





Note: The mean of the sample proportions, $\bar{x}$, estimates the population proportion.

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When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

Thus, when samples of size $n = 100$ are selected from a population in which the proportion of people with blue eyes is $p = 0.55$, the distribution of the sample proportion $\hat{P}$ is approximately normal, with mean and standard deviation given by

$$\mu = \mathrm{E}(\hat{P}) = 0.55 \quad \text{and} \quad \sigma = \mathrm{sd}(\hat{P}) = \sqrt{\frac{0.55 \times 0.45}{100}} = 0.0497$$

Example 11

Assume that 60% of people have a driver's licence. Using the normal approximation, find the approximate probability that, in a randomly selected sample of size 200, more than 65% of people have a driver's licence.

Solution

Here $n = 200$ and $p = 0.6$. Since n is large, the distribution of $\hat{P}$ is approximately normal, with mean $\mu = p = 0.6$ and standard deviation

$$\sigma = \sqrt{\frac{0.6(1-0.6)}{200}} = 0.0346$$

Thus the probability that more than 65% of people in the sample have a driver's licence is

$$\mathrm{P}(\hat{P} > 0.65) = 0.0745 \quad \text{(correct to four decimal places)}$$

Summary 12C

When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

Exercise 12C

In each of the following questions, use the normal approximation to the binomial distribution.

Example 11

1 Find the approximate probability that, in the next 50 tosses of a fair coin, the proportion of heads observed will be less than or equal to 0.46.

2 In a large city, 12% of the workforce are unemployed. If 300 people from the workforce are selected at random, find the approximate probability that more than 10% of the people surveyed are unemployed.

3 It is known that on average 50% of the children born at a particular hospital are female. Find the approximate probability that more than 60% of the next 25 children born at that hospital will be female.

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4 A car manufacturer expects 10% of cars produced to require minor adjustments before they are certified as ready for sale. What is the approximate probability that more than 15% of the next 200 cars inspected will require minor adjustments?

5 Past records show that on average 30% of the workers at a particular company have had one or more accidents in the workplace. What is the approximate probability that less than 20% of a random sample of 50 workers have had one or more accidents?

6 Sacha is shooting at a target which she has a probability of 0.6 of hitting. What is the approximate probability that:

a the proportion of times she hits the target in her next 100 attempts is less than 0.8

b the proportion of times she hits the target in her next 100 attempts is between 0.6 and 0.8

c the proportion of times she hits the target in her next 100 attempts is between 0.7 and 0.8, given that it is more than 0.6?

7 Find the approximate probability that, in the next 100 tosses of a fair coin, the proportion of heads will be between 0.4 and 0.6.

8 A machine has a probability of 0.1 of producing a defective item.

a What is the approximate probability that, in the next batch of 1000 items produced, the proportion of defective items will be between 0.08 and 0.12?

b What is the approximate probability that, in the next batch of 1000 items produced, the proportion of defective items will be between 0.08 and 0.12, given that we know that it is greater than 0.10?

9 The proportion of voters in the population who favour Candidate A is 52%. Of a random sample of 400 voters, 230 indicated that they would vote for Candidate A at the next election.

a What is the value of the sample proportion, $\hat{p}$?

b Find the approximate probability that, in a random sample of 400 voters, the proportion who favour Candidate A is greater than or equal to the value of $\hat{p}$ observed in this particular sample.

10 A manufacturer claims that 90% of their batteries will last more than 100 hours. Of a random sample of 250 batteries, 212 lasted more than 100 hours.

a What is the value of the sample proportion, $\hat{p}$?

b Find the approximate probability that, in a random sample of 250 batteries, the proportion lasting more than 100 hours is less than or equal to the value of $\hat{p}$ observed in this particular sample.

c Does your answer to part **b** cause you to doubt the manufacturer's claim?

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12D Confidence intervals for the population proportion

In practice, the reason we analyse samples is to further our understanding of the population from which they are drawn. That is, we know what is in the sample, and from that knowledge we would like to infer something about the population.

Point estimates

Suppose, for example, we wish to know the proportion of primary school children in Australia who regularly use social media. The value of the population proportion p is unknown. As already mentioned, collecting information about the whole population is generally not feasible, and so a random sample must suffice. What information can be obtained from a single sample? Certainly, the sample proportion $\hat{p}$ gives some indication of the value of the population proportion p, and can be used when we have no other information.

The value of the sample proportion $\hat{p}$ can be used to estimate the population proportion p. Since this is a single-valued estimate, it is called a **point estimate** of p.

Thus, if we select a random sample of 20 Australian primary school children and find that the proportion who use social media is 0.7, then the value $\hat{p} = 0.7$ serves as an estimate of the unknown population proportion p.

Interval estimates

The value of the sample proportion $\hat{p}$ obtained from a single sample is going to change from sample to sample, and while sometimes the value will be close to the population proportion p, at other times it will not. To use a single value to estimate p can be rather risky. What is required is an interval that we are reasonably sure contains the parameter value p.

An **interval estimate** for the population proportion p is called a **confidence interval** for p.

We have already seen that, when the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution with $\mu = p$ and $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

By standardising, we can say that the distribution of the random variable

$$\frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximated by that of a standard normal random variable Z.

We know that $\mathrm{P}(-1.96 < Z < 1.96) = 0.95$, correct to two decimal places, and therefore

$$\mathrm{P}\left(-1.96 < \frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}} < 1.96\right) \approx 0.95$$

Multiplying through gives

$$\mathrm{P}\left(-1.96\sqrt{\frac{p(1-p)}{n}} < \hat{P} - p < 1.96\sqrt{\frac{p(1-p)}{n}}\right) \approx 0.95$$

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Further simplifying, we obtain

$$\mathrm{P}\left(\hat{P} - 1.96\sqrt{\frac{p(1-p)}{n}} < p < \hat{P} + 1.96\sqrt{\frac{p(1-p)}{n}}\right) \approx 0.95$$

Remember that what we want to do is to use the value of the sample proportion $\hat{p}$ obtained from a single sample to calculate an interval that we are fairly certain (say 95% certain) contains the true population proportion p (which we do not know).

In order to do this, we need to make one further approximation, and substitute $\hat{p}$ for p in our estimate of the standard deviation σ of $\hat{P}$.

An approximate **95% confidence interval** for p is given by

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where:

- p is the population proportion (unknown)
- $\hat{p}$ is a value of the sample proportion
- n is the size of the sample from which $\hat{p}$ was calculated.

Note: In order to use this rule to calculate a confidence interval, the criteria for the normal approximation to the binomial distribution must apply. Therefore, we require both np and $n(1 - p)$ to be greater than 5.

Example 12

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 20 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

Since $\hat{p} = 0.7$ and $n = 20$, we have

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.7 \times 0.3}{20}} = 0.1025$$

and so a 95% confidence interval for p is

$$(0.7 - 1.96 \times 0.1025,\ 0.7 + 1.96 \times 0.1025) = (0.499, 0.901)$$

Thus, based on a sample of size 20 and a sample estimate of 0.7, an approximate 95% confidence interval for the population proportion p is (0.499, 0.901).

Interpretation of confidence intervals

The confidence interval found in Example 12 should not be interpreted as meaning that $\mathrm{P}(0.499 < p < 0.901) = 0.95$. In fact, such a statement is meaningless, as p is a constant and either does or does not lie in the stated interval.

The particular confidence interval found is just one of any number of confidence intervals which could be found for the population proportion p, each one depending on the particular value of the sample proportion $\hat{p}$.

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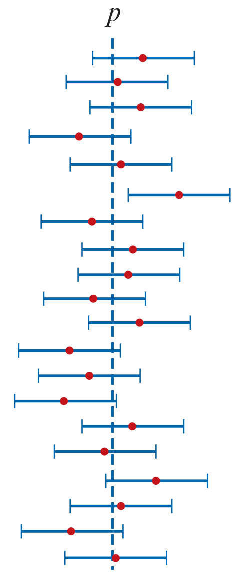
The correct interpretation of the confidence interval is that we expect approximately 95% of such intervals to contain the population proportion p. Whether or not the particular confidence interval obtained contains the population proportion p is generally not known.

If we were to repeat the process of taking a sample and calculating a confidence interval many times, the result would be something like that indicated in the diagram.



The diagram shows the confidence intervals obtained when 20 different samples were drawn from the same population. The round dot indicates the value of the sample estimate in each case. The intervals vary, because the samples themselves vary. The value of the population proportion p is indicated by the vertical line, and it is of course constant.

It is quite easy to see from the diagram that none of the values of the sample estimate is exactly the same as the population proportion, but that all the intervals except one (19 out of 20, or 95%) have captured the value of the population proportion, as would be expected in the case of a 95% confidence interval.

Using a calculator to determine confidence intervals

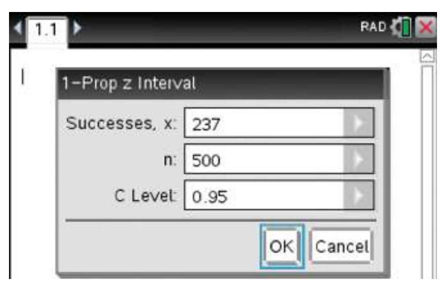
Example 13

A survey found that 237 out of 500 undergraduate university students questioned intended to take a postgraduate course in the future. Find a 95% confidence interval for the proportion of undergraduates intending to take a postgraduate course.

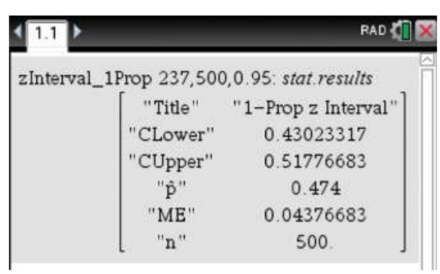
Using the TI-Nspire

In a **Calculator** page:

- Use (Menu) > **Statistics** > **Confidence Intervals** > **1–Prop z Interval**.
- Enter the values $x = 237$ and $n = 500$ as shown.



- The 'CLower' and 'CUpper' values give the 95% confidence interval (0.43, 0.52).



Note: 'ME' stands for margin of error, which is covered in the next subsection.

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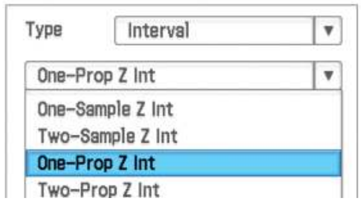
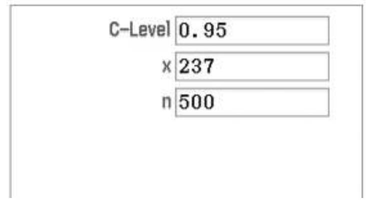
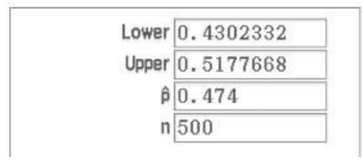
Using the Casio ClassPad

- In Statistics, go to **Calc** > **Interval**.
- Select **One–Prop Z Int** and tap Next.
- Enter the values C-Level = 0.95, $x = 237$ and $n = 500$ as shown below. Tap Next.





- The 'Lower' and 'Upper' values give the 95% confidence interval (0.43, 0.52).



Precision and margin of error

In Example 12, we found an approximate 95% confidence interval (0.499, 0.901) for the proportion p of primary school children in Australia who use social media, based on a sample of size 20. Therefore we predict that the population proportion p is somewhere in the range of approximately 50% to 90%! But this interval is so wide as to be not very helpful.

Example 14

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

Since $\hat{p} = 0.7$ and $n = 200$, we have

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.7 \times 0.3}{200}}$$

$$= 0.0324$$

and so a 95% confidence interval for p is

$$(0.7 - 1.96 \times 0.0324,\ 0.7 + 1.96 \times 0.0324) = (0.636, 0.764)$$

Thus, based on a sample of size 200 and a sample estimate of 0.7, an approximate 95% confidence interval for the population proportion p is (0.636, 0.764).

Note: This interval is much narrower than the one determined in Example 12, which was based on a sample of size 20.

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Often we discuss the confidence interval in terms of its width or, more formally, in terms of the distance between the sample estimate and the endpoints of the confidence interval.

That is, we find it useful to make statements such as 'we predict the proportion of people who will vote Labor in the next election as 52% ± 2%'. Here the sample estimate is 52%, and the distance between the sample estimate and the endpoints is 2%.

The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (M). For a 95% confidence interval,

$$M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

We can see from this rule that the margin of error is a function of the sample size n, and that one way to make the interval narrower (that is, to increase the precision of the estimate) is to increase the sample size.

Example 15

Determine the sample size required to achieve a margin of error of 2% in an approximate 95% confidence interval for the proportion p of primary school children in Australia who use social media, if the sample proportion $\hat{p}$ is found to be 0.7.

Solution

Substituting $M = 0.02$ and $\hat{p} = 0.7$ in the expression for the margin of error gives

$$0.02 = 1.96\sqrt{\frac{0.7 \times 0.3}{n}}$$

Solving for n:

$$\left(\frac{0.02}{1.96}\right)^2 = \frac{0.7 \times 0.3}{n}$$

$$\therefore \quad n = 0.7 \times 0.3 \times \left(\frac{1.96}{0.02}\right)^2 \approx 2016.84$$

Thus, to achieve a margin of error of 2%, we need a sample of size 2017.

Of course, it is highly unlikely that we will know the value of the sample proportion $\hat{p}$ before we have selected the sample. Thus it is usual to substitute an estimated value into the equation in order to determine the sample size before we select the sample. This estimate can be based on our prior knowledge of the population or on a pilot study. If we denote this estimated value for the sample proportion by p^*, we can write

$$M = 1.96\sqrt{\frac{p^*(1-p^*)}{n}}$$

Rearranging to make n the subject of the equation, we find

$$M^2 = 1.96^2\left(\frac{p^*(1-p^*)}{n}\right)$$

$$\therefore \quad n = \left(\frac{1.96}{M}\right)^2 p^*(1-p^*)$$

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A 95% confidence interval for a population proportion p will have margin of error approximately equal to a specified value of M when the sample size is

$$n = \left(\frac{1.96}{M}\right)^2 p^*(1 - p^*)$$

where p^* is an estimated value for the population proportion p.

Changing the level of confidence

So far we have only considered 95% confidence intervals, but in fact we can choose any level of confidence for a confidence interval. What is the effect of changing the level of confidence?

Consider again a 95% confidence interval:

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

From our knowledge of the normal distribution, we can say that a 99% confidence interval will be given by

$$\left(\hat{p} - 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

In general, a C% confidence interval is given by

$$\left(\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where k is such that

$$\mathrm{P}(-k < Z < k) = \frac{C}{100}$$

Example 16

Calculate and compare 90%, 95% and 99% confidence intervals for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

From Example 14, we know that the 95% confidence interval is $(0.636, 0.764)$.

The 90% confidence interval is

$$\left(0.7 - 1.64\sqrt{\frac{0.7 \times 0.3}{200}},\ 0.7 + 1.64\sqrt{\frac{0.7 \times 0.3}{200}}\right) = (0.647, 0.753)$$

The 99% confidence interval is

$$\left(0.7 - 2.58\sqrt{\frac{0.7 \times 0.3}{200}},\ 0.7 + 2.58\sqrt{\frac{0.7 \times 0.3}{200}}\right) = (0.616, 0.784)$$

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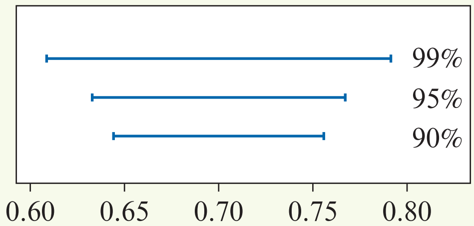
It is helpful to use a diagram to compare these confidence intervals.

From the diagram, it can be clearly seen that the effect of being more confident that the confidence interval captures the true value of the population proportion means that a wider interval is required.



In general, the width of the confidence interval (and hence the margin of error) will increase as the level of confidence increases. To be more confident that the interval will capture the true value of the population proportion, a wider confidence interval will be required.

Example 17

Suppose that we toss a fair coin 20 times, and determine the proportion of heads observed in this sample. Suppose further that this is repeated 10 times.

a Use your calculator to generate 10 values of the sample proportion $\hat{p}$ of heads in 20 coin tosses.

b Use your calculator to find an approximate 90% confidence interval for the population proportion p from each of these values of the sample proportion $\hat{p}$.

c How many of these intervals contain the value of the population proportion p?

d How many of these intervals would you expect to contain the value of the population proportion p?

Solution

a One set of simulations gave the following values for $\hat{p}$:

- 0.5
- 0.6
- 0.4
- 0.55
- 0.5
- 0.6
- 0.45
- 0.7
- 0.55
- 0.55

b Confidence intervals based on these values of $\hat{p}$ are

- $(0.32, 0.68)$
- $(0.42, 0.78)$
- $(0.22, 0.58)$
- $(0.37, 0.73)$
- $(0.32, 0.68)$
- $(0.42, 0.78)$
- $(0.27, 0.63)$
- $(0.53, 0.87)$
- $(0.37, 0.73)$
- $(0.37, 0.73)$

c Here we find that 9 of the 10 intervals contain $p = 0.5$.

d On average, we would expect $0.9 \times 10 = 9$ intervals to contain the value of p.

Summary 12D

- The value of the sample proportion $\hat{p}$ can be used to estimate the population proportion p. Since this is a single-valued estimate, it is called a **point estimate** of p.
- An **interval estimate** for the population proportion p is called a **confidence interval** for p.

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- An approximate **95% confidence interval** for p is given by

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where:

  - p is the population proportion (unknown)
  - $\hat{p}$ is a value of the sample proportion
  - n is the size of the sample from which $\hat{p}$ was calculated.

- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (M) and, for a 95% confidence interval,

$$M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- A 95% confidence interval for a population proportion p will have margin of error approximately equal to a specified value of M when the sample size is

$$n = \left(\frac{1.96}{M}\right)^2 p^*(1-p^*)$$

where p^* is an estimated value for the population proportion p.

- In general, a ***C*% confidence interval** is given by

$$\left(\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where k is such that $\mathrm{P}(-k < Z < k) = \dfrac{C}{100}$.

Skill-sheet

Exercise 12D

Example 12

1 A quality-control engineer in a factory needs to estimate the proportion of bags of potato chips packed by a certain machine that are underweight. The engineer takes a random sample of 100 bags and finds that eight of them are underweight.

a Find a point estimate for p, the proportion of bags packed by the machine that are underweight.

b Calculate a 95% confidence interval for p.

2 A newspaper wants to estimate the proportion of its subscribers who believe that the government should be allowed to tap telephones without a court order. It selects a random sample of 250 subscribers, and finds that 48 of them believe that the government should have this power.

a Find a point estimate for p, the proportion of subscribers who believe that the government should be allowed to tap telephones without a court order.

b Calculate a 95% confidence interval for p.

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3 The lengths of stay in hospital among patients is of interest to health planners. A random sample of 100 patients was investigated, and 20 were found to have stayed longer than 7 days.

a Find a point estimate for p, the proportion of patients who stay in hospital longer than 7 days.

b Calculate a 95% confidence interval for p.

Example 13

4 Given that 132 out of 400 randomly selected adult males are cigarette smokers, find a 95% confidence interval for the proportion of adult males in the population who smoke.

5 Of a random sample of 400 voters in a particular electorate, 210 indicated that they would vote for the Labor party at the next election.

a Use this information to find a 95% confidence interval for the proportion of Labor voters in the electorate.

b A random sample of 4000 voters from the same electorate was taken, and this time 2100 indicated that they would vote for Labor at the next election. Find a 95% confidence interval for the proportion of Labor voters in the electorate.

c Compare your answers to parts **a** and **b**.

6 A manufacturer claims that 90% of their batteries will last more than 50 hours.

a Of a random sample of 250 batteries, 212 lasted more than 50 hours. Use this information to find a 95% confidence interval for the proportion of batteries lasting more than 50 hours.

b An inspector requested further information. A random sample of 2500 batteries was selected and this time 2120 lasted more than 50 hours. Use this information to find a 95% confidence interval for the proportion of batteries lasting more than 50 hours.

c Compare your answers to parts **a** and **b**.

Example 15

7 Determine the size of sample required to achieve a margin of error of 2% in an approximate 95% confidence interval when the sample proportion $\hat{p}$ is 0.8.

8 Determine the size of sample required to achieve a margin of error of 5% in an approximate 95% confidence interval when the sample proportion $\hat{p}$ is 0.2.

9 Samar is conducting a survey to estimate the proportion of people in Australia who would support reducing the driving age to 16. He knows from previous studies that this proportion is about 30%.

a Determine the size of sample required for the survey to achieve a margin of error of 3% in an approximate 95% confidence interval for this proportion.

b Determine the size of sample required for the survey to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.

c Compare your answers to parts **a** and **b**.

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10 Bob is thinking of expanding his pizza delivery business to include a range of desserts. He would like to know the proportion of his clients who would order dessert from him, and so he intends to ask a number of his clients what they think.

a Bob thinks that the proportion of his clients who would order dessert is around 0.3. Determine the size of sample required for Bob to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.

b Bob's business partner Phil thinks that the proportion of clients who would order dessert is around 0.5. Determine the size of sample required to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.

c What is the effect on the margin of error if:

i Bob is correct, but they use the sample size from Phil's estimate

ii Phil is correct, but they use the sample size from Bob's estimate?

d What sample size would you recommend that Bob and Phil use?

Example 16

11 When a coin thought to be biased was tossed 100 times, it came up heads 60 times. Calculate and compare 90%, 95% and 99% confidence intervals for the probability of observing a head when that coin is tossed.

12 In a survey of attitudes to climate change, a total of 537 people from a random sample of 1000 people answered no to the question 'Do you think the government is doing enough to address global warming?' Calculate and compare 90%, 95% and 99% confidence intervals for the proportion of people in Australia who would answer no to that question.

Example 17

13 Jelly beans are packed in boxes of 50, and the overall proportion of black jelly beans is set by the manufacturer to be 0.2. Suppose that 10 boxes of jelly beans are selected at random, and the proportion of black jelly beans in each box determined.

a Use your calculator to generate 10 values of the sample proportion $\hat{p}$ of black jelly beans in a box.

b Use your calculator to find an approximate 80% confidence interval for the population proportion p from each of these values of the sample proportion $\hat{p}$.

c How many of these intervals contain the value of the population proportion p?

d How many of these intervals would you expect to contain the value of the population proportion p?

e Suppose that we generate 50 approximate 80% confidence intervals for p. How many of these intervals would you expect to contain the value of the population proportion p?

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Chapter summary

Assign-ment

Nrich

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute, and is constant.
- The **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing the attribute, and varies from sample to sample.
- The sample proportion $\hat{P} = \dfrac{X}{n}$ is a random variable, where X is the number of favourable outcomes in a sample of size n. The distribution of the random variable $\hat{P}$ is known as the **sampling distribution** of the sample proportion.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using our knowledge of selections.
- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined by assuming that X is a binomial random variable with parameters n and p. In this case, the mean and standard deviation of $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

- When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.
- If the value of the sample proportion $\hat{p}$ is used as an estimate of the population proportion p, then it is called a **point estimate** of p.
- An approximate **95% confidence interval** for p is given by

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  where:
  - p is the population proportion (unknown)
  - $\hat{p}$ is a value of the sample proportion
  - n is the size of the sample from which $\hat{p}$ was calculated.
- In general, a ***C*% confidence interval** is given by

$$\left(\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  where k is such that $\mathrm{P}(-k < Z < k) = \dfrac{C}{100}$.

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Review

- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (M) and, for a 95% confidence interval,
$$M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
- A 95% confidence interval for a population proportion p will have margin of error approximately equal to a specified value of M when the sample size is
$$n = \left(\frac{1.96}{M}\right)^2 p^*(1-p^*)$$
where p^* is an estimated value for the population proportion p.

Short-answer questions

1 A company has 2000 employees, 700 of whom are female. A random sample of 100 employees was selected, and 40 of them were female. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

2 To study the effectiveness of yoga for reducing stress levels, a researcher measured the stress levels of 50 people who had just enrolled in a 10-week introductory yoga course, and then measured their stress levels at the end the course.

a Do you think that this sample will be representative of the general population? Explain your answer.

b How would you suggest that the sample could be chosen?

3 A coin is tossed 100 times, and k heads observed.

a Give a point estimate for p, the probability of observing a head when the coin is tossed.

b Write down an expression for a 95% confidence interval for p.

4 A sample of n people were asked whether they thought that income tax in Australia was too high, and 90% said yes.

a What is the value of the sample proportion $\hat{p}$?

b Write down an expression for M, the margin of error for this estimate at the 95% confidence level, in terms of n.

c If the number of people in the sample were doubled, what would be the effect on the margin of error M?

5 Suppose that 40 independent random samples are taken from a large population, and a 95% confidence interval for the population proportion p is computed from each sample.

a How many of the 95% confidence intervals would you expect to contain the population proportion p?

b Write down an expression for the probability that all 40 confidence intervals contain the population proportion p.

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6 Suppose that 50 independent random samples were taken from a large population, and that a 90% confidence interval for the population proportion p was computed from each of these samples.

a How many of the 90% confidence intervals would you expect to contain the population proportion p?

b Write down an expression for the probability that at least 49 of the 50 confidence intervals contain the population proportion p.

7 A newspaper determined that an approximate 95% confidence interval for the proportion of people in Australia who regularly read the news online was $(0.50, 0.70)$.

a What was the value of $\hat{p}$ which was used to determine this confidence interval?

b What is the margin of error?

c How could the newspaper increase the precision of their study?

Extended-response questions

1 A survey is being planned to estimate the proportion of people in Australia who think that university fees should be abolished. The organisers of the survey want the error in the approximate 95% confidence interval for this proportion to be no more than ±2%. They have no prior information about the value of the proportion.

a Plot the sample size, $n = \left(\frac{1.96}{M}\right)^2 p^*(1 - p^*)$, against p^* for $0 \le p^* \le 1$.

b For what value of p^* is the sample size the maximum?

c What value of n would you recommend be used for the survey?

d Show that the maximum sample size required for the error in an approximate 95% confidence interval to be no more than M is approximately $n = \frac{1}{M^2}$.

2 It is known that 60% of the voters in a particular electorate support the Liberal party. A sample of 100 voters is taken. Let $\hat{P}$ be the random variable for the sampling distribution of the sample proportion. Use the normal approximation to find:

a $\mathrm{P}(\hat{P} > 0.65)$

b $\mathrm{P}(0.5 < \hat{P} < 0.65)$

3 **a** Summer is investigating the probability that a drawing pin will land point-up when tossed. She tosses the drawing pin 100 times, and finds that it lands point-up 57 times. Determine an approximate 95% confidence interval for the probability that the drawing pin lands point-up when tossed.

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b Four of Summer's friends decide to repeat her investigation, each tossing the drawing pin 100 times. They each calculate an approximate 95% confidence interval based on their own data, making five confidence intervals in all.

  i What is the probability that all five confidence intervals contain the true value of p, the probability that the drawing pin will land point-up when tossed?

  ii What is the probability that none of the confidence intervals contain p?

  iii What is the probability that at least one of the confidence intervals does not contain p?

  iv How many of these five confidence intervals would you expect to contain p?

c Summer's four friends obtained the following results, each based on tossing the drawing pin 100 times and counting the number of times that it lands point-up:

- Emma 67
- Chloe 72
- Maddie 55
- Regan 60

Summer suggests that the best estimate of p would be obtained by pooling their results. Based on all the data collected, determine an approximate 95% confidence interval for p.

4 A landscape gardener wishes to estimate how many carp live in his very large ornamental lake. He is advised that the best way to do this is through capture–recapture sampling.

a Suppose that there are N carp in the lake and he captures 500 of them, tags them and then releases them back into the lake. Write down an expression for the proportion of tagged carp in the lake.

b The next day, a sample of 400 carp is captured from the lake, and he finds that there are 60 tagged carp in this sample. What is the proportion of tagged carp in the second sample?

c If the second sample is representative of the population, we expect the proportion of tagged carp in the second sample to be the same as the proportion of tagged carp in the lake. That is,

$$\frac{60}{400} \approx \frac{500}{N}$$

Use this equation to find an estimate for the number of carp in the lake.

d Show that an expression for a 95% confidence interval for the proportion of tagged carp in the lake can be written as

$$0.15 - 1.96\sqrt{\frac{0.1275}{400}} < \frac{500}{N} < 0.15 + 1.96\sqrt{\frac{0.1275}{400}}$$

e Use this inequality to find an approximate 95% confidence interval for the number of carp in the lake.

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13 Revision of Chapters 8–12

13A Short-answer questions

1 The function

$$f(x) = \begin{cases} k\cos(\pi x) & \text{if } \frac{3}{2} < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for the continuous random variable X.

a Find the value of k.

b Find the median of X.

c Find $\mathrm{P}\left(X < \frac{7}{4} \,\middle|\, X < 2\right)$.

d Find $\mathrm{P}\left(X > \frac{9}{4} \,\middle|\, X > \frac{7}{4}\right)$.

2 The random variable X has the following probability distribution.

x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.3	0.2	0.1	0.3	0.1

Find:

a $\mathrm{P}(X > 3 \mid X > 1)$

b $\mathrm{P}(X > 1 \mid X \le 3)$

c the mean of X

d the variance of X

3 The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} kx(6 - x) & \text{if } 0 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Find:

a the value of k

b $\mathrm{P}(X < 4)$

c the median of X

d the mean of X

e $\mathrm{P}(X < 2 \mid X < 3)$

f $\mathrm{P}(X > 2 \mid X < 4)$

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4 Eight coloured balls are placed in a box: 3 red balls, 2 black balls, 2 green balls and a yellow ball. A ball is randomly withdrawn from the box and not returned, and then a second ball is randomly withdrawn.

a What is the probability of withdrawing a red ball first and a green ball second?

b What is the probability of obtaining one green and one red ball?

c What is the probability that the second ball withdrawn is not red, given that the first ball withdrawn is red?

d What is the probability that neither of the two balls withdrawn is red?

e What is the probability of obtaining two balls of the same colour?

5 Two events A and B are such that $\mathrm{P}(A) = \frac{4}{7}$ and $\mathrm{P}(B) = \frac{1}{3}$. Find $\mathrm{P}(A' \cap B)$ if:

a $\mathrm{P}(A \cup B) = \frac{5}{7}$

b A and B are mutually exclusive.

6 Two events A and B are such that $\mathrm{P}(A) = \frac{3}{4}$, $\mathrm{P}(B \mid A) = \frac{1}{5}$ and $\mathrm{P}(B' \mid A') = \frac{4}{7}$. Find:

a $\mathrm{P}(A \cap B)$

b $\mathrm{P}(B)$

c $\mathrm{P}(A \mid B)$

7 Janette and four friends each have an independent probability of 0.45 of winning a prize. Find the probability that:

a exactly two of the friends win a prize

b Janette and only one friend win a prize.

8 The random variable X has probability distribution:

x	1	2	3	4	5
$\mathrm{P}(X = x)$	a	0.3	0.1	0.2	b

a Given that $\mathrm{E}(X) = 2.34$, find the values of a and b.

b Find $\mathrm{Var}(X)$.

9 The probability that a certain football club has all its first-team players fit is 0.7. When the club has a fully fit team, the probability of it winning a home game is 0.9. When the team is not fully fit, the probability of winning a home game is 0.4.

a Find the probability that the team will win its next home game.

b Given that the team did not win its last home game, find the probability that the team was fully fit.

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10 The random variable X has probability density function:

$$f(x) = \begin{cases} (x-a)(2a-x) & \text{if } a \le x \le 2a \\ 0 & \text{otherwise} \end{cases}$$

a Show that $a^3 = 6$.

b Find $\mathrm{E}(X)$.

11 The random variable X is normally distributed with mean 40 and standard deviation 2. If $\mathrm{P}(36 < X < 44) = q$, find $\mathrm{P}(X > 44)$ in terms of q.

12 The probability density function of a random variable X is given by

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Find the value a of X such that $\mathrm{P}(X \le a) = \frac{3}{4}$.

13 A biased coin is tossed three times. On each toss, the probability of a head is p.

a Find, in terms of p, the probability that all three tosses show tails.

b If the probability of three tails is equal to 8 times the probability of three heads, find p.

14 Suppose that an approximate 95% confidence interval for the population proportion is given by the interval (a, b).

a Write down an expression for the sample proportion in terms of a and b.

b Write down an expression for the margin of error for this confidence interval in terms of a and b.

13B Extended-response questions

1 In each of a sequence of trials, the probability of the occurrence of a certain event is $\frac{1}{2}$, except that this event cannot occur in two consecutive trials.

a Show that the probability of this event occurring:

i exactly twice in three trials is $\frac{1}{4}$

ii exactly twice in four trials is $\frac{1}{2}$.

b What is the probability of this event occurring exactly twice in five trials?

2 Katia and Mikki play a game in which a fair six-sided die is thrown five times:

- Katia will receive \$1 from Mikki if there is an odd number of sixes
- Mikki will receive \$$x$ from Katia if there is an even number of sixes.

Find the value of x so that the game is fair. (Note that the number 0 is even.)

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3 A newspaper seller buys papers for 50 cents and sells them for 75 cents, and cannot return unsold papers. Daily demand has the following distribution, and each day's demand is independent of the previous day's demand.

Number of customers	24	25	26	27	28	29	30
Probability	0.05	0.10	0.10	0.25	0.25	0.15	0.10

If the newspaper seller stocks too many papers, a loss is incurred. If too few papers are stocked, potential profit is lost because of the excess demand. Let s represent the number of newspapers stocked, and X the daily demand.

a If P is the newspaper seller's profit for a particular stock level s, find an expression for P in terms of s and X.

b Find the expected value of the profit, $\mathrm{E}(P)$, when $s = 26$.

c Hence find an expression for the expected profit when s is unknown.

d By evaluating the expression for expected profit for different values of s, determine how many papers the newspaper seller should stock.

4 Let X be a random variable with mean μ and variance σ^2. Show that, if $Z = \dfrac{X - \mu}{\sigma}$, then $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

5 Anne and Jane play a game against each other, which starts with Anne aiming to throw a bean bag into a circle marked on the ground.

a The probability that the bean bag lands entirely inside the circle is $\frac{1}{2}$, and the probability that it lands on the rim of the circle is $\frac{1}{3}$.

i Show that the probability that the bean bag lands entirely outside the circle is $\frac{1}{6}$.

ii What is the probability that two successive throws land outside the circle?

iii What is the probability that, for two successive throws, the first lands on the rim of the circle and the second inside the circle?

b Jane then shoots at a target on which she can score 10, 5 or 0. With any one shot, Jane scores 10 with probability $\frac{2}{5}$, scores 5 with probability $\frac{1}{10}$, and scores 0 with probability $\frac{1}{2}$. With exactly two shots, what is the probability that her total score is:

i 20 **ii** 10?

c If the bean bag thrown by Anne lands outside the circle, then Jane is allowed two shots at her target; if the bean bag lands on the rim of the circle, then Jane is allowed one shot; if it lands inside the circle, then Jane is not allowed any shots. Find the probability that Jane scores a total of 10 as a result of any one throw from Anne.

6 The lifetime of a certain brand of light bulb is normally distributed with a mean of $\mu = 400$ hours and a standard deviation of $\sigma = 50$ hours.

a Find the probability that a randomly chosen light bulb will last more than 375 hours.

b The light bulbs are sold in boxes of 10. Find the probability that at least nine of the bulbs in a randomly selected box will last more than 375 hours.

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7 A large taxi company determined that the distance travelled annually by each taxi is normally distributed with a mean of 80 000 km and a standard deviation of 20 000 km.

a What is the probability that a randomly selected taxi will travel between 56 000 km and 60 000 km in a year?

b What percentage of taxis can be expected to travel either less than 48 000 km or more than 96 000 km in a year?

c How many of the 250 taxis in the fleet are expected to travel between 48 000 km and 96 000 km in a year?

d At least how many kilometres would be travelled by 85% of the taxis?

8 The weight of cereal in boxes, packed by a particular machine, is normally distributed with a mean of μ g and a standard deviation of $\sigma = 5$ g.

a A box is considered underweight if it weighs less than 500 g.

i Find the proportion of boxes that will be underweight if $\mu = 505$ g.

ii Find the value of μ required to ensure that only 1% of boxes are underweight.

b As a check on the setting of the machine, a random sample of five boxes is chosen and the setting changed if more than one of them is underweight. Find the probability that the setting of the machine is changed if $\mu = 505$ g.

9 A factory has two machines that produce widgets. The time taken, X seconds, to produce a widget using machine A is normally distributed, with a mean of 10 seconds and a standard deviation of 2 seconds. The time taken, Y seconds, to produce a widget using machine B has probability density function given by

$$f(y) = \begin{cases} k(y-8) & \text{if } 8 < y < 12 \\ 0 & \text{otherwise} \end{cases}$$

a **i** Find the value of k.

ii Show that machine A has a greater probability of producing a widget in less than 11 seconds than machine B.

iii Find which machine, on average, is quicker in producing widgets.

b Suppose that 60% of the widgets manufactured at the factory are produced by machine A, and 40% by machine B. If a widget selected at random is known to have been produced in less than 10 seconds, what is the probability that it was produced by machine A?

10 The random variable X has probability function given by

$$P(X = x) = \begin{cases} \dfrac{c}{x} & \text{if } x = 1, 2, \dots, 6 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find the value of:

a c **b** $E(X)$ **c** $\mathrm{Var}(X)$

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11 The queuing time, X minutes, at the box office of a movie theatre has probability density function:

$$f(x) = \begin{cases} kx(100 - x^2) & \text{if } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a Find:

i the value of k **ii** the mean of X

iii the probability that a moviegoer will have to queue for more than 3 minutes

iv the probability that a moviegoer will have to queue for more than 3 minutes, given that she queues for less than 7 minutes.

b If 10 moviegoers go independently to the theatre, find the probability that at least five of them will be required to queue for more than 3 minutes.

12 Electronic sensors of a certain type fail when they become too hot. The temperature at which a randomly chosen sensor fails is T°C, where T is modelled as a normal random variable with mean μ and standard deviation σ.

a In a laboratory test, 98% of a random sample of sensors continued working at a temperature of 80°C, but only 4% continued working at 104°C.

i Show the given information on a sketch of the distribution of T.

ii Determine estimates for the values of μ and σ.

b More extensive tests confirmed that T is normally distributed, but with $\mu = 94.5$ and $\sigma = 5.7$. Use these values in the rest of the question.

i What proportion of sensors will operate in boiling water (i.e. at 100°C)?

ii The manufacturers wish to quote a safe operating temperature at which 99% of the sensors will work. What temperature should they quote?

13 A flight into an airport is declared to be 'on time' if it touches down within 3 minutes either side of the advertised arrival time; otherwise, it is declared early or late. On any one occasion, the probability that a fight is on time is 0.5 and the probability that it is late is 0.3. The time of arrival of a particular flight on any one day is independent of the time of arrival on any other day.

a Calculate the probability that:

i on any given day, the flight arrives early

ii on any given day, the flight does not arrive late

iii the flight arrives on time on three consecutive days

iv in any given week, the flight arrives late on Monday, but is on time for all the remaining four weekdays.

b In a given week of five days, find the probability that:

i the flight is late exactly once

ii the flight is early exactly twice.

c The airline is reported to the authority if the flight is late on more than two occasions in a five-day week. Find the probability that this happens.

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14 Jam is packed in tins of nominal net weight 1 kg. The actual weight of jam delivered to a tin by the filling machine is normally distributed about the mean weight set on the machine, with a standard deviation of 12 g.

a If the machine is set to 1 kg, find the probability that a tin chosen at random contains less than 985 g.

b It is a legal requirement that no more than 1% of tins contain less than the nominal weight. Find the minimum setting of the filling machine which will meet this requirement.

15 In a factory, machines A, B and C are all producing springs of the same length. Of the total production of springs in the factory, machine A produces 35% and machines B and C produce 25% and 40% respectively. Of their production, machines A, B and C produce 3%, 6% and 5% defective springs respectively.

a Find the probability that:

i a randomly selected spring is produced by machine A and is defective

ii a randomly selected spring is defective.

b Given that a randomly selected spring is defective, find the probability that it was produced by machine C.

c Given that a randomly selected spring is not defective, find the probability that it was produced by either machine A or machine B.

16 An electronic game comes with five batteries. The game only needs four batteries to work. But because the batteries are sometimes faulty, the manufacturer includes five of them with the game. Suppose that X is the number of good batteries included with the game. The probability distribution of X is given in the following table.

x	0	1	2	3	4	5
$\Pr(X = x)$	0.01	0.02	0.03	0.04	0.45	0.45

a Use the information in the table to:

i find μ, the expected value of X

ii find σ, the standard deviation of X, correct to one decimal place

iii find, exactly, the proportion of the distribution that lies within two standard deviations of the mean

iv find the probability that a randomly selected game works, i.e. find $\Pr(X \geq 4)$.

b The electronic games are packed in boxes of 20. Whether or not an electronic game in a box will work is independent of any other game in the box working. Let Y be the number of working games in a box.

i Name the distribution of Y.

ii Find the expected number of working games in a box.

iii Find the standard deviation of the number of working games in a box.

iv Find the probability that a randomly chosen box will contain at least 19 working games.

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17 There are n identical black balls and n identical white balls. A blue box contains 3 black balls and $n - 3$ white balls. A red box contains $n - 3$ black balls and 3 white balls. A ball is taken at random from the red box and put in the blue box. A ball is then taken at random from the blue box.

a Find the probability, in terms of n, that the ball taken from the blue box is:

i black **ii** white.

b Find the probability, in terms of n, that the first ball is black given that the second is white.

18 In a study of the prevalence of red hair in a certain country, researchers collected data from a random sample of 1800 adults.

a Of the 1000 females in the sample, they found that 10% had red hair. Calculate an approximate 95% confidence interval for the proportion with red hair in the female population.

b Of the 800 males in the sample, they found that 10% had red hair. Calculate an approximate 95% confidence interval for the proportion with red hair in the male population.

c Why is the width of the confidence interval for males different from the width of the confidence interval for females?

d How should the sample of 1800 adults be chosen to ensure that the widths of the two confidence intervals are the same when the sample proportions are the same?

e Assume that there are 1000 females and 800 males in the sample, and that the proportion of females in the sample with red hair is 10%. What sample proportion of red-headed males would result in the 95% confidence interval for the proportion with red hair in the female population and the 95% confidence interval for the proportion with red hair in the male population being of the same width?

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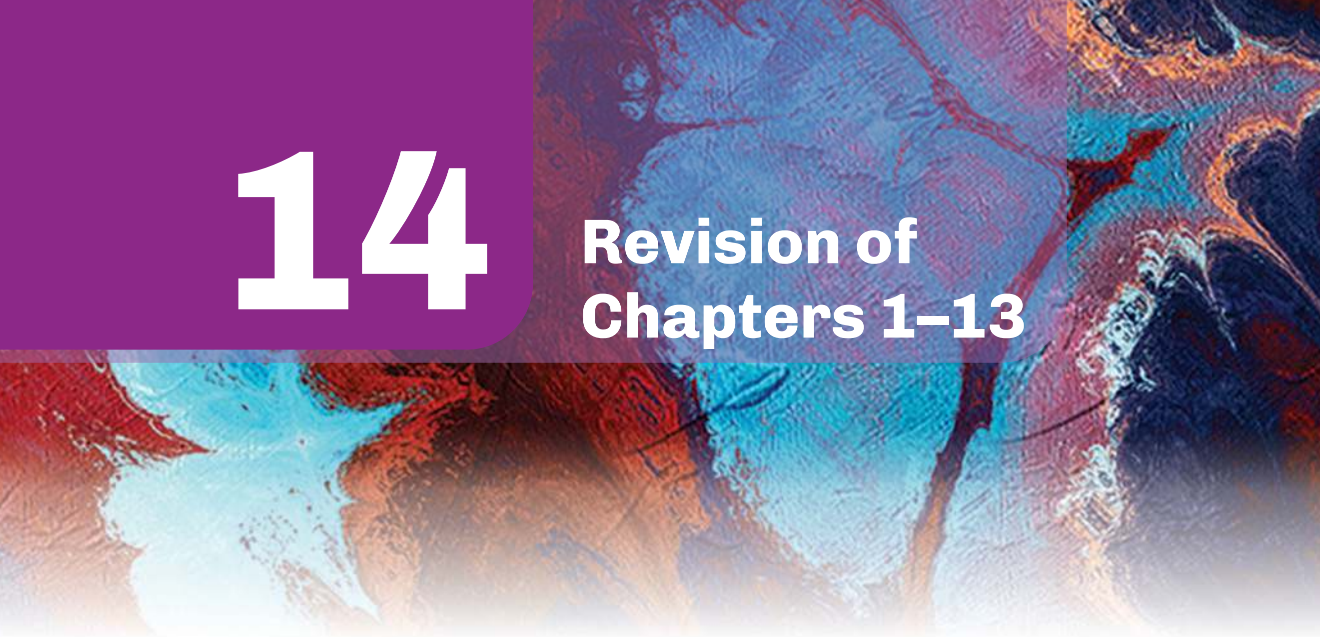

14 Revision of Chapters 1–13

14A Short-answer questions

1 Let $f(x) = x^2 + 6$ and $g(x) = 3x + 1$. Write down the rule of $f(g(x))$.

2 **a** Let $f(x) = (5x^3 - 3x)^7$. Find $f'(x)$. **b** Let $f(x) = 2xe^{4x}$. Evaluate $f'(0)$.

3 **a** For $f(x) = \dfrac{\sin x}{2x + 1}$, find $f'\left(\dfrac{\pi}{2}\right)$. **b** For $f(x) = 3x\tan(2x)$, find $f'\left(\dfrac{\pi}{3}\right)$.

4 **a** Find the second derivative of $x^2 \ln(2x)$ with respect to x.

b Let $f(x) = e^{\sin(2x)}$. Find $f''(x)$.

5 For each of the following, solve for x in terms of y:

a $y = 5e^{x-1} - 3$ **b** $y = \ln(\sqrt{3 - x})$

6 Find the average value of $y = e^x$ over the interval $[0, 4]$.

7 Using linear approximation, find an estimate for the change in the area of a circle when the radius increases from 3 cm to 3.02 cm.

8 Find the coordinates of the points of inflection on the graph of $f(x) = x^2(4 - x^2)$.

9 Find an antiderivative of $\dfrac{1}{1 - 3x}$ with respect to x, for $x < \dfrac{1}{3}$.

10 Let X be a normally distributed random variable with a mean of 84 and a standard deviation of 6. Let Z be the standard normal random variable.

a Find the probability that X is greater than 84.

b Use the result that $\Pr(Z < 1) = 0.84$ to find the probability that $78 < X < 90$.

c Find the probability that $X < 78$ given that $X < 84$.

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11 The probability density function of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x}{24} & \text{if } 1 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\text{P}(X < 3)$.

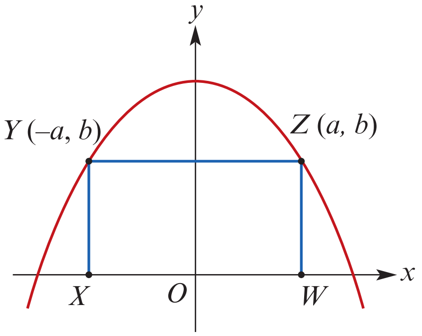
b If $b \in [1, 7]$ and $\text{P}(X \ge b) = \dfrac{3}{8}$, find b.

12 A tangent to the graph of $y = x^{\frac{1}{3}}$ has equation $y = \dfrac{1}{3}x + a$. Find the value(s) of a.

13 A rectangle $XYZW$ has two vertices on the x-axis and the other two vertices on the graph of $y = 16 - 4x^2$, as shown in the diagram.

a Find the area, A, of rectangle $XYZW$ in terms of a.

b Find the maximum value of A and the value of a for which this occurs.



14 Let $f(x) = -3x^2 + 2bx + 9$ with $\int_{-1}^{3} f(x)\,dx = 32$. Find the value of b.

15 Simone has either a sandwich or pasta for lunch. If she has a sandwich, the probability that she has a sandwich again the next day is 0.6. If she has pasta, the probability that she has pasta again the next day is 0.3. Suppose that Simone has a sandwich for lunch on a Monday. What is the probability that she has pasta for lunch on the following Wednesday?

16 A player in a game of chance can win \$0, \$1, \$2 or \$3. The amount won, \$$X$, is a random variable with probability distribution given by:

x	0	1	2	3
$\text{P}(X = x)$	0.6	0.2	0.15	0.05

a Find the mean of X.

b What is the probability that a player wins the same amount from two games?

17 Every Thursday night, Chris either goes to the gym or goes for a run. If he goes to the gym one Thursday, the probability that he goes to the gym the next Thursday is 0.5. If he goes for a run one Thursday, the probability that he goes for a run the next Thursday is 0.6. If Chris goes to the gym one Thursday, what is the probability that he goes for a run on exactly two of the next three Thursdays?

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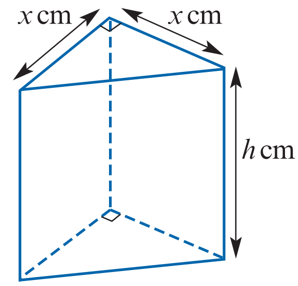
18 A brick is made in the shape of a right triangular prism. The triangular end is a right-angled isosceles triangle, with the equal sides of length x cm. The height of the brick is h cm. The volume of the brick is 2000 cm^3.



a Find an expression for h in terms of x.

b Show that the total surface area, $A\text{ cm}^2$, of the brick is given by $A = \dfrac{4000\sqrt{2} + 8000}{x} + x^2$.

c Find the value of x^3 if the brick has minimum surface area.

19 In order to measure the effect of alcohol on reaction time, an investigator selects a random sample of subjects from a group of diners in a restaurant.

a Do you think this sample will be representative of the general population? Explain your answer.

b How would you suggest that the sample could be chosen?

20 A coin is tossed 100 times, and 53 heads observed.

a Give a point estimate for p, the probability of a head when the coin is tossed.

b Write down an expression for a 95% confidence interval for p.

21 A sample of n people were asked whether they thought that Australians had access to adequate hospital care, and 37% said no.

a What is the value of the sample proportion, $\hat{p}$?

b Write down an expression for M, the margin of error for this estimate at the 95% confidence level, in terms of n.

c If the number of people in the sample were halved, what would be the effect on M?

14B Extended-response questions

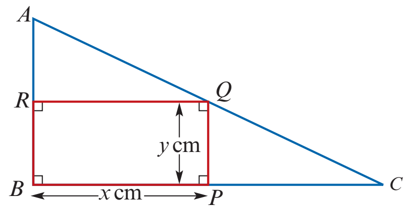
1 a i Find the coordinates of the stationary point for the curve with equation

$$y = \frac{16x^3 + 4x^2 + 1}{2x^2}$$

ii Determine the nature of this stationary point.

b The right-angled triangle ABC shown in the diagram has side lengths $AB = 5$ cm and $AC = 13$ cm.

The rectangle $BPQR$ is such that its vertices P, Q and R lie on the line segments BC, CA and AB respectively.



i Given that $BP = x$ cm and $PQ = y$ cm, show that $y = \dfrac{60 - 5x}{12}$.

ii Find the area of the rectangle, $A\text{ cm}^2$, in terms of x.

iii Find the maximum value of this area as x varies.

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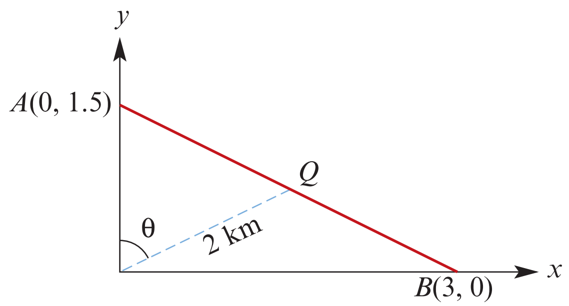
Revision

2 **a** Find the equation of the straight line joining the points $A(0, 1.5)$ and $B(3, 0)$.



b Let $y = \sin\theta + 2\cos\theta$.

i Find $\dfrac{dy}{d\theta}$.

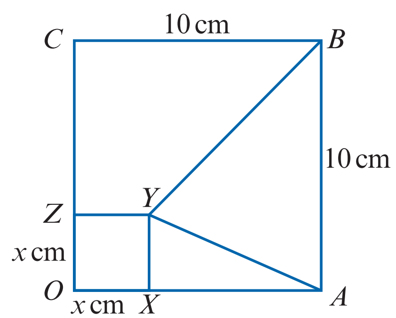
ii Solve the equation $\dfrac{dy}{d\theta} = 0$ for θ, where $0° \le \theta \le 90°$.

iii State the coordinates of the stationary point of $y = \sin\theta + 2\cos\theta$, where $0° \le \theta \le 90°$.

iv It can be shown that $\sin\theta + 2\cos\theta$ can be written in the form $r\sin(\theta + \alpha)$. Use the result of part **iii** and the fact that $y = 2$ when $\theta = 0$ to find the values of r and α.

v Use addition of ordinates and the result of part **iv** to sketch the graph of y against θ for $0° \le \theta \le 90°$.

3 A square piece of card $OABC$, of side length 10 cm, is cut into four pieces by removing a square $OXYZ$ of side length x cm as shown, and then cutting out the triangle ABY.



a **i** Find A cm^2, the sum of the areas of $OXYZ$ and ABY, in terms of x.

ii Find the domain of the function which determines this area.

iii Sketch the graph of the function, with domain determined in part **ii**.

iv State the minimum value of this area.

b **i** Find the rule for the function of x which represents the area of triangle AXY.

ii Sketch the graph of this function for a suitable domain.

c Find the ratio of the areas of the four pieces when the area of triangle AXY is a maximum.

4 A curve C has equation $y = ax - x^2$, where a is a positive constant.

a Sketch C, showing clearly the coordinates of the axis intercepts.

b Calculate the area of the finite region bounded by C and the x-axis, giving your answer in terms of a.

c The lines $x = \dfrac{1}{3}a$ and $x = \dfrac{2}{3}a$ intersect C at the points A and B respectively.

i Find, in terms of a, the y-coordinates of A and B.

ii Calculate the area of the finite region bounded by C and the straight line AB, giving your answer in terms of a.

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5 The number of people unemployed in a particular population can be modelled by the function

$$f(t) = 1000(t^2 - 10t + 44)e^{\frac{-t}{10}}$$

where t is the number of months after January 2012 and $0 \le t \le 35$.

a Use this function to find an expression for:

i the rate of increase of the number unemployed

ii the rate of increase of this rate of increase.

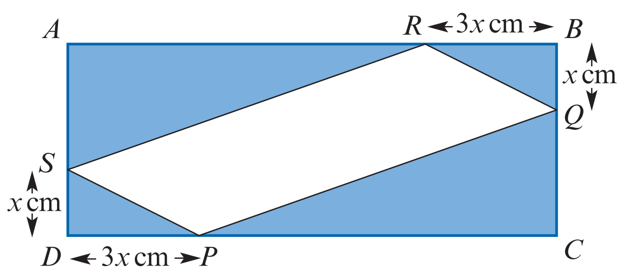
b Find the values of t for which:

i the number unemployed was increasing

ii the rate of increase of the number unemployed was going down

iii the number unemployed was increasing and the rate of increase of the number unemployed was going down.

6 In the figure, $ABCD$ is a rectangle with $AB = 30$ cm and $AD = 10$ cm. The shaded portions are cut away, leaving the parallelogram $PQRS$, where $BQ = SD = x$ cm and $RB = DP = 3x$ cm.



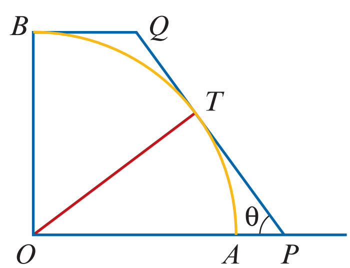
a Find the area, S cm^2, of the parallelogram in terms of x.

b Find the allowable values of x.

c Find the value of x for which S is a maximum.

d Sketch the graph of S against x for a suitable domain.

7 In the figure, OAB is a quadrant of a circle of radius 1 unit. The line segment OA is extended to a point P. From P, a tangent to the quadrant is drawn, touching it at T and meeting another tangent, BQ, at Q. Let $\angle OPQ = \theta$.



a **i** Find the length OP as a function of θ.

ii Find the length BQ as a function of θ.

b Show that the area, S, of trapezium $OPQB$ is given by $\dfrac{2 - \cos\theta}{2\sin\theta}$.

c Show that $\dfrac{dS}{d\theta} = \dfrac{2 - 4\cos\theta}{4\sin^2\theta}$.

d Find the minimum value of S and the distance AP when S is a minimum.

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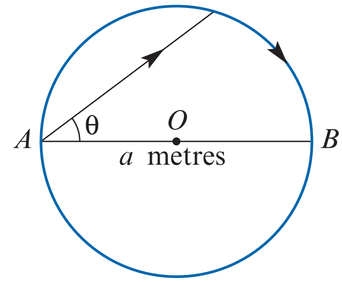
Revision

8 A dog is at point A on the edge of a circular lake of diameter a metres, and she wishes to reach her owner who is at the diametrically opposite point B.

The dog can swim at $\frac{1}{2}$ m/s and run at 1 m/s.



a If she swims in a direction making an angle of θ with AB and then runs round the edge of the lake to B, show that the time taken, T s, is given by

$$T = a(\theta + 2\cos\theta)$$

b On one set of axes, sketch the graphs of $y = 200\,\theta$ and $y = 400\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Using addition of ordinates, sketch the graph of

$$y = 200(\theta + 2\cos\theta)$$

(Find the maximum value of y for $0 \le \theta \le \frac{\pi}{2}$ by finding $\frac{dy}{d\theta}$ and then solving the equation $\frac{dy}{d\theta} = 0$.)

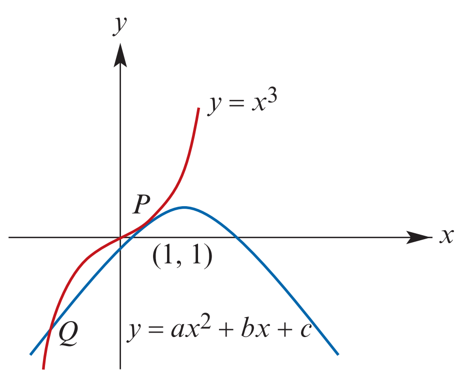
c Sketch the graph of $T = a(\theta + 2\cos\theta)$ for $0 \le \theta \le \frac{\pi}{2}$ and state the minimum value of T.

9 **a** **i** Show that, if $f(x) = (x-1)g(x)$ and $f'(x) = (x-1)h(x)$, where $g(x)$ and $h(x)$ are polynomials, then $(x-1)$ must be a factor of $g(x)$.

ii Let $F(x) = x^3 - kx^2 - (3-2k)x - (k-2)$. Show that $F(1) = F'(1) = 0$.

iii Using the results of parts **i** and **ii**, solve the equation $F(x) = 0$.

b The parabola $y = ax^2 + bx + c$ and the cubic $y = x^3$ touch at $P(1, 1)$ (and have the same gradient at this point). The curves also meet at Q.



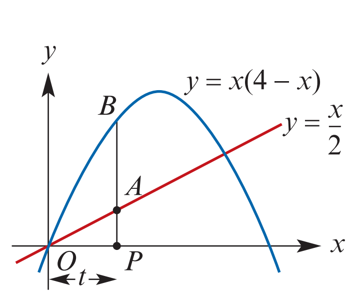
i Find b and c in terms of a.

ii If the coordinates of Q are (h, k), find h in terms of a. (Use the result of **a**.)

iii If Q has coordinates $(-2, -8)$, find the values of a, b and c.

iv If Q has coordinates $(-3, -27)$, find the values of a, b and c.

10 The point P has coordinates $(t, 0)$, where $0 < t < \frac{7}{2}$. The line PAB is parallel to the y-axis.



a Let Z be the length of AB. Find Z in terms of t.

b Sketch the graph of Z against t.

c State the maximum value of Z and the value of t for which it occurs.

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11 A study is being conducted of the numbers of male and female children in families in a certain population.

a A simple model is that each child in any family is equally likely to be male or female, and that the sex of each child is independent of the sex of any previous children in the family. Using this model, calculate the probability that in a randomly chosen family of four children:

i there will be two males and two females

ii there will be exactly one female, given that there is at least one female.

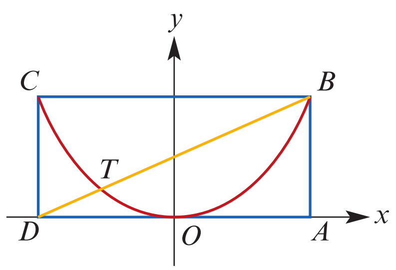
b An alternative model is that the first child in any family is equally likely to be male or female, but that, for any subsequent children, the probability that they will be of the same sex as the previous child is $\frac{3}{5}$. Using this model, calculate the probability that in a randomly chosen family of four children:

i all four will be of the same sex

ii no two consecutive children will be of the same sex

iii there will be two males and two females.

12 In the figure, $ABCD$ is a rectangle with $OA = OD = a$ and $AB = b$.
The equation of the parabola BOC is $y = kx^2$.



a Express k in terms of a and b.

b If BD cuts the parabola at T, find:

i the equation of the straight line BD

ii the coordinates of T.

c Show that the area bounded by the parabola and the line BC is $\frac{4}{3}ab$ square units.

d Let S_1 be the area of the region bounded by the line segment BT and the curve BOT. Let S_2 be the area of the region bounded by the curve CT and the line segments BC and BT. Find the ratio $S_1 : S_2$.

13 A certain type of brass washer is manufactured as follows. A length of brass rod is cut cross-sectionally into pieces of mean thickness 0.25 cm, with a standard deviation of 0.002 cm. These brass slices are then put through a machine that punches out a circular hole of mean diameter 0.5 cm through the middle of the slice, with a standard deviation of 0.05 cm. The thickness of the washers and the diameters of the holes are known to be normally distributed, and do not depend on each other.

a Find the probability that a randomly selected washer will:

i have a thickness of less than 0.253 cm

ii have a thickness of less than 0.247 cm

iii have a hole punched with a diameter greater than 0.56 cm

iv have a hole punched with a diameter less than 0.44 cm.

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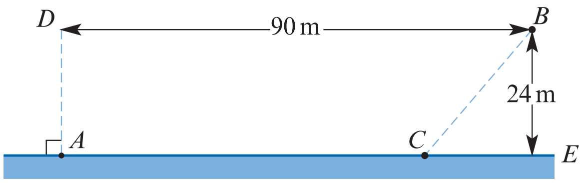
b The brass washers are acceptable only if they are between 0.247 cm and 0.253 cm in thickness with a hole of diameter between 0.44 cm and 0.56 cm. Find:

i the percentage of washers that are rejected

ii the expected number of acceptable washers in a batch of 1000 washers

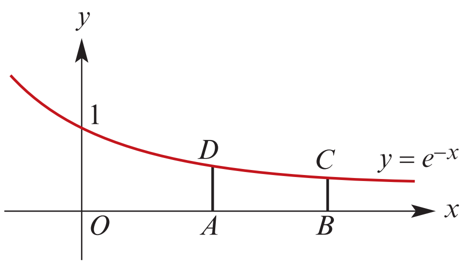
iii the expected number of acceptable washers that will be rejected in a batch of 1000 washers.

14 A ditch is to be dug to connect the points A and B in the figure. The earth on the same side of AE as B is hard, and the earth on the other side is soft.



The cost of digging hard earth is \$200 per metre and soft earth is \$100 per metre. Find the position of point C, where the turn is made, that will minimise the cost.

15 The diagram shows the graph of $y = e^{-x}$. The points A and B have coordinates $(n, 0)$ and $(n + 1, 0)$ respectively, and the points C and D on the curve are such that AD and BC are parallel to the y-axis.



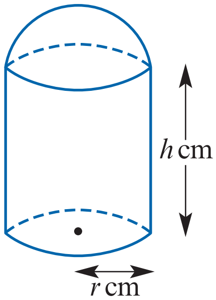
a **i** Find the equation of the tangent to $y = e^{-x}$ at the point D.

ii Find the intercept of the tangent with the x-axis.

b **i** Find the area of the region $ABCD$.

ii The line BD divides the region into two parts. Find the ratio of the areas of these two parts.

16 A closed capsule is to be constructed as shown in the diagram. It consists of a circular cylinder of height h cm with a flat base of radius r cm. It is surmounted by a hemispherical cap.



a **i** Show that the volume of the capsule, V cm^3, is given by $V = \dfrac{\pi r^2}{3}(3h + 2r)$.

ii Show that the surface area of the capsule, S cm^2, is given by $S = \pi r(2h + 3r)$.

b **i** If $V = \pi a^3$, where a is a positive constant, find h in terms of a and r.

ii Hence find S in terms of a and r.

c **i** By using addition of ordinates, sketch the graph of S against r for a suitable domain.

ii Find the coordinates of the turning point by first finding $\dfrac{dS}{dr}$, and then solving the equation $\dfrac{dS}{dr} = 0$ for r and determining the corresponding value of S.

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17 A manufacturer sells cylinders whose diameters are normally distributed with mean 3 cm and standard deviation 0.002 cm. The selling price is $\$s$ per cylinder and the cost of manufacture is \$1 per cylinder. A cylinder is returned and the purchase money is refunded if the diameter of the cylinder is found to differ from 3 cm by more than d cm. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is 0.25.

- **a** Find d.
- **b** The profit, $\$Q$, per cylinder is a random variable. Give the possible values of Q in terms of s, and the probabilities of these values.
- **c** Express the mean and standard deviation of Q in terms of s.

18 The length of a certain species of worm has a normal distribution with mean 20 cm and standard deviation 1.5 cm.

- **a** Find the probability that a randomly selected worm has a length greater than 22 cm.
- **b** If the lengths of the worms are measured to the nearest centimetre, find the probability that a randomly selected worm has its length measured as 20 cm.
- **c** If five worms are randomly selected, find the probability that exactly two will have their lengths measured as 20 cm (to the nearest centimetre).

19 The amount of coal, P tonnes, produced by x miners in one shift is given by the rule:

$$P = \frac{x^2}{90}(56 - x) \quad \text{where } 1 \le x \le 40$$

- **a** Find $\frac{dP}{dx}$.
- **b** **i** Sketch the graph of P against x for $1 \le x \le 40$.
  ii State the maximum value of P.
- **c** Write down an expression in terms of x for the average production per miner in the shift. Denote the average production per miner by A (in tonnes).
  i Sketch the graph of A against x for $1 \le x \le 40$.
  ii State the maximum value of A and the value of x for which it occurs.

20 Consider the family of quadratic functions with rules of the form

$$f(x) = (k + 2)x^2 + (6k - 4)x + 2$$

where k is an arbitrary constant.

- **a** Sketch the graph of f when:
  i $k = 0$ **ii** $k = -2$ **iii** $k = -4$
- **b** Find the coordinates of the turning point of the graph of $y = f(x)$ in terms of k. If the coordinates of the turning point are (a, b), find:
  i $\{k : a > 0\}$ **ii** $\{k : a = 0\}$ **iii** $\{k : b > 0\}$ **iv** $\{k : b < 0\}$
- **c** For what values of k is the turning point a local maximum?

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d By using the discriminant, state the values of k for which:

i $f(x)$ is a perfect square

ii there are no solutions to the equation $f(x) = 0$.

21 **a** Find the solution to the equation $e^{2-2x} = 2e^{-x}$.

b Let $y = e^{2-2x} - 2e^{-x}$.

i Find $\dfrac{dy}{dx}$. **ii** Solve the equation $\dfrac{dy}{dx} = 0$.

iii State the coordinates of the turning points of $y = e^{2-2x} - 2e^{-x}$.

iv Sketch the graph of $y = e^{2-2x} - 2e^{-x}$ for $x \geq 0$.

c State the set of values of k for which the equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive solutions.

22 **a** Sketch, on a single clear diagram, the graphs of:

i $y = x^2$ **ii** $y = (x + a)^2$ **iii** $y = b(x + a)^2$ **iv** $y = b(x + a)^2 + c$

where a, b and c are positive constants with $b > 1$.

b Show that $\dfrac{2x^2 + 4x + 5}{x^2 + 2x + 1} = \dfrac{3}{(x + 1)^2} + 2$, for all values except $x = -1$.

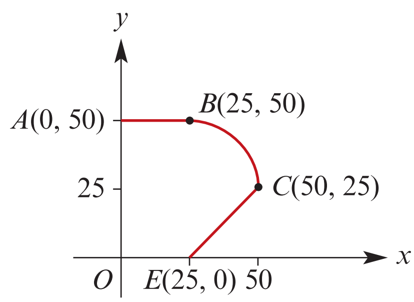
c Hence state precisely a sequence of transformations by which the graph of $y = \dfrac{2x^2 + 4x + 5}{x^2 + 2x + 1}$ may be obtained from the graph of $y = \dfrac{1}{x^2}$.

d Evaluate $\displaystyle\int_0^1 \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}\,dx$.

e Sketch the graphs of $y = \dfrac{1}{x^2}$ and $y = \dfrac{3}{(x + 1)^2} + 2$ on the one set of axes, and indicate the region for which the area has been determined in part **d**.

23 A real-estate agent has a block of land to sell. An x–y coordinate grid is placed with the origin at O, as shown in the diagram.

The block of land is $OABCE$, where OA, AB, CE and EO are straight line segments and the curve through points B and C is part of a parabola with equation of the form $y = ax^2 + 4x + c$.



a Find the equation of line segments:

i AB **ii** EC

b Find the values of a and c and hence find the equation of the parabola through points B and C.

c Find the area of:

i the rectangle $OEBA$ **ii** the region EBC (with boundaries as defined above)

iii the block of land.

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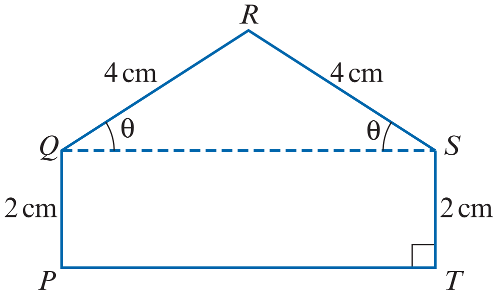
24 In the diagram, $PQRST$ is a thin metal plate, where $PQST$ is a rectangle with $PQ = 2$ cm and QRS is an isosceles triangle with $QR = RS = 4$ cm.



a Show that the area of the metal plate, A cm^2, is given by

$$A = 16(\cos\theta + \cos\theta\sin\theta)$$

for $0 < \theta < \dfrac{\pi}{2}$.

b Show that $\dfrac{dA}{d\theta} = 16(1 - \sin\theta - 2\sin^2\theta)$.

c Solve the equation $\dfrac{dA}{d\theta} = 0$ for $0 < \theta < \dfrac{\pi}{2}$ by first solving $16(1 - a - 2a^2) = 0$ for a.

d Hence sketch the graph of A against θ for $0 < \theta < \dfrac{\pi}{2}$ and state the maximum value of A.

25 The length of an engine part must be between 4.81 cm and 5.20 cm. In mass production, it is found that 0.8% are too short and 3% are too long. Assume that the lengths are normally distributed.

a Find the mean and standard deviation of this distribution.

b Each part costs \$4 to produce; those that turn out to be too long are shortened at an extra cost of \$2, and those that turn out to be too short are rejected. Find the expected total cost of producing 100 parts that meet the specifications.

26 The temperature, T°C, of water in a kettle at time t minutes is given by the formula

$$T = \theta + Ae^{-kt}$$

where θ°C is the temperature of the room in which the kettle sits.

a Assume that the room is of constant temperature 21°C. At 2:23 p.m., the water in the kettle boils at 100°C. After 10 minutes, the temperature of the water in the kettle is 84°C. Use this information to find the values of k and A, giving your answer correct to two decimal places.

b At what time will the temperature of the water in the kettle be 70°C?

c Sketch the graph of T against t for $t \geq 0$.

d Find the average rate of change of temperature for the time interval [0, 10].

e Find the instantaneous rate of change of temperature when:

i $t = 6$ **ii** $T = 60$

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27 Large batches of similar components are delivered to a company. A sample of five articles is taken at random from each batch and tested. If at least four of the five articles are found to be good, the batch is accepted. Otherwise, the batch is rejected.

a If the fraction of defectives in a batch is $\frac{1}{2}$, find the probability of the batch being accepted.

b If the fraction of defectives in a batch is p, show that the probability of the batch being accepted is given by a function of the form

$$A(p) = (1 - p)^4(1 + bp), \quad 0 \le p \le 1$$

and find the value of b.

c Sketch the graph of A against p for $0 \le p \le 1$. (Using a calculator would be appropriate.)

d Find correct to two decimal places:

i the value of p for which $A(p) = 0.95$

ii the value of p for which $A(p) = 0.05$.

e **i** Find $A'(p)$, for $0 \le p \le 1$.

ii Sketch the graph of $A'(p)$ against p.

iii For what value of p is $A'(p)$ a minimum?

iv Describe what the result of part **iii** means.

28 A machine produces ball-bearings with a mean diameter of 3 mm. It is found that 6.3% of the production is being rejected as below the lower tolerance limit of 2.9 mm, and a further 6.3% is being rejected as above the upper tolerance limit of 3.1 mm. Assume that the diameters are normally distributed.

a Calculate the standard deviation of the distribution.

b A sample of eight ball-bearings is taken. Find the probability that:

i at least one is rejected **ii** two are rejected.

c The setting of the machine now 'wanders' such that the standard deviation remains the same, but the mean changes to 3.05 mm.

i Calculate the total percentage of the production that will now fall outside the given tolerance limits.

ii Find the value of c such that the probability that the diameter lies in the interval $(3.05 - c, 3.05 + c)$ is 0.9.

29 There is a probability of 0.8 that a boarding student will miss breakfast if he oversleeps. There is a probability of 0.3 that the student will miss breakfast even if he does not oversleep. The student has a probability of 0.4 of oversleeping.

a On a random day, what is the probability of:

i the student oversleeping and missing breakfast

ii the student not oversleeping and still missing breakfast

iii the student not missing breakfast?

b Given that the student misses breakfast, find the probability that he overslept.

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c It is found that 10 students in the boarding house have identical probabilities for sleeping in and missing breakfast to the student mentioned above. Find the probability that:

i exactly two of the 10 students miss breakfast

ii at least one of the 10 students misses breakfast

iii at least eight of the students don't miss breakfast.

30 A coin is tossed 1000 times, and 527 heads observed.

a Give a point estimate for p, the probability of observing a head when the coin is tossed.

b Determine an approximate 95% confidence interval for p.

c What level of confidence would be given by a confidence interval for p which is half the width of the approximate 95% confidence interval?

d What level of confidence would be given by a confidence interval for p which is twice the width of the approximate 95% confidence interval?

31 **a** On the one set of axes, sketch the graphs of $y = \frac{1}{x}$ and $y = e^x$ for $x > 0$.

b Using addition of ordinates, sketch the graph of $y = \frac{1}{x} + e^x$ for $x > 0$.

(Do not attempt at this stage to find the coordinates of the turning points.)

c Find $\frac{dy}{dx}$ for $y = \frac{1}{x} + e^x$.

d **i** Show that $\frac{dy}{dx} = 0 \Leftrightarrow 2\ln x = -x$, for $x > 0$.

ii Explain why this implies that the local minimum of $y = \frac{1}{x} + e^x$ lies in the interval $(0, 1)$.

iii Using a calculator, show that the point of intersection of the graphs of $y = 2\ln x$ and $y = -x$ is at $(0.70, -0.70)$, correct to two decimal places.

iv Hence find the coordinates of the local minimum of $y = \frac{1}{x} + e^x$, correct to one decimal place.

32 A section of a creek bank can be modelled by the function:

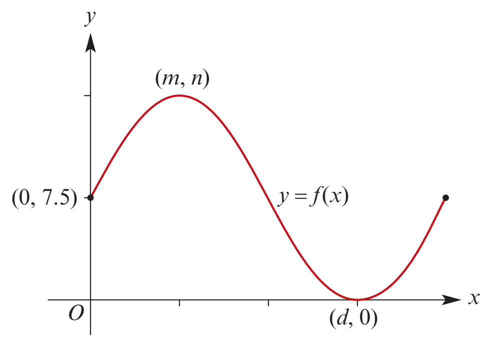
$$f(x) = a + b\sin\left(\frac{2\pi x}{50}\right),\ \{x \in \mathbb{R} : 0 \le x \le 50\}$$

where units are in metres.

a **i** Find the values of a, b, d, m and n.

ii The other bank of the creek can be modelled by the function $y = f(x) + 4$. Sketch the graph of this new function.

b Find the coordinates of the points on the first bank with y-coordinate 10.



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Revision

c A particular river has a less severe bend than this creek. It is found that a section of the bank of the river can be modelled by the function:

$$g(x) = 2f\left(\frac{x}{5}\right), \ \{ x \in \mathbb{R} : 0 \le x \le 250 \}$$

Sketch the graph of this function; label the turning points with their coordinates.

d Over the years, the river bank moves. The shape of the bends are maintained, but there is a translation of 10 metres in the positive direction of the x-axis.

i Give the rule that describes this section of the river bank after the translation (relative to the original axes).

ii Sketch the graph of this new function.

33 The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} k(5-2x) & \text{if } 2 < x \le \dfrac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of k.

b **i** Find $\mathrm{E}(X)$.

ii Find the median of X.

iii Find σ, the standard deviation of X, correct to two decimal places.

c Find $\mathrm{P}(X < \mu - \sigma)$, where $\mu = \mathrm{E}(X)$.

34 The lifetime, X days, of a particular type of computer component has a probability density function given by

$$f(x) = \begin{cases} k(a-x) & \text{if } 0 < x \le a \\ 0 & \text{if } x \le 0 \text{ or } x > a \end{cases}$$

where k and a are positive constants.

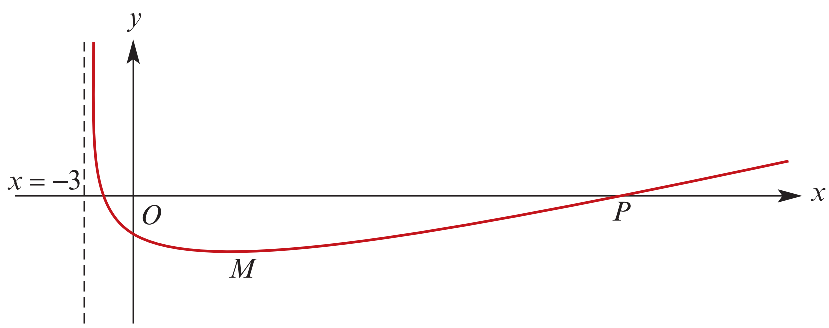
a Find k in terms of a.

b Find the mean, μ, and the variance, σ^2, of X in terms of a.

c Find $\mathrm{P}(X > \mu + 2\sigma)$.

d Find the value of a if the median lifetime is 1000 days.

35 The diagram shows a sketch graph of $y = \dfrac{x}{10} - \ln(x+3), \quad x > -3$



a Find the x-coordinate of the local minimum at M.

b Show that the gradient of the curve is always less than $\dfrac{1}{10}$.

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c Find the equation of the straight line through M with a gradient of $\frac{1}{10}$.

d **i** Hence show that the value of the x-axis intercept at P is greater than $10 \ln 10$.

ii Find, correct to three decimal places, the value of the intercept at P.

36 A particle is moving along a path with equation $y = \sqrt{x^2 + 24}$.

a Find $\frac{dy}{dx}$.

b Find the coordinates of the local minimum of the curve.

c Does this rule define an even function?

d As $x \to \infty$, $y \to x$ and as $x \to -\infty$, $y \to -x$. Sketch the graph of $y = \sqrt{x^2 + 24}$, showing the asymptotes.

e Find the equation of the normal to the curve at the point with coordinates $(1, 5)$, and sketch the graph of this normal with the graph of part **d**.

f Show that

$$\frac{d}{dx}\left(12 \ln\left(\sqrt{x^2 + 24} + x\right) + \frac{x\sqrt{x^2 + 24}}{2}\right) = \sqrt{x^2 + 24}$$

for $x > 0$.

g Use this result to find the area of the region bounded by the curve, the x-axis and the lines $x = 2$ and $x = 5$.

37 The boxplot is a display used to describe the distribution of a data set. Located on the boxplot are the minimum, the lower quartile, the median, the upper quartile and the maximum. Boxplots also show outliers. These are values which are more than 1.5 interquartile ranges below the lower quartile or above the upper quartile.

a Suppose that a random variable Z is normally distributed with a mean of 0 and a standard deviation of 1.

i Find the value of the median, i.e. find m such that $P(Z \le m) = 0.5$.

ii Find the value of the lower quartile, i.e. find q_1 such that $P(Z \le q_1) = 0.25$.

iii Find the value of the upper quartile, i.e. find q_3 such that $P(Z \le q_3) = 0.75$.

iv Hence find the interquartile range (IQR) for this distribution.

v Find $P(q_1 - 1.5 \times \text{IQR} < Z < q_3 + 1.5 \times \text{IQR})$.

vi What percentage of data values would you expect to be designated as outliers for this distribution?

b Suppose that a random variable X is normally distributed with a mean of μ and a standard deviation of σ.

i Find the value of the median, i.e. find m such the $P(X \le m) = 0.5$.

ii Find the value of the lower quartile, i.e. find q_1 such that $P(X \le q_1) = 0.25$.

iii Find the value of the upper quartile, i.e. find q_3 such that $P(X \le q_3) = 0.75$.

iv Hence find the interquartile range (IQR) for this distribution.

v Find $P(q_1 - 1.5 \times \text{IQR} < X < q_3 + 1.5 \times \text{IQR})$.

vi What percentage of data values would you expect to be designated as outliers for this distribution?

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38 The random variable X has probability density function given by

$$f(x) = \begin{cases} kx^n & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

where n and k are constants with $n > 0$. Find in terms of n:

a k

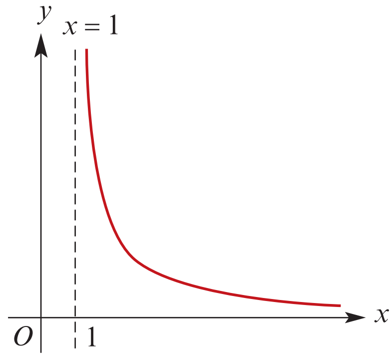
b $\mathrm{E}(X)$

c $\mathrm{Var}(X)$

d the median of X

39 The diagram shows the graph of the function

$$g(x) = \frac{1}{x-1}, \ \{ x \in \mathbb{R} : x > 1 \}$$



The line segment AB is drawn from the point $A(2, 1)$ to the point $B(b, g(b))$, where $b > 2$.

a **i** What is the gradient of AB?

ii At what value of x between 1 and b does the tangent to the graph of g have the same gradient as AB?

b **i** Calculate $\int_2^{e+1} g(x)\,dx$.

ii Let c be a real number with $1 < c < 2$. Find the exact value of c such that $\int_c^{e+1} g(x)\,dx = 8$.

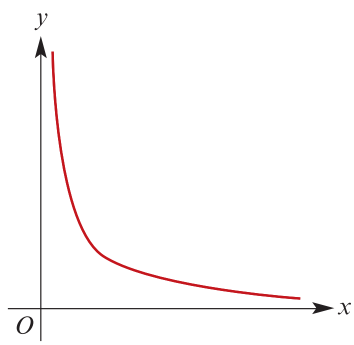
c **i** What is the area of the trapezium bounded by the line segment AB, the x-axis and the lines $x = 2$ and $x = b$?

ii For what exact value of b does this area equal 8?

d Given that $\int_2^{mn+1} g(x)\,dx + \int_2^{\frac{m}{n}+1} g(x)\,dx = 2$, where $n > 0$, find the value of m.

40 The diagram shows the graph of the function

$$f(x) = \frac{1}{x^2}, \ \{ x \in \mathbb{R} : x > 0 \}$$



The line segment AB is drawn from the point $A(1, 1)$ to the point $B(b, f(b))$, where $b > 1$.

a **i** What is the gradient of AB?

ii At what value of x between 1 and b does the tangent to the graph of f have the same gradient as AB?

b **i** What is the area, $S(b)$, of the trapezium bounded by the line segment AB, the x-axis and the lines $x = 1$ and $x = b$?

ii For what exact value of b does this area equal $\frac{10}{9}$?

iii Show that $\int_1^b f(x)\,dx < 1$ for $b > 1$.

c Show that the function $D(b) = S(b) - \int_1^b f(x)\,dx$ is strictly increasing for $b > 1$.

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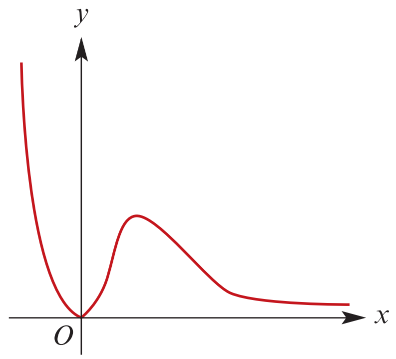
41 Define the function $f(x) = x^m e^{-nx+n}$, where m and n are positive integers. The graph of $y = f(x)$ is as shown.



a Find the coordinates of the stationary point not at the origin in terms of n, and state its nature.

b Find the coordinates of the point on the graph at which the tangent of f passes through the origin.

c Consider the continuous probability density function with rule

$$f(x) = \begin{cases} kx^2 e^{-2x+2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a positive real number.

i Find the value of k.

ii Find $\text{P}(X < 1)$, where X is the associated random variable.

42 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} ke^{-qx} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where q is a positive real number.

a **i** Find the value of k in terms of q.

ii Find $\text{E}(X)$ in terms of q.

iii Find $\text{Var}(X)$ in terms of q.

iv Show that the median of the distribution is $m = \frac{1}{q} \ln 2$.

b Find $\text{P}\left(X > \frac{1}{q} \ln 3 \;\middle|\; X > \frac{1}{q} \ln 2\right)$.

c The distance, X metres, between flaws in a certain type of yarn is a continuous random variable with probability density function $f(x) = 0.01e^{-0.01x}$ for $x \geq 0$.

i Sketch the graph of $y = f(x)$.

ii Find the probability, correct to two decimal places, that the distance between consecutive flaws is more than 100 m.

iii Find the median value of this distribution, correct to two decimal places.

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Glossary

A

Absolute maximum and minimum [p. 183] For a continuous function f defined on an interval $[a, b]$:

- the *absolute maximum* is the value M of the function f such that $f(x) \le M$ for all $x \in [a, b]$
- the *absolute minimum* is the value N of the function f such that $f(x) \ge N$ for all $x \in [a, b]$.

Acceleration [p. 150] the rate of change of a particle's velocity with respect to time

Acceleration, average [MM1&2] The average acceleration of a particle for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Acceleration, instantaneous [p. 150] $a = \dfrac{dv}{dt}$

Addition formulas [MM1&2]

- $\cos(u + v) = \cos u \cos v - \sin u \sin v$
- $\cos(u - v) = \cos u \cos v + \sin u \sin v$
- $\sin(u + v) = \sin u \cos v + \cos u \sin v$
- $\sin(u - v) = \sin u \cos v - \cos u \sin v$
- $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$
- $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

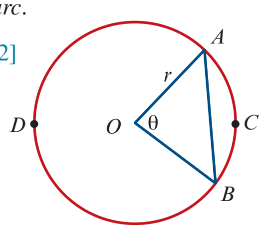
Addition rule for probability [p. 283] The probability of A or B or both occurring is given by $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Amplitude of trigonometric functions [MM1&2] The distance between the mean position and the maximum position is called the amplitude.
The graph of $y = \sin x$ has an amplitude of 1.

Antiderivative [p. 223] To find the general antiderivative of $f(x)$: If $F'(x) = f(x)$, then
$\int f(x)\,dx = F(x) + c$
where c is an arbitrary real number.

Arc [MM1&2] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Arc length, ℓ [MM1&2] The length of arc ACB is given by $\ell = r\theta$, where $\theta^c = \angle AOB$.



Asymptote [MM1&2] A straight line is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to the straight line. An asymptote can be horizontal, vertical or oblique.

Average value [p. 252] The average value of a continuous function f for an interval $[a, b]$ is defined as $\dfrac{1}{b - a}\displaystyle\int_a^b f(x)\,dx$.

Note: The glossary contains some terms from Mathematical Methods Year 11 [MM1&2].

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B

Bernoulli random variable [p. 326] a random variable that takes only the values 1 (indicating a success) and 0 (indicating a failure)

Bernoulli sequence [p. 326] a sequence of repeated trials with the following properties:

- Each trial results in one of two outcomes, usually designated as a success or a failure.
- The probability of success on a single trial, p, is constant for all trials.
- The trials are independent. (The outcome of a trial is not affected by outcomes of other trials.)

Binomial distribution [p. 328] The probability of observing x successes in n independent trials, each with probability of success p, is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

Binomial experiment [p. 328] a Bernoulli sequence of n independent trials, each with probability of success p

C

Chain rule [p. 110] The chain rule can be used to differentiate a complicated function $y = f(x)$ by transforming it into two simpler functions, which are 'chained' together:

$$x \xrightarrow{h} u \xrightarrow{g} y$$

Using Leibniz notation, the chain rule is stated as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Change of base [p. 57] $\log_a b = \dfrac{\log_c b}{\log_c a}$

Circle, general equation [p. 6] The general equation for a circle is $(x-h)^2 + (y-k)^2 = r^2$, where the centre is (h, k) and the radius is r.

Coefficient [MM1&2] the number that multiplies a power of x in a polynomial; e.g. for $2x^5 - 7x^2 + 4$, the coefficient of x^2 is -7

Complement, A' [p. 283] the set of outcomes that are in the sample space, ε, but not in A. The probability of the event A' is $\Pr(A') = 1 - \Pr(A)$.

Complementary relationships [MM1&2]

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

Composite function [p. 23] For functions f and g such that $\operatorname{ran} f \subseteq \operatorname{dom} g$, the composite function of g with f is defined by $g \circ f(x) = g(f(x))$, where $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Compound angle formulas [MM1&2] *see* addition formulas

Concavity [p. 176]

- If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval; the curve is said to be *concave up*.
- If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval; the curve is said to be *concave down*.

Conditional probability [p. 291] the probability of an event A occurring when it is known that some event B has occurred, given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Confidence interval [p. 433] an interval estimate for the population proportion p based on the value of the sample proportion $\hat{p}$

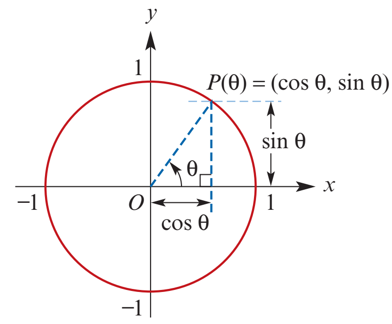
Constant function [MM1&2] a function $f(x) = a$

Continuous function [MM1&2] A function f is continuous at the point $x = a$ if $f(x)$ is defined at $x = a$ and $\lim_{x \to a} f(x) = f(a)$.

Continuous random variable [p. 347] a random variable X that can take any value in an interval of the real number line

Coordinates [MM1&2] an ordered pair of numbers that identifies a point in the Cartesian plane; the first number identifies the position with respect to the x-axis, and the second number identifies the position with respect to the y-axis

Cosine function [MM1&2] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



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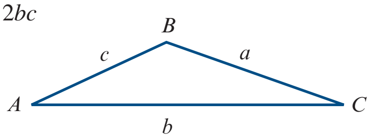
Cosine rule [MM1&2] For triangle ABC:
$a^2 = b^2 + c^2 - 2bc\cos A$
or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



The cosine rule is used to find unknown quantities in a triangle given two sides and the included angle, or given three sides.

Cubic function [MM1&2] a polynomial of degree 3. A cubic function f has a rule of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

Cumulative distribution function [p. 374] gives the probability that the random variable X takes a value less than or equal to x; that is, $F(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$

D

Definite integral [pp. 219, 232] $\int_a^b f(x)\, dx$ denotes the signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

Degree of a polynomial [MM1&2] given by the highest power of x with a non-zero coefficient; e.g. the polynomial $2x^5 - 7x^2 + 4$ has degree 5

Derivative function [p. 90] also called the gradient function. The derivative f' of a function f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives, basic [pp. 114–126]

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^a	ax^{a-1}	where $\{a \in \mathbb{R} : a \neq 0\}$
e^{kx}	ke^{kx}	
$\ln(kx)$	$\frac{1}{x}$	
$\sin(kx)$	$k\cos(kx)$	
$\cos(kx)$	$-k\sin(kx)$	
$\tan(kx)$	$k\sec^2(kx)$	

Difference of two cubes [MM1&2]
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Difference of two squares [MM1&2]
$x^2 - y^2 = (x - y)(x + y)$

Differentiable [MM1&2] A function f is said to be differentiable at the point $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Differentiation rules [p. 93]

- Sum: $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$
- Multiple: $f(x) = k\,g(x)$, $f'(x) = k\,g'(x)$

see also chain rule, product rule, quotient rule

Dilation parallel to the *x*-axis [MM1&2] A dilation of factor a parallel to the x-axis is described by the rule $(x, y) \to (ax, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$.

Dilation parallel to the *y*-axis [MM1&2] A dilation of factor b parallel to the y-axis is described by the rule $(x, y) \to (x, by)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = bf(x)$.

Discontinuity [MM1&2] A function is said to be discontinuous at a point if it is not continuous at that point.

Discrete random variable [p. 299] a random variable X which can take only a countable number of values, usually whole numbers

Discriminant, Δ, of a quadratic [MM1&2] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint [p. 2] Two sets A and B are disjoint if they have no elements in common, i.e. $A \cap B = \emptyset$.

Distance between two points [MM1&2] the distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ as $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Domain [p. 6] the set of all the first coordinates of the ordered pairs in a relation

Double angle formulas [MM1&2]

- $\cos(2u) = \cos^2 u - \sin^2 u$
  $= 2\cos^2 u - 1$
  $= 1 - 2\sin^2 u$
- $\sin(2u) = 2\sin u\,\cos u$
- $\tan(2u) = \dfrac{2\tan u}{1 - \tan^2 u}$

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E

Element [p. 2] a member of a set.

- If x is an element of a set A, we write $x \in A$.
- If x is *not* an element of a set A, we write $x \notin A$.

Empty set, ∅ [p. 2] the set that has no elements

Euler's number, *e* [p. 43] the natural base for exponential and logarithmic functions:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.718\,281\ldots$$

Even function [p. 17] A function f is even if $f(-x) = f(x)$ for all x in the domain of f; the graph is symmetric about the y-axis.

Event [p. 281] a subset of the sample space (that is, a set of outcomes)

Expected value of a random variable, E(*X*) [pp. 307, 360] also called the mean, μ.
For a discrete random variable X:

$$\mathrm{E}(X) = \sum_x x \cdot \mathrm{P}(X = x) = \sum_x x \cdot p(x)$$

For a continuous random variable X:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Exponential function [p. 36] a function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1

F

Factor [MM1&2] a number or expression that divides another number or expression without remainder

Factor theorem [MM1&2] If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$. Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Factorise [MM1&2] express as a product of factors

Formula [MM1&2] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

Function [p. 8] a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$.

Function, vertical-line test [p. 8] used to identify whether a relation is a function or not. If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a *function*.

Fundamental theorem of calculus [pp. 233, 259] If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\, dx = G(b) - G(a)$$

where G is any antiderivative of f and $\int_a^b f(x)\, dx$ is the definite integral from a to b.

G

Gradient function *see* derivative function

Gradient of a line [MM1&2] The gradient is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of two points on the line. The gradient of a vertical line (parallel to the y-axis) is undefined.

I

Implied domain *see* natural domain

Increments formula *see* linear approximation

Indefinite integral *see* antiderivative

Independence [p. 294] Two events A and B are independent if and only if $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

Index laws [p. 46]

- To multiply two powers with the same base, add the indices: $a^x \times a^y = a^{x+y}$
- To divide two powers with the same base, subtract the indices: $a^x \div a^y = a^{x-y}$
- To raise a power to another power, multiply the indices: $(a^x)^y = a^{x \times y}$
- Rational indices: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$
- For base $a \in (0, 1) \cup (1, \infty)$, if $a^x = a^y$, then $x = y$.

Inequality [MM1&2] a mathematical statement that contains an inequality symbol rather than an equals sign; e.g. $2x + 1 < 4$

Integers [p. 3] $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Integrals, basic [pp. 227–230, 240]

$f(x)$	$\int f(x)\, dx$	
x^r	$\frac{x^{r+1}}{r+1} + c$	where $\{ r \in \mathbb{Q} : r \neq -1 \}$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b) + c$	for $ax + b > 0$
e^{kx}	$\frac{1}{k}e^{kx} + c$	
$\sin(kx)$	$-\frac{1}{k}\cos(kx) + c$	
$\cos(kx)$	$\frac{1}{k}\sin(kx) + c$	

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Integration, properties [p. 225]

- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int kf(x)\,dx = k \int f(x)\,dx$

Integration (definite), properties [p. 235]

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Intersection of sets [pp. 2, 281] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 3] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Irrational number [p. 3] a real number that is not rational; e.g. π and $\sqrt{2}$

K

Karnaugh map [p. 286] a probability table

L

Law of total probability [p. 292] In the case of two events, A and B:
$\mathrm{P}(A) = \mathrm{P}(A \mid B)\mathrm{P}(B) + \mathrm{P}(A \mid B')\mathrm{P}(B')$

Limit [MM1&2] The notation $\lim_{x\to a} f(x) = p$ says that the limit of $f(x)$, as x approaches a, is p. We can also say: 'As x approaches a, $f(x)$ approaches p.'

Limits, properties [MM1&2]

- Sum: $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- Multiple: $\lim_{x\to a}(kf(x)) = k \lim_{x\to a} f(x)$
- Product: $\lim_{x\to a}(f(x)\,g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- Quotient: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

Linear approximation (increments formula) [p. 154]

- For a small number h, the change in the value of $y = f(x)$ as x changes from a to $a + h$ can be approximated using $f(a + h) - f(a) \approx hf'(a)$.
- Using Leibniz notation, we can write this as $\delta y \approx \dfrac{dy}{dx} \times \delta x$.

Linear equation [MM1&2] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear function [MM1&2] a function $f(x) = mx + c$; e.g. $f(x) = 3x + 1$

Linear function of a random variable [p. 372]

- $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
- $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Literal equation [MM1&2] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

Logarithm [p. 48] If $a \in (0, 1) \cup (1, \infty)$ and $x \in \mathbb{R}$, then the statements $a^x = y$ and $\log_a y = x$ are equivalent.

Logarithm, natural [p. 49] The natural logarithm function is given by
$\ln x = \log_e x$
where the base e is Euler's number.

Logarithm laws [p. 50]

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a\left(\dfrac{1}{x}\right) = -\log_a x$
- $\log_a(x^p) = p \log_a x$

Logarithmic scale [p. 73] a scale of measurement that uses the logarithm of a quantity; e.g. Richter scale for earthquakes

M

Margin of error, *M* [p. 437] the distance between the sample estimate and the endpoints of the confidence interval

Mean of a random variable, μ [pp. 307, 360] *see* expected value of a random variable, $\mathrm{E}(X)$

Median of a random variable, *m* [p. 363] the middle value of the distribution. For a continuous random variable, the median is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0.5$.

Midpoint of a line segment [MM1&2] If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, then
$x = \dfrac{x_1 + x_2}{2}$ and $y = \dfrac{y_1 + y_2}{2}$

Multiplication rule for probability [p. 291] the probability of events A and B both occurring is
$\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times \mathrm{P}(B)$

Multi-stage experiment [p. 292] an experiment that could be considered to take place in more than one stage; e.g. tossing two coins

Mutually exclusive [p. 283] Two events are said to be mutually exclusive if they have no outcomes in common.

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N

Natural domain [p. 16] When the rule for a relation is given and no domain is specified, then the domain taken is the largest for which the rule has meaning.

Natural logarithm [p. 49] *see* logarithm

Natural numbers [p. 3] $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

Normal, equation of [p. 143] Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, the equation of the normal at (x_1, y_1) is
$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

Normal distribution [p. 386] a symmetric, bell-shaped distribution that often occurs for a measure in a population (e.g. height, weight, IQ); its centre is determined by the mean, μ, and its width by the standard deviation, σ

O

Odd function [p. 17] A function f is odd if $f(-x) = -f(x)$ for all x in the domain of f; the graph has rotational symmetry about the origin.

Ordered pair [p. 6] a pair of elements, denoted (x, y), where x is the first coordinate and y is the second coordinate

P

Percentile [p. 362] For a continuous random variable X, the value p such that $P(X \le p) = q\%$ is called the qth percentile of X, and is found by solving $\int_{-\infty}^{p} f(x)\,dx = \dfrac{q}{100}$.

Period of a function [MM1&2] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f.

- Sine and cosine have period 2π.
- Tangent has period π.
- A function of the form $y = a\cos(nx + \varepsilon) + b$ or $y = a\sin(nx + \varepsilon) + b$ has period $\dfrac{2\pi}{n}$.

Point estimate [p. 433] If the value of the sample proportion $\hat{p}$ is used as an estimate of the population proportion p, then it is called a point estimate of p.

Point of inflection [p. 176] a point where a curve changes from concave up to concave down or from concave down to concave up; i.e. a point of inflection occurs where the sign of the second derivative changes

Polynomial function [MM1&2] A polynomial has a rule of the type
$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N} \cup \{0\}$$
where $a_0, a_1, \ldots, a_n$ are numbers called coefficients.

Population [p. 412] the set of all eligible members of a group which we intend to study

Population parameter [p. 416] a statistical measure that is based on the whole population; the value is constant for a given population

Population proportion, p [p. 415] the proportion of individuals in the entire population possessing a particular attribute

Position [MM1&2] For a particle moving in a straight line, the position of the particle relative to a point O on the line is determined by its distance from O and whether it is to the right or left of O. The direction to the right of O is positive.

Power function [MM1&2] a function of the form $f(x) = x^r$, where r is a non-zero real number

Probability [p. 281] a numerical value assigned to the likelihood of an event occurring. If the event A is impossible, then $P(A) = 0$; if the event A is certain, then $P(A) = 1$; otherwise $0 < P(A) < 1$.

Probability density function [p. 350] usually denoted $f(x)$; describes the probability distribution of a continuous random variable X such that $P(a < X < b) = \int_a^b f(x)\,dx$

Probability function (discrete) [p. 300] denoted by $p(x)$ or $P(X = x)$, a function that assigns a probability to each value of a discrete random variable X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

Probability table [p. 286] a table used for illustrating a probability problem diagrammatically

Product of functions [p. 19] $(fg)(x) = f(x)\,g(x)$ and $\operatorname{dom}(fg) = \operatorname{dom} f \cap \operatorname{dom} g$

Product rule [p. 128] Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then
$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
In Leibniz notation:
If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Pythagorean identity [MM1&2] $\cos^2\theta + \sin^2\theta = 1$

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Q

$\mathbb{Q}$ [p. 3] the set of all rational numbers

Quadratic, turning point form [MM1&2] The turning point form of a quadratic function is $y = a(x-h)^2 + k$, where (h, k) is the turning point.

Quadratic formula [MM1&2] $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the solution of the quadratic equation $ax^2 + bx + c = 0$

Quadratic function [MM1&2] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quartic function [MM1&2] a polynomial of degree 4. A quartic function f has a rule of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$.

Quotient rule [p. 132] Let $F(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

In Leibniz notation:

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

R

$\mathbb{R}^2$ [MM1&2] $\{(x, y) : x, y \in \mathbb{R}\}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [MM1&2] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

Random experiment [p. 281] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Random sample [p. 412] A sample of size n is called a *simple random sample* if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.

Random variable [p. 299] a variable that takes its value from the outcome of a random experiment; e.g. the number of heads observed when a coin is tossed three times

Range [p. 6] the set of all the second coordinates of the ordered pairs in a relation

Rational number [p. 3] a number that can be written as $\dfrac{p}{q}$, for some integers p and q with $q \neq 0$

Rational-root theorem [MM1&2]
Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1.
If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

Rectangular hyperbola [MM1&2] The basic rectangular hyperbola has equation $y = \dfrac{1}{x}$.

Relation [p. 6] a set of ordered pairs; e.g. $\{(x, y) : y = x^2\}$

Remainder theorem [MM1&2]
When a polynomial $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\dfrac{\alpha}{\beta}\right)$.

S

Sample [p. 412] a subset of the population which we select in order to make inferences about the whole population

Sample proportion, $\hat{p}$ [p. 415] the proportion of individuals in a particular sample possessing a particular attribute. The sample proportions $\hat{p}$ are the values of a random variable $\hat{P}$.

Sample space, ε [p. 281] the set of all possible outcomes for a random experiment

Sample statistic [p. 416] a statistical measure that is based on a sample from the population; the value varies from sample to sample

Sampling distribution [p. 420] the distribution of a statistic which is calculated from a sample

Scientific notation [MM1&2] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}.

Secant [p. 87] a straight line that passes through two points $(a, f(a))$ and $(b, f(b))$ on the graph of a function $y = f(x)$

Second derivative [p. 135]

- The second derivative of a function f with rule $f(x)$ is denoted by f'' and has rule $f''(x)$.
- The second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Second derivative test [p. 177]

- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
- If $f''(a) = 0$, then further investigation is necessary.

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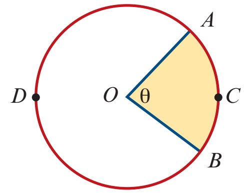
Sector [MM1&2] Two radii and an arc define a region called a sector. In this diagram, the shaded region is a *minor sector* and the unshaded region is a *major sector*.

Area of sector $= \frac{1}{2}r^2\theta$

where $\theta^c = \angle AOB$



Set notation [p. 2]

$\in$ means 'is an element of'

$\notin$ means 'is not an element of'

$\subseteq$ means 'is a subset of'

$\cap$ means 'intersection'

$\cup$ means 'union'

$\varnothing$ is the empty set, containing no elements

Sets of numbers [p. 3]

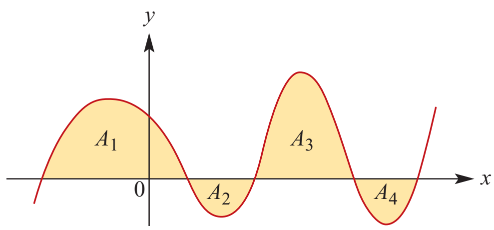
$\mathbb{N}$ is the set of natural numbers

$\mathbb{Z}$ is the set of integers

$\mathbb{Q}$ is the set of rational numbers

$\mathbb{R}$ is the set of real numbers

Signed area [p. 232] The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.



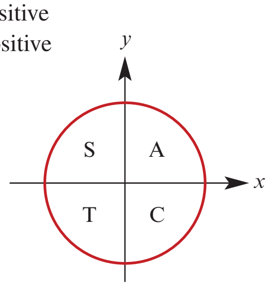
Signs of trigonometric functions [MM1&2]

1st quadrant all are positive

2nd quadrant sin is positive

3rd quadrant tan is positive

4th quadrant cos is positive

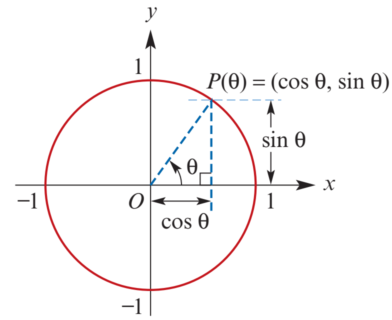


Simulation [MM1&2] the process of finding an approximate solution to a probability problem by repeated trials using a simulation model

Simulation model [MM1&2] a simple model which is analogous to a real-world situation; e.g. the outcomes from a toss of a coin (head, tail) could be used as a simulation model for the sex of a child (male, female) under the assumption that in both situations the probabilities are 0.5 for each outcome

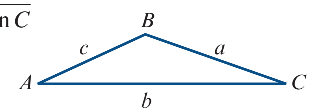
Simultaneous equations [MM1&2] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [MM1&2] Sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Sine rule [MM1&2] For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle given one side and two angles, or given two sides and a non-included angle.

Standard deviation of a random variable, σ [pp. 310, 367] a measure of the spread or variability, given by $\text{sd}(X) = \sqrt{\text{Var}(X)}$

Standard normal distribution [p. 385] a special case of the normal distribution where $\mu = 0$ and $\sigma = 1$

Stationary point [p. 160] A point $(a, f(a))$ on a curve $y = f(x)$ is a stationary point if $f'(a) = 0$.

Straight line, equation given two points [MM1&2] $y - y_1 = m(x - x_1)$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$

Straight line, gradient–intercept form [MM1&2] $y = mx + c$, where m is the gradient and c is the y-axis intercept

Straight lines, parallel [MM1&2] Two non-vertical straight lines are parallel to each other if and only if they have the same gradient.

Straight lines, perpendicular [MM1&2] Two straight lines are perpendicular if and only if the product of their gradients is −1 (or if one is vertical and the other horizontal).

Glossary S

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Strictly decreasing [pp. 18, 98] A function f is strictly decreasing on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

Strictly increasing [pp. 18, 98] A function f is strictly increasing on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Subset [p. 2] A set B is called a subset of set A if every element of B is also an element of A. We write $B \subseteq A$.

Sum of functions [p. 19] $(f+g)(x) = f(x) + g(x)$ and $\text{dom}(f+g) = \text{dom}\, f \cap \text{dom}\, g$

Sum of two cubes [MM1&2] $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

T

Tangent, equation of [p. 143] Let (x_1, y_1) be a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

Tangent function [MM1&2] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Translation [MM1&2] A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x+h, y+k)$, where $h, k > 0$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$.

Tree diagram [p. 292] a diagram representing the outcomes of a multi-stage experiment

Total change [p. 255] Given the rule for $f'(x)$, the total change in the value of $f(x)$ between $x = a$ and $x = b$ can be found using

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

Trigonometric functions [MM1&2] the sine, cosine and tangent functions

Trigonometric functions, exact values [MM1&2]

θ^c	θ°	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined

U

Uniform distribution [pp. 324, 383]

- A discrete random variable X with values $1, 2, 3, \dots, n$ has a uniform distribution if each value of X is equally likely, and therefore

$$P(X = x) = \frac{1}{n}, \quad \text{for } x = 1, 2, 3, \dots, n$$

- A continuous random variable X has a uniform distribution if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where a and b are real constants with $a < b$.

Union of sets [pp. 2, 281] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

V

Variance of a random variable, σ^2 [pp. 310, 367] a measure of the spread or variability, defined by $\text{Var}(X) = E[(X - \mu)^2]$. An alternative (computational) formula is $\text{Var}(X) = E(X^2) - [E(X)]^2$

Velocity [p. 150] the rate of change of a particle's position with respect to time

Velocity, average [MM1&2]

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Velocity, instantaneous [p. 150] $v = \dfrac{dx}{dt}$

Vertical-line test [p. 8] *see* function

Z

$\mathbb{Z}$ [p. 3] the set of all integers

Zero polynomial [MM1&2] The number 0 is called the zero polynomial.

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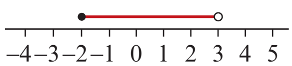
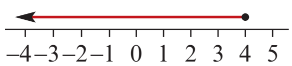
Answers

Chapter 1

Exercise 1A

1 a {8, 11} **b** {8, 11}
c {1, 3, 8, 11, 18, 22, 23, 24, 25, 30}
d {3, 8, 11, 18, 22, 23, 24, 25, 30, 32}
e {1, 3, 8, 11, 18, 22, 23, 24, 25, 30, 32}
f {1, 8, 11, 25, 30}

2 a


b


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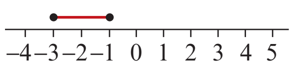

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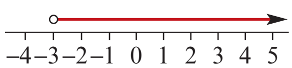
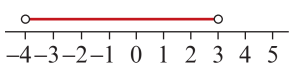
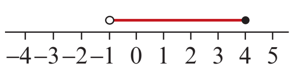


e


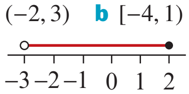
f


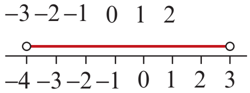
3 a {7, 9} **b** {7, 9}
c {2, 3, 5, 7, 9, 11, 15, 19, 23} **d** {2, 7, 9}
e {2, 3, 5, 7} **f** {7}
g {7, 9, 15, 19, 23} **h** $(3, \infty)$

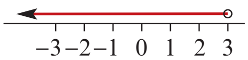
4 a $\{a, e\}$ **b** $\{a, b, c, d, e, i, o, u\}$
c $\{b, c, d\}$ **d** $\{i, o, u\}$

5 a $[-3, 1)$ **b** $(-4, 5]$ **c** $(-\sqrt{2}, 0)$
d $\left(-\frac{1}{\sqrt{2}}, \sqrt{3}\right)$ **e** $(-\infty, -3)$ **f** $[-2, \infty)$

6 a $(-2, 3)$ **b** $[-4, 1)$ **c** $[-1, 5]$ **d** $(-3, 2]$

7 a


b


c


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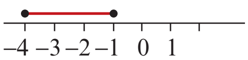

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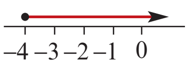
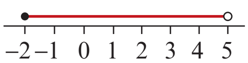
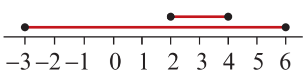


f


8 a


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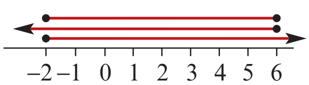
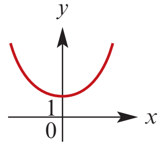
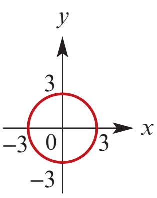


Exercise 1B

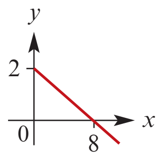
1 a Domain = $\mathbb{R}$ Range = $[-2, \infty)$
b Domain = $(-\infty, 2]$ Range = $\mathbb{R}$
c Domain = $(-2, 3)$ Range = $[0, 9)$
d Domain = $(-3, 1)$ Range = $(-6, 2)$
e Domain = $[-4, 0]$ Range = $[0, 4]$
f Domain = $\mathbb{R}$ Range = $(-\infty, 2)$

2 a


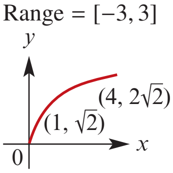
Domain = $\mathbb{R}$
Range = $[1, \infty)$

b


Domain = $[-3, 3]$
Range = $[-3, 3]$

c


Domain = $[0, \infty)$
Range = $(-\infty, 2]$

d


Domain = $[0, \infty)$
Range = $[0, \infty)$

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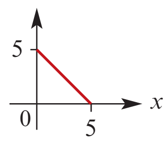
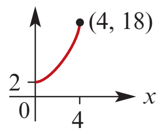
e



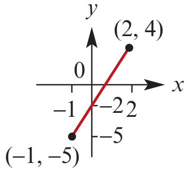
Domain = [0, 5]
Range = [0, 5]

f



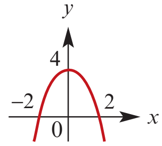
Domain = [0, 4]
Range = [2, 18]

g



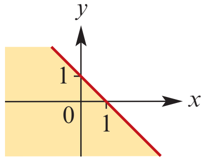
Domain = [−1, 2]
Range = [−5, 4]

h



Domain = ℝ
Range = (−∞, 4]

i



Domain = ℝ
Range = ℝ

3 a Not a function; Domain = {−1, 1, 2, 3}; Range = {1, 2, 3, 4}
b A function; Domain = {−2, −1, 0, 1, 2}; Range = {−4, −1, 0, 3, 5}
c Not a function; Domain = {−2, −1, 2, 4}; Range = {−2, 1, 2, 4, 6}
d A function; Domain = {−1, 0, 1, 2, 3}; Range = {4}

4 Functions: a, c

5 a A function; Domain = ℝ; Range = {4}
b Not a function; Domain = {2}; Range = ℤ
c A function; Domain = ℝ; Range = ℝ
d Not a function; Domain = ℝ; Range = ℝ
e Not a function; Domain = [−4, 4]; Range = [−4, 4]

6 $y = \sqrt{x+2},\ x \geq -2;$ Range = $[0, \infty)$
$y = -\sqrt{x+2},\ x \geq -2;$ Range = $(-\infty, 0]$

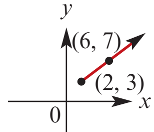
7 a $f(-1) = -2,\quad f(2) = 16,\quad f(-3) = 6,\quad f(2a) = 8a^2 + 8a$
b $g(-1) = -10,\quad g(2) = 14,\quad g(3) = 54,\quad g(a-1) = 2a^3 - 6a^2 + 8a - 10$

8 a $g(-2) = 10,\quad g(4) = 46$
b **i** $12x^2 - 2$
ii $3x^2 - 12x + 10$
iii $3x^2 + 12x + 10$
iv $3x^4 - 2$

9 a 3 **b** 7 **c** $\left\{-\frac{3}{2}\right\}$ **d** $(3, \infty)$

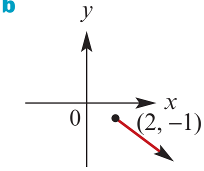
10 a $x = -3$ **b** $x > -3$ **c** $x = \frac{2}{3}$

11 a



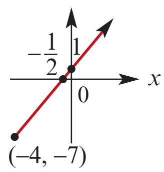
Range = $[3, \infty)$

b



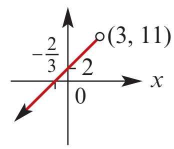
Range = $(-\infty, -1]$

c



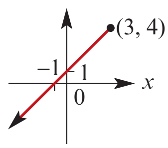
Range = $[-7, \infty)$

d



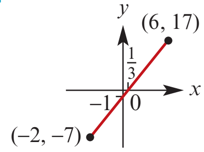
Range = $(-\infty, 11)$

e



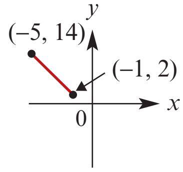
Range = $(-\infty, 4]$

f



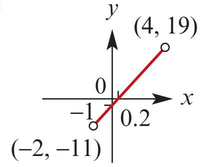
Range = $[-7, 17]$

g



Range = $[2, 14]$

h



Range = $(-11, 19)$

12 a $f(2) = -3,\quad f(-3) = 37,\quad f(-2) = 21$
b $g(-2) = 7,\quad g(1) = 1,\quad g(-3) = 9$
c **i** $f(a) = 2a^2 - 6a + 1$
ii $f(a+2) = 2a^2 + 2a - 3$
iii $g(-a) = 3 + 2a$ **iv** $g(2a) = 3 - 4a$
v $f(5-a) = 21 - 14a + 2a^2$
vi $f(2a) = 8a^2 - 12a + 1$
vii $g(a) + f(a) = 2a^2 - 8a + 4$
viii $g(a) - f(a) = 2 + 4a - 2a^2$

13 a $\left\{\frac{2}{3}, -1\right\}$ **b** $\left\{-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right\}$
c $\{0, -\frac{1}{3}\}$ **d** $(-\infty, -1) \cup \left(\frac{2}{3}, \infty\right)$
e $\left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$ **f** $[-\frac{1}{3}, 0]$

14 a $f(-2) = 2$ **b** $f(2) = 6$
c $f(-a) = a^2 - a$ **d** $f(a) + f(-a) = 2a^2$
e $f(a) - f(-a) = 2a$ **f** $f(a^2) = a^4 + a^2$

15 a $\{2\}$ **b** $\{\, x : x > 2 \,\}$ **c** $\left\{\frac{a+2}{3}\right\}$
d $\left\{-\frac{8}{3}\right\}$ **e** $\{1\}$ **f** $\left\{\frac{13}{18}\right\}$

16 a $\frac{4}{3}$ **b** 6 **c** $-\frac{7}{3}$ **d** 9 **e** $\frac{1}{3}$ **f** $-\frac{2}{9}$

17 a $\frac{6}{5}$ **b** $\frac{1}{5}$ **c** $\pm\frac{1}{3}$ **d** 1 **e** $-1, 2$

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Exercise 1C

1 a Domain = $\mathbb{R}$ Range = $\mathbb{R}$

b Domain = $[0, \infty)$ Range = $[0, \infty)$

c Domain = $\mathbb{R}$ Range = $[-2, \infty)$

d Domain = $[-4, 4]$ Range = $[0, 4]$

e Domain = $\{ x \in \mathbb{R} : x \neq 0 \}$
Range = $\{ y \in \mathbb{R} : y \neq 0 \}$

f Domain = $\mathbb{R}$ Range = $(-\infty, 4]$

g Domain = $[3, \infty)$ Range = $[0, \infty)$

2 a Domain = $\mathbb{R}$ Range = $\mathbb{R}$

b Domain = $\mathbb{R}$ Range = $[-2, \infty)$

c Domain = $[-3, 3]$ Range = $[0, 3]$

d Domain = $\{ x \in \mathbb{R} : x \neq 1 \}$
Range = $\{ y \in \mathbb{R} : y \neq 0 \}$

3 a $\{ x \in \mathbb{R} : x \neq 3 \}$

b $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$ **c** $\mathbb{R}$

d $[4, 11]$ **e** $\{ x \in \mathbb{R} : x \neq -1 \}$

f $(-\infty, -1] \cup [2, \infty)$

g $\{ x \in \mathbb{R} : x \neq -1, x \neq 2 \}$

h $(-\infty, -2) \cup [1, \infty)$ **i** $\left[0, \frac{1}{3}\right]$

j $[-5, 5]$ **k** $[3, 12]$

4 a Even **b** Odd **c** Neither

d Even **e** Odd **f** Neither

5 a Even **b** Even **c** Odd

d Odd **e** Neither **f** Even

g Neither **h** Neither **i** Even

Exercise 1D

1 a $(f + g)(x) = 4x + 2$
$(fg)(x) = 3x^2 + 6x$
dom = $\mathbb{R}$

b $(f + g)(x) = 1$
$(fg)(x) = x^2 - x^4$
dom = $(0, 2]$

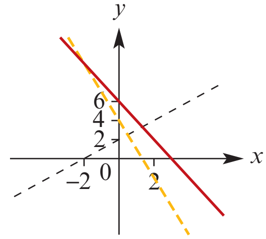
c $(f + g)(x) = \dfrac{x + 1}{\sqrt{x}}$
$(fg)(x) = 1$
dom = $[1, \infty)$

d $(f + g)(x) = x^2 + \sqrt{4 - x}$
$(fg)(x) = x^2\sqrt{4 - x}$
dom = $[0, 4]$

2 a i Even **ii** Odd **iii** Even **iv** Odd

b $(f + h)(x) = x^2 + 1 + \dfrac{1}{x^2}$, even;
$(fh)(x) = 1 + \dfrac{1}{x^2}$, even;
$(g + k)(x) = x + \dfrac{1}{x}$, odd; $(gk)(x) = 1$, even;
$(f + g)(x) = x^2 + x + 1$, neither;
$(fg)(x) = x^3 + x$, odd

3



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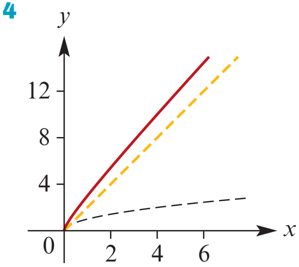

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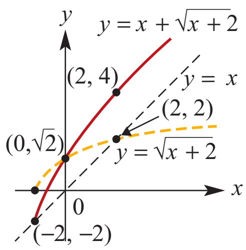

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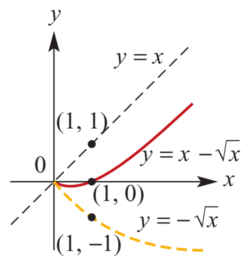

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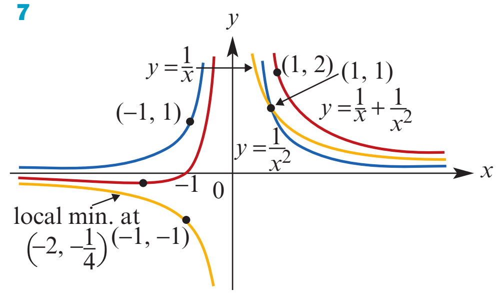
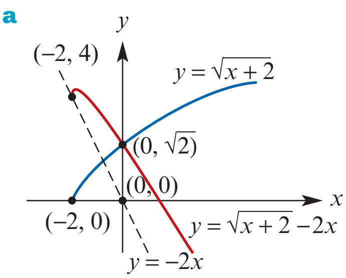


8 a



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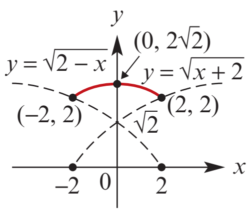
b



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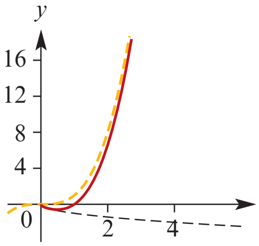

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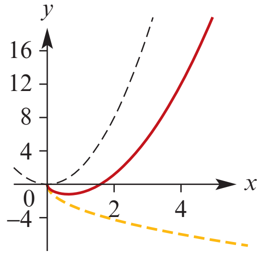
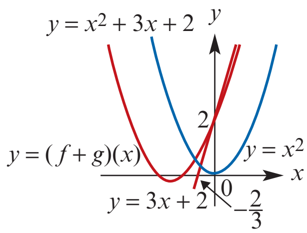
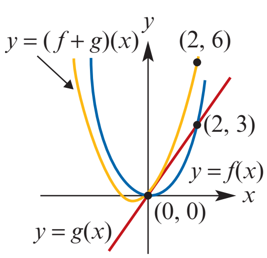


11



12 a



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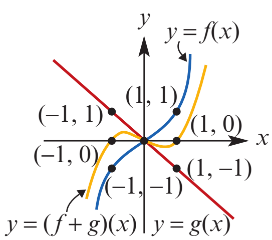
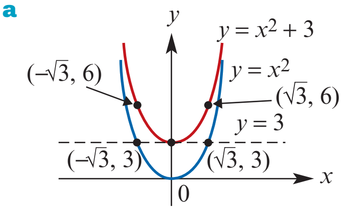
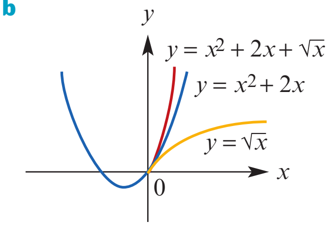


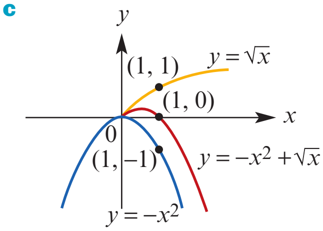
13 a



b



c



Exercise 1E

1 a $f(g(x)) = 4x - 1$, $g(f(x)) = 4x - 2$
b $f(g(x)) = 8x + 5$, $g(f(x)) = 8x + 3$
c $f(g(x)) = 4x - 7$, $g(f(x)) = 4x - 5$
d $f(g(x)) = 2x^2 - 1$, $g(f(x)) = (2x - 1)^2$
e $f(g(x)) = 2(x - 5)^2 + 1$, $g(f(x)) = 2x^2 - 4$
f $f(g(x)) = 2x^2 + 1$, $g(f(x)) = (2x + 1)^2$

2 a $f(h(x)) = 6x + 3$ **b** $h(f(x)) = 6x - 1$
c 15 **d** 11 **e** 21 **f** −7 **g** 3

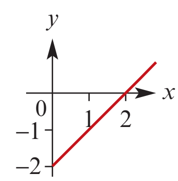
3 a $9x^2 + 12x + 3$ **b** $3x^2 + 6x + 1$
c 120 **d** 46 **e** 3 **f** 1

4 a $h(g(x)) = \dfrac{1}{(3x + 2)^2}$, $\text{dom}(h \circ g) = (0, \infty)$
b $g(h(x)) = \dfrac{3}{x^2} + 2$,
$\text{dom}(g \circ h) = \{x \in \mathbb{R} : x \neq 0\}$
c $\dfrac{1}{25}$ **d** 5

5 a $\text{ran } f = [-4, \infty)$, $\text{ran } g = [0, \infty)$
b $f \circ g(x) = x - 4$, $\text{ran}(f \circ g) = [-4, \infty)$
c $\text{ran } f \not\subseteq \text{dom } g$

6 a $f \circ g(x) = x$, $\text{dom} = \left\{x \in \mathbb{R} : x \neq \dfrac{1}{2}\right\}$,
$\text{ran} = \left\{y \in \mathbb{R} : y \neq \dfrac{1}{2}\right\}$
b $g \circ f(x) = x$, $\text{dom} = \{x \in \mathbb{R} : x \neq 0\}$,
$\text{ran} = \{y \in \mathbb{R} : y \neq 0\}$

7 a $\text{ran } f = [-2, \infty) \not\subseteq \text{dom } g = [0, \infty)$
b $f \circ g(x) = x - 2$, $x \geq 0$



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8 a ran $g = [-1, \infty) \not\subseteq$ dom $f = (-\infty, 3]$

b $g^*(x) = x^2 - 1, \{x \in \mathbb{R} : -2 \le x \le 2\}$

$f \circ g^*(x) = 4 - x^2, \{x \in \mathbb{R} : -2 \le x \le 2\}$

9 a ran $g = \mathbb{R} \not\subseteq$ dom $f = (0, \infty)$

b $g_1(x) = 3 - x, \{x \in \mathbb{R} : x < 3\}$

10 a $h(x) = g \circ f(x), f(x) = x^2 - 1, g(x) = x^4$

b $h(x) = g \circ f(x), f(x) = x^4 + 3,$

$g(x) = \sqrt{x}$ for $x \ge 0$

c $h(x) = g \circ f(x), f(x) = x^2 - 2x, g(x) = x^n$

d $h(x) = g \circ f(x), f(x) = 2x + 3$ for $x \ne -\dfrac{3}{2}$,

$g(x) = \dfrac{1}{x}$ for $x \ne 0$

e $h(x) = g \circ f(x), f(x) = x^2 - 2x,$

$g(x) = x^3 - 2x$

f $h(x) = f \circ f(x), f(x) = 2x^2 + 1$

11

	Domain	Range
f	$\mathbb{R}$	$[0, \infty)$
g	$(-\infty, 3]$	$[0, \infty)$

a ran $g \subseteq$ dom f, so $f \circ g$ exists

b ran $f \not\subseteq$ dom g, so $g \circ f$ does not exist

12 a $S = [-2, 2]$

b ran $f = [0, 2]$, ran $g = [1, \infty)$

c ran $f \subseteq$ dom g, so $g \circ f$ is defined

ran $g \not\subseteq$ dom f, so $f \circ g$ is not defined

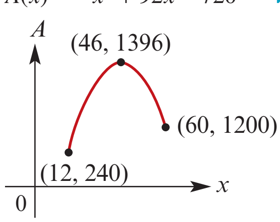
13 $a \in [2, 3]$

Exercise 1F

1 $V(x) = 4x(10 - x)(18 - x)$, domain $= [0, 10]$

2 a $A(x) = -x^2 + 92x - 720$ **b** $12 \le x \le 60$

c



d Maximum area 1396 m^2 occurs when $x = 46$ and $y = 34$

3 a i $S = 2x^2 + 6xh$ **ii** $S = 2x^2 + \dfrac{3V}{x}$

b Maximal domain $= (0, \infty)$

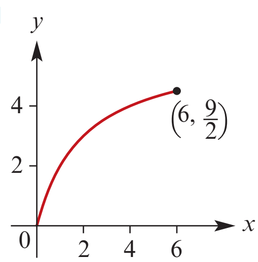
c Maximum value of $S = 1508$ m^2

4 Area $= x\sqrt{4a^2 - x^2}$, domain $= [0, 2a]$

5 a $A = \dfrac{6a}{a + 2}$

b Domain $= [0, 6]$; Range $= \left[0, \dfrac{9}{2}\right]$

c $\dfrac{9}{2}$ **d**

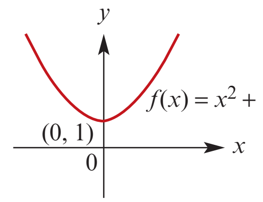


Chapter 1 review

Short-answer questions

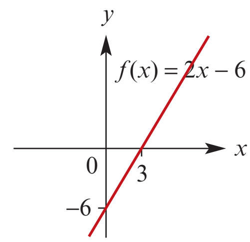
1 a Domain $= \mathbb{R}$

Range $= [1, \infty)$



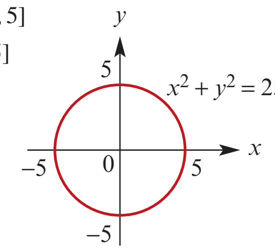
b Domain $= \mathbb{R}$

Range $= \mathbb{R}$



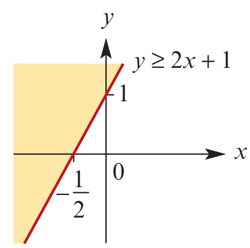
c Domain $= [-5, 5]$

Range $= [-5, 5]$



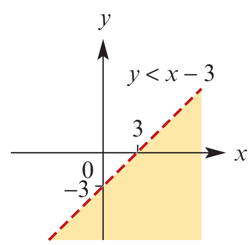
d Domain $= \mathbb{R}$

Range $= \mathbb{R}$

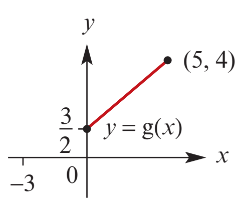


e Domain $= \mathbb{R}$

Range $= \mathbb{R}$



2 a



b ran $g = \left[\dfrac{3}{2}, 4\right]$ **c** $\{5\}$

3 a $\left\{\dfrac{1}{5}\right\}$ **b** $\left(-\infty, \dfrac{1}{5}\right]$ **c** $\left\{-\dfrac{1}{10}\right\}$

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4 a $\{x \in \mathbb{R} : x \neq 3\}$
b $\{x \in \mathbb{R} : x < -\sqrt{5},\ x > \sqrt{5}\}$
c $\{x \in \mathbb{R} : x \neq 1, x \neq -2\}$ **d** $[-5, 5]$
e $[5, 15]$ **f** $\{x \in \mathbb{R} : x \neq 2\}$

5 $(f+g)(x) = x^2 + 5x + 1$,
$(fg)(x) = (x-3)(x+2)^2$

6 $(f+g)(x) = x^2 + 1, \{x \in \mathbb{R} : 1 \leq x \leq 5\}$
$(fg)(x) = 2x(x-1)^2, \{x \in \mathbb{R} : 1 \leq x \leq 5\}$

7 a $(f+g)(x) = -x^2 + 2x + 3$
b $(fg)(x) = -x^2(2x+3)$ **c** $\{-1, 3\}$

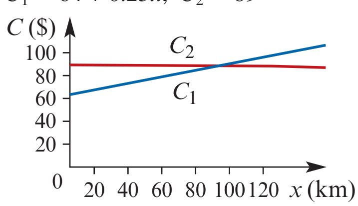
8 a $f \circ g(x) = -2x^3 + 3$
b $g \circ f(x) = -(2x+3)^3$
c $g \circ g(x) = x^9$ **d** $f \circ f(x) = 4x + 9$
e $f \circ (f+g)(x) = -2x^3 + 4x + 9$
f $f \circ (f-g)(x) = 2x^3 + 4x + 9$
g $f \circ (f \cdot g)(x) = -4x^4 - 6x^3 + 3$

9 a $h(x) = g \circ f(x),\ f(x) = x^3 - 1,\ g(x) = x^{\frac{1}{3}}$
b $h(x) = g \circ f(x),\ f(x) = x^2 + 1$,
$g(x) = \dfrac{1}{x}$ for $x \neq 0$
c $h(x) = g \circ f(x),\ f(x) = x^2 - 1$ for $x \neq \pm 1$,
$g(x) = \dfrac{1}{x}$ for $x \neq 0$

10 $x \geq -1$ or $x \leq -9$

Extended-response questions

1 a $C_1 = 64 + 0.25x,\ C_2 = 89$
b

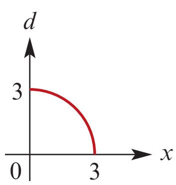


c $x > 100$ km

2 a $S = 6x^2$ **b** $S = 6V^{\frac{2}{3}}$

3 a $A = \dfrac{\sqrt{3}s^2}{4}$ **b** $A = \dfrac{\sqrt{3}h^2}{3}$

4 a $d(x) = \sqrt{9 - x^2}$ **b** dom $= [0, 3]$
ran $= [0, 3]$



5 $S(x) = \dfrac{160x}{x + 80}$

6 a $V_1(h) = \pi h\left(36 - \dfrac{h^2}{4}\right), \{h \in \mathbb{R} : 0 < h < 12\}$
b $V_2(r) = 2\pi r^2\sqrt{36 - r^2}, \{r \in \mathbb{R} : 0 < r < 6\}$

7 a i $YB = r$ **ii** $ZB = r$
iii $AZ = x - r$ **iv** $CY = 3 - r$
b $r = \dfrac{x + 3 - \sqrt{x^2 + 9}}{2}$
c i $r = 1$ **ii** $x = 1.25$

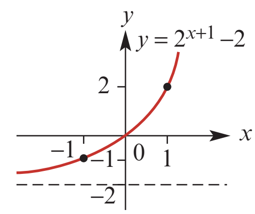
8 b $f(x) = \dfrac{q}{x}$ **c** $x = 3 \pm \sqrt{17}$

9 a i $f(2) = 3,\ f(f(2)) = 2,\ f(f(f(2))) = 3$
ii $f(f(x)) = x$
b $f(f(x)) = \dfrac{-x-3}{x-1},\ f(f(f(x))) = x$

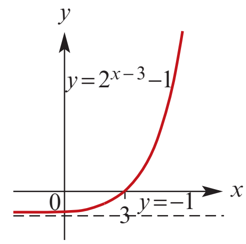
Chapter 2

Exercise 2A

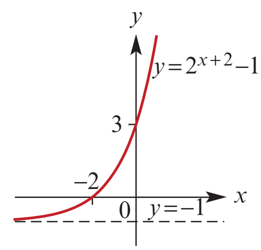
1 a Range $= (-2, \infty)$



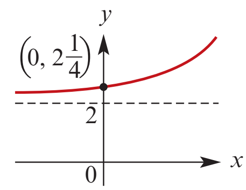
b Range $= (-1, \infty)$



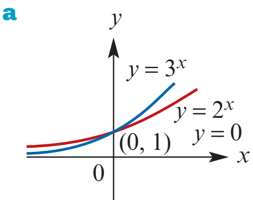
c Range $= (-1, \infty)$



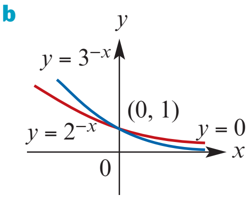
d Range $= (2, \infty)$



2 a



b



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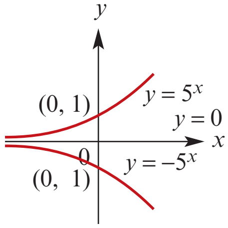
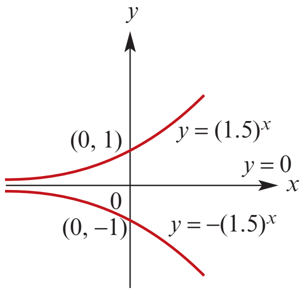
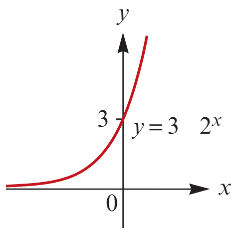
c



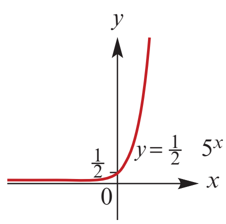
d



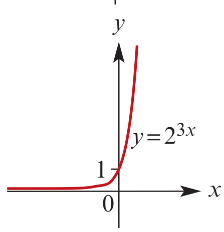
3 a Range = $(0, \infty)$



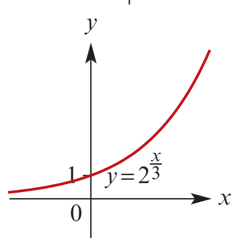
b Range = $(0, \infty)$



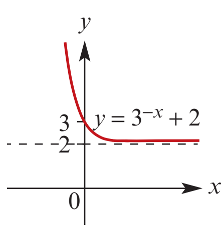
c Range = $(0, \infty)$



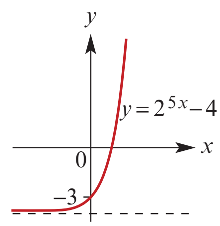
d Range = $(0, \infty)$



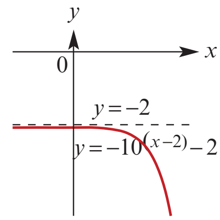
4 a Range = $(2, \infty)$



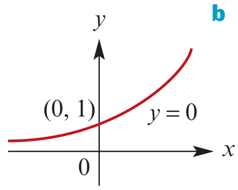
b Range = $(-4, \infty)$



c Range = $(-\infty, -2)$

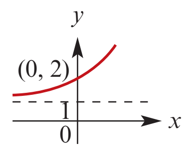


5 a



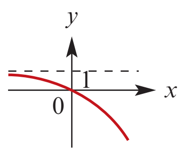
Range = $(0, \infty)$

b



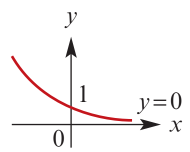
Range = $(1, \infty)$

c



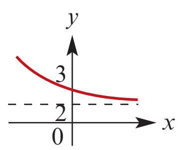
Range = $(-\infty, 1)$

d



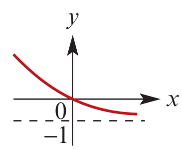
Range = $(0, \infty)$

e



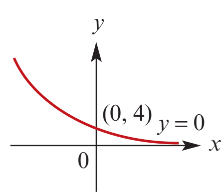
Range = $(2, \infty)$

f

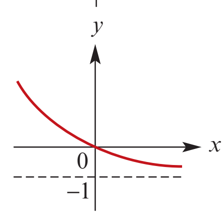


Range = $(-1, \infty)$

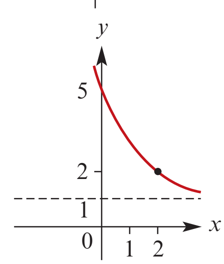
6 a Range = $(0, \infty)$



b Range = $(-1, \infty)$



c Range = $(1, \infty)$



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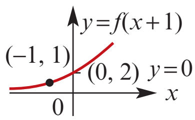
7 a



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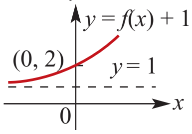
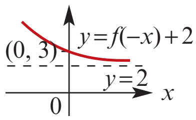


c



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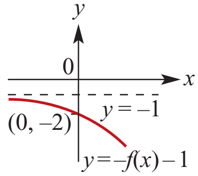
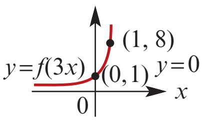


e



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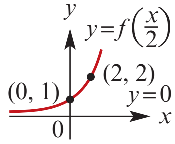
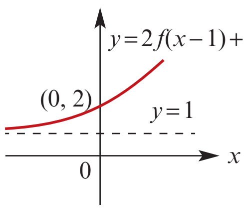
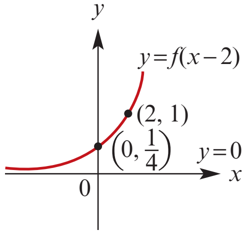


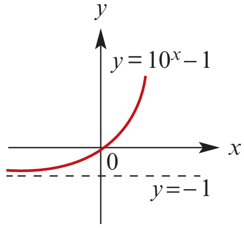
g



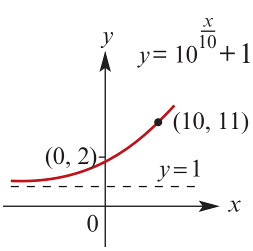
h



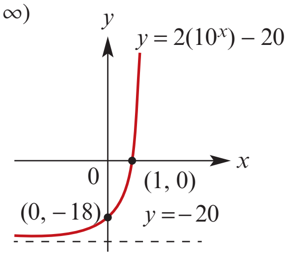
8 a Range = $(-1, \infty)$



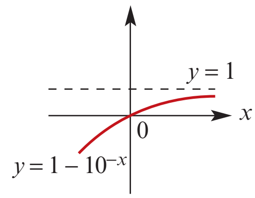
b Range = $(1, \infty)$



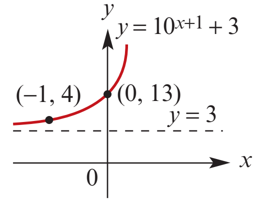
c Range = $(-20, \infty)$



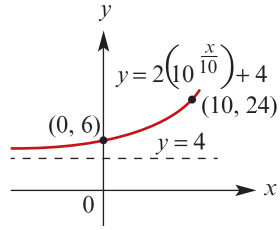
d Range = $(-\infty, 1)$



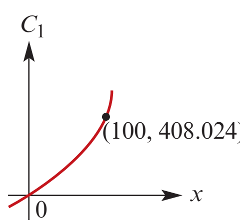
e Range = $(3, \infty)$



f Range = $(4, \infty)$



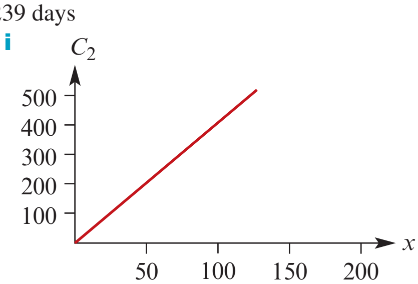
9 a



b **i** \$408.02 **ii** \$1274.70

c 239 days

d **i**



ii 302 days

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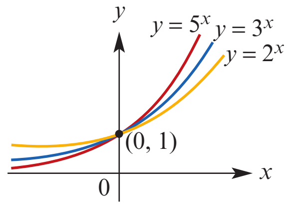
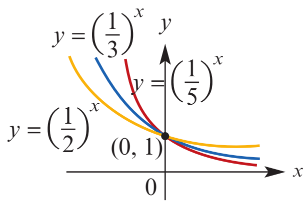
10 36 days

11 a i



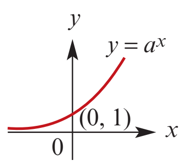
ii $x < 0$ **iii** $x > 0$ **iv** $x = 0$

b i

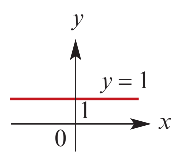


ii $x > 0$ **iii** $x < 0$ **iv** $x = 0$

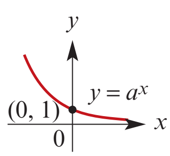
c i $a > 1$



ii $a = 1$

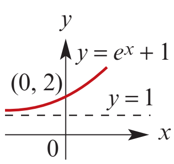


iii $0 < a < 1$



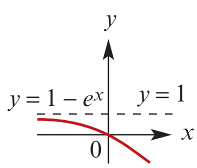
Exercise 2B

1 a



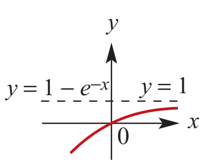
Range = $(1, \infty)$

b



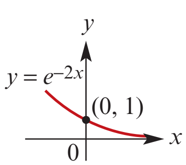
Range = $(-\infty, 1)$

c



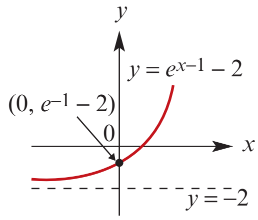
Range = $(-\infty, 1)$

d



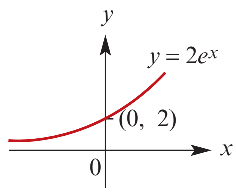
Range = $(0, \infty)$

e



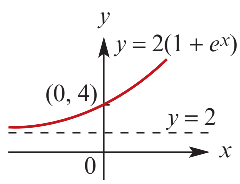
Range = $(-2, \infty)$

f



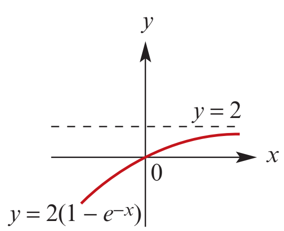
Range = $(0, \infty)$

g



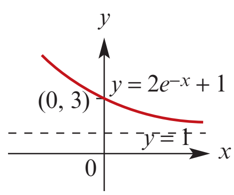
Range = $(2, \infty)$

h



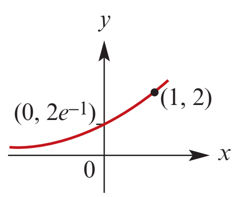
Range = $(-\infty, 2)$

i



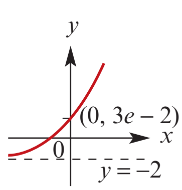
Range = $(1, \infty)$

j



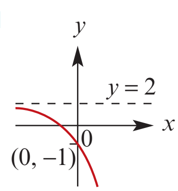
Range = $(0, \infty)$

k



Range = $(-2, \infty)$

l



Range = $(-\infty, 2)$

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2 a Translation 2 units to the left and 3 units down

b Dilation of factor 3 parallel to the y-axis, then translation 1 unit to the left and 4 units down

c Dilation of factor 5 parallel to the y-axis and factor $\frac{1}{2}$ parallel to the x-axis, then translation $\frac{1}{2}$ unit to the left

d Reflection in the x-axis, then translation 1 unit to the right and 2 units up

e Dilation of factor 2 parallel to the y-axis, reflection in the x-axis, then translation 2 units to the left and 3 units up

f Dilation of factor 4 parallel to the y-axis and factor $\frac{1}{2}$ parallel to the x-axis, then translation 1 unit down

3 a $y = -2e^{x-3} - 4$ **b** $y = 4 - 2e^{x-3}$

c $y = -2e^{x-3} - 4$ **d** $y = -2e^{x-3} - 8$

e $y = 8 - 2e^{x-3}$ **f** $y = -2e^{x-3} + 8$

4 a Translation 2 units to the right and 3 units up

b Translation 1 unit to the right and 4 units up, then dilation of factor $\frac{1}{3}$ parallel to the y-axis

c Translation $\frac{1}{2}$ unit to the right, then dilation of factor $\frac{1}{5}$ parallel to the y-axis and factor 2 parallel to the x-axis

d Translation 1 unit to the left and 2 units down, then reflection in the x-axis

e Translation 2 units to the right and 3 units down, then dilation of factor $\frac{1}{2}$ parallel to the y-axis and reflection in the x-axis

f Translation 1 unit up, then dilation of factor $\frac{1}{4}$ parallel to the y-axis and factor 2 parallel to the x-axis

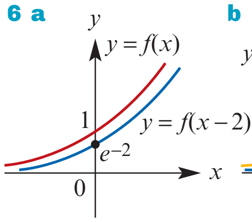
5 a $x = 1.146$ or $x = -1.841$

b $x = -0.443$

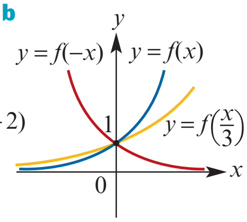
c $x = -0.703$

d $x = 1.857$ or $x = 4.536$

6 a



b



Exercise 2C

1 a $6x^6y^9$ **b** $3x^6$ **c** $\frac{6y^2}{x^2}$ **d** 8

e 16 **f** $\frac{5x^{28}}{y^6}$ **g** $24x^5y^{10}$ **h** $2xy^2$

i x^2y^2

2 a 4 **b** $\frac{1}{2}$ **c** 8 **d** $\frac{1}{4}$ **e** $\frac{3}{5}$ **f** 3

g $\frac{5}{2}$ **h** 6 **i** 4

3 a 1 **b** 1 **c** $-\frac{3}{2}$ **d** 3 **e** -2 **f** 4

g $-\frac{10}{3}$ **h** $-\frac{3}{2}$ **i** 6 **j** $\frac{3}{5}$ **k** $\pm\frac{1}{2}$

4 a 1 **b** 2 **c** 1 **d** 1, 2 **e** 0, 1

f 2, 4 **g** 0, 1 **h** -1, 2 **i** -1, 0

Exercise 2D

1 a 3 **b** -4 **c** -3 **d** 6 **e** 6 **f** -7

2 a $\log_{10} 6$ **b** $\log_{10} 4$ **c** $\ln(10^6) = 6\ln 10$

d $\ln 7$ **e** $\ln\frac{1}{60} = -\ln 60$

f $\ln(u^3v^6) = 3\ln(uv^2)$ **g** $\ln(x^7) = 7\ln x$

h $\ln 1 = 0$

3 a $x = 100$ **b** $x = 16$ **c** $x = 6$ **d** $x = 64$

e $x = e^3 - 5 \approx 15.086$ **f** $x = \frac{1}{2}$ **g** $x = -1$

h $x = 10^{-3} = \frac{1}{1000}$ **i** $x = 36$

4 a $x = 15$ **b** $x = 5$ **c** $x = 4$

d $x = 1$ ($x = -\frac{1}{2}$ is not an allowable solution)

e $x = \frac{3}{2}$

5 a $\log_{10} 27$ **b** $\log_2 4 = 2$

c $\frac{1}{2}\log_{10}\left(\frac{a}{b}\right) = \log_{10}\sqrt{\frac{a}{b}}$ **d** $\log_{10}\left(\frac{10a}{b^{\frac{1}{3}}}\right)$

e $\log_{10}\left(\frac{1}{8}\right) = -3\log_{10} 2$

6 a 1 **b** 1 **c** $2\frac{1}{2}$ **d** 3 **e** 0

7 a $-x$ **b** $2\log_2 x$

8 a $x = 4$ **b** $x = \frac{3e}{5 + 2e} \approx 0.7814$

9 a $x = \frac{-1 + \sqrt{1 + 12e}}{6}$, i.e. $x \approx 0.7997$

b $x = \ln 2 \approx 0.6931$

10 a $x = 3$ **b** $x = \frac{1}{2}$

11 $\frac{1}{4}$, 2

12 $N = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

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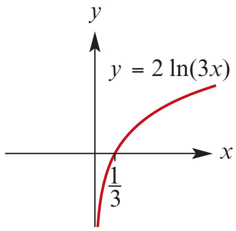
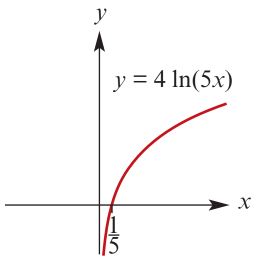
Exercise 2E

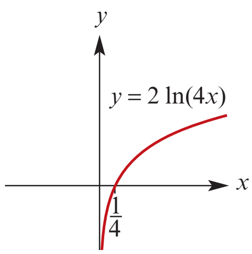
1 a



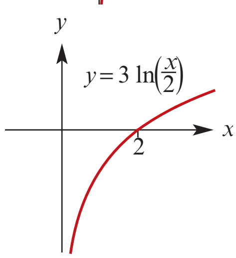
b



c

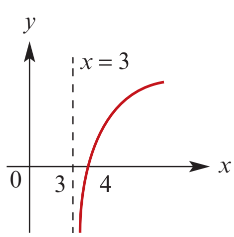


d



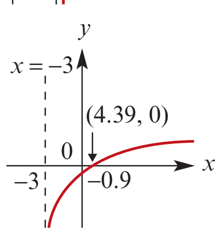
2 a Domain = $(3, \infty)$

Range = $\mathbb{R}$



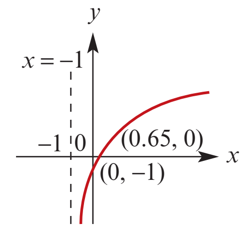
b Domain = $(-3, \infty)$

Range = $\mathbb{R}$



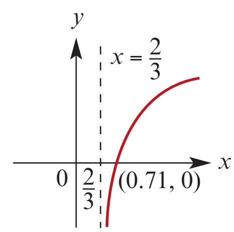
c Domain = $(-1, \infty)$

Range = $\mathbb{R}$



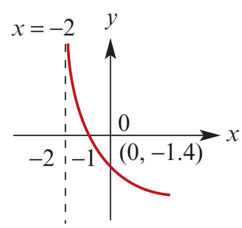
d Domain = $\left(\frac{2}{3}, \infty\right)$

Range = $\mathbb{R}$



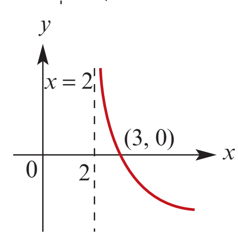
e Domain = $(-2, \infty)$

Range = $\mathbb{R}$



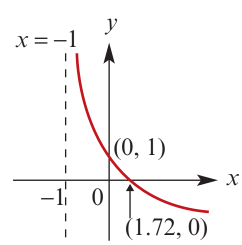
f Domain = $(2, \infty)$

Range = $\mathbb{R}$



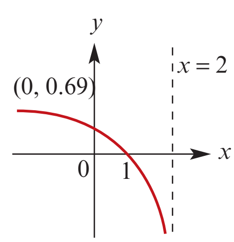
g Domain = $(-1, \infty)$

Range = $\mathbb{R}$



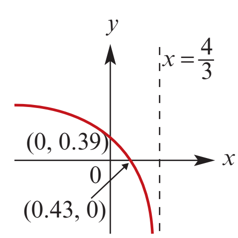
h Domain = $(-\infty, 2)$

Range = $\mathbb{R}$



i Domain = $\left(-\infty, \frac{4}{3}\right)$

Range = $\mathbb{R}$



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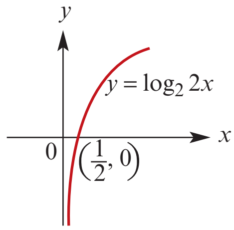
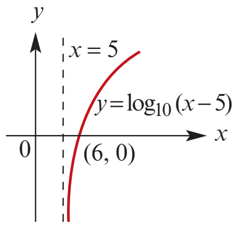
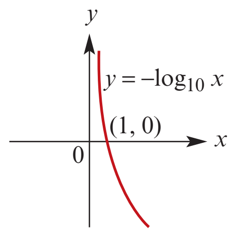
3 a Domain = $(0, \infty)$



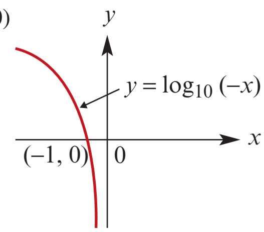
b Domain = $(5, \infty)$



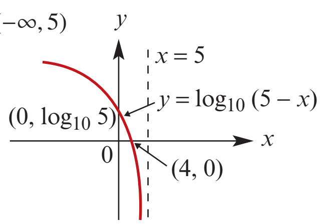
c Domain = $(0, \infty)$



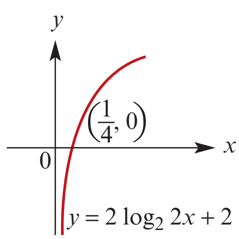
d Domain = $(-\infty, 0)$



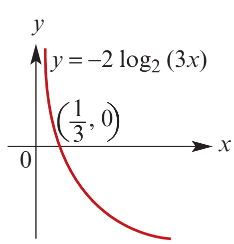
e Domain = $(-\infty, 5)$



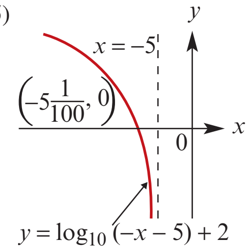
f Domain = $(0, \infty)$



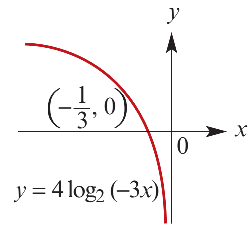
g Domain = $(0, \infty)$



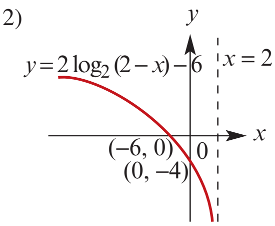
h Domain = $(-\infty, -5)$



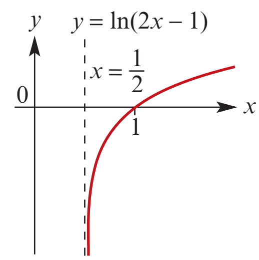
i Domain = $(-\infty, 0)$



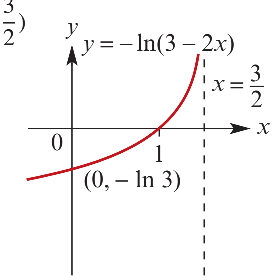
j Domain = $(-\infty, 2)$



k Domain = $(\frac{1}{2}, \infty)$

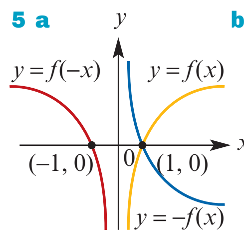


l Domain = $(-\infty, \frac{3}{2})$

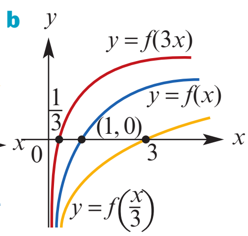


4 a $x = 1.557$ **b** $x = 1.189$

5 a



b



6 A dilation of factor $\ln 3$ parallel to the x-axis

7 A dilation of factor $\frac{1}{\ln 2}$ parallel to the x-axis

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Exercise 2F

1 $a = \dfrac{6}{e^4 - 1}$, $b = \dfrac{5e^4 - 11}{e^4 - 1}$

2 $a = \dfrac{2}{\ln 6}$, $b = -4$ **3** $a = 2, b = 4$

4 $a = \dfrac{14}{e - 1}$, $b = \dfrac{14}{1 - e}$ $(a \approx 8.148, b \approx -8.148)$

5 $a = 250, b = \dfrac{1}{3}\ln 5$ **6** $a = 200, b = 500$

7 $a = 2, b = 4$ **8** $a = 3, b = 5$

9 $a = 2, b = \dfrac{1}{3}\ln 5$ **10** $a = 2, b = 3$

11 $b = 1, a = \dfrac{2}{\ln 2}, c = 8$ $(a \approx 2.885)$

12 $a = \dfrac{2}{\ln 2}, b = 4$

Exercise 2G

1 a $k = \dfrac{1}{\log_2 7}$ **b** $x = \dfrac{\log_2 7 - 4}{\log_2 7}$
c $x = \dfrac{\ln 7 - 1}{\ln 14}$

2 a 2.58 **b** −0.32 **c** 2.18 **d** 1.16
e −2.32 **f** −0.68 **g** −2.15 **h** −1.38
i 2.89 **j** −1.70 **k** −4.42 **l** 5.76
m −6.21 **n** 2.38 **o** 2.80

3 a $x < 2.81$ **b** $x > 1.63$
c $x < -0.68$ **d** $x \le 3.89$
e $x \ge 0.57$

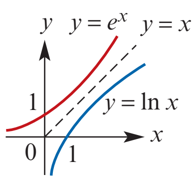
4 a $\log_2 5$ **b** $\dfrac{1}{2}(\log_3 8 + 1)$
c $\dfrac{1}{3}(\log_7 20 - 1)$ **d** $\log_3 7$ **e** $\log_3 6$
f $\log_5 6$ **g** $x = \log_3 8$ or $x = 0$ **h** $x = 1$

5 a $x > \log_7 52$ **b** $x < \dfrac{1}{2}\log_3 120$
c $x \ge \dfrac{1}{6}\log_2\left(\dfrac{5}{4}\right)$ **d** $x \le \log_3 7290$
e $x < \log_3 106$ **f** $x < \log_5\left(\dfrac{3}{5}\right)$

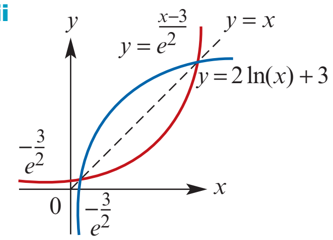
6 $x = e^{\frac{y+4}{3}}$

7 a $x = \dfrac{1}{2}e^y$ **b** $x = \dfrac{1}{2}e^{\frac{y-1}{3}}$
c $x = \ln(y - 2)$ **d** $x = \ln(y) - 2$
e $x = \dfrac{1}{2}(e^y - 1)$ **f** $x = \dfrac{1}{3}\left(e^{\frac{y}{4}} - 2\right)$
g $x = 10^y - 1$ **h** $x = \ln\left(\dfrac{y}{2}\right) + 1$

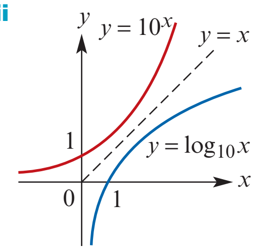
8 a i



ii



iii



b The graphs of $y = f(x)$ and $y = g(x)$ are reflections of each other in the line $y = x$

9 $t = \dfrac{-1}{k}\ln\left(\dfrac{P - b}{A}\right)$

10 a $x = e^{\frac{y-5}{2}}$ **b** $x = -\dfrac{1}{6}\ln\left(\dfrac{P}{A}\right)$
c $n = \dfrac{\ln\left(\frac{y}{a}\right)}{\ln x}$ **d** $x = \log_{10}\left(\dfrac{y}{5}\right)$
e $x = \dfrac{1}{2}e^{\frac{5-y}{3}}$ **f** $n = \dfrac{\ln\left(\frac{y}{6}\right)}{2\ln x}$
g $x = \dfrac{1}{2}(e^y + 1)$ **h** $x = \ln\left(\dfrac{5}{5 - y}\right)$

11 a 0.544 **b** 549.3

13 a 9^u **b** $u + \dfrac{1}{2}$ **c** $\dfrac{2}{u}$

14 $625, \dfrac{1}{625}$ **15** $\dfrac{2}{p}$

Exercise 2H

1 a $N = 1000 \times 2^{\frac{t}{15}}$ **b** 50 minutes

2 $d_0 = 52\left(\dfrac{13}{20}\right)^{\frac{1}{2}}$, $m = \dfrac{1}{2}\log_{10}\left(\dfrac{20}{13}\right)$

3 a i $N_0 = 20\,000$ **ii** −0.223
b 6.2 years

4 a $M_0 = 10$, $k = 4.95 \times 10^{-3}$
b 7.07 grams **c** 325 days

5 a $k = \dfrac{1}{1690}\ln 2$ **b** 3924 years

6 55 726 years

7 7575 years

8 a 16 471 **b** 35 years on from 2002

9 18.4 years

10 a 607 millibars **b** 6.389 km

11 21.82 hours

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12 6.4°C

13 $k = 0.349$, $N_0 = 50.25$

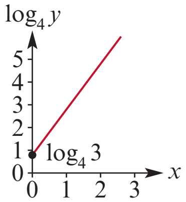
14 a $k = \ln\left(\frac{5}{4}\right)$ **b** 7.21 hours

15 a $a = 1000$, $b = 15^{\frac{1}{5}}$ **b** 3 hours **c** 13 hours
d 664 690

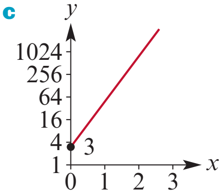
Exercise 2I

1 a $m = 2$ and $c = \log_4 3$

b

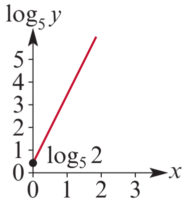


c

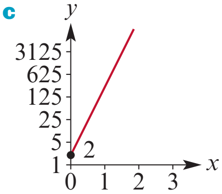


2 a $m = 3$ and $c = \log_5 2$

b



c



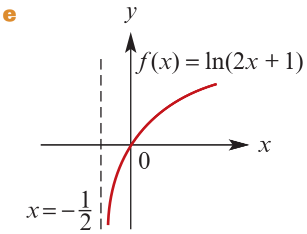
3 Street 10^{-9} watt/cm^2; Quiet car 10^{-11} watt/cm^2

4 a Increases by $10\log_{10} 2 \approx 3$ dB
b Increases by $10\log_{10} 10 = 10$ dB
c $P_{\text{new}} = (P_{\text{old}})^3 \times 10^{32}$
d $P = 10^{-16}$ **e** $P = 10^{-6}$

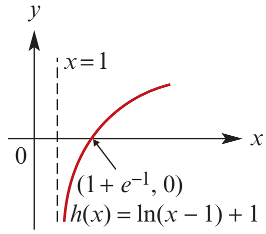
5 $P_1 = 10^{\frac{\lambda}{10}} \times P_2$ **6** $5 + \log_{10} 5$

7 $7.3 - \log_{10} 4$ **8** $[10^{-4}, 10^{-2}]$

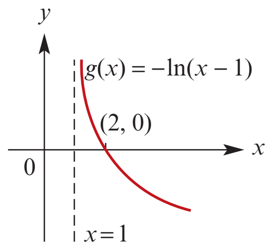
e



f



g



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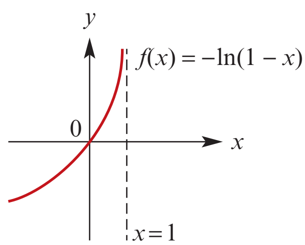
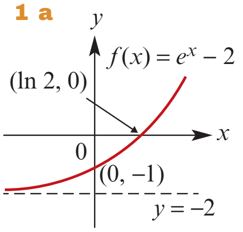


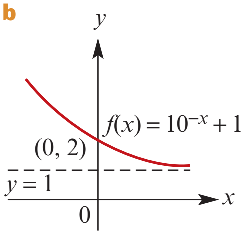
Chapter 2 review

Short-answer questions

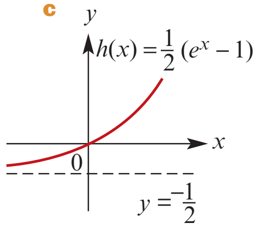
1 a



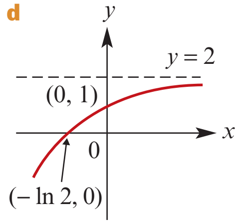
b



c



d



2 a $y = e^2 x$ **b** $y = 10x$ **c** $y = 16x^3$
d $y = \frac{x^5}{10}$ **e** $y = \frac{e^3}{x}$ **f** $y = e^{2x-3}$

3 a $x = \frac{\ln 11}{\ln 3}$ **b** $x = \frac{\ln 0.8}{\ln 2}$ **c** $x = \frac{\ln 3}{\ln(\frac{2}{3})}$

4 a $x = 1$ **b** $x = \frac{2}{3}$ **c** $x = \frac{1}{20}$
d $x = \log_{10} 3$ or $x = \log_{10} 4$

5 $a = 2$, $b = 2^{-\frac{2}{3}} - 1$ **6** $10^{\frac{6}{5}} - 1$

7 $\frac{1}{3}\ln\left(\frac{287}{4}\right)$ **8** $2a$

9 $x = 3^y + 4$

10 $a = \ln 5$, $b = 5$, $k = 2$

11 $x = \frac{1}{3}\ln(y + 4)$ **13** 3

14 a $k = \frac{1}{7}$ **b** $q = \frac{3}{2}$

15 a $f(-x) = f(x)$ **b** $2(e^u + e^{-u})$
c 0 **d** $e^{2u} + e^{-2u}$
e $g(-x) = -g(x)$ **f** $2e^x$, $2e^{-x}$, $e^{2x} - e^{-2x}$

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Extended-response questions

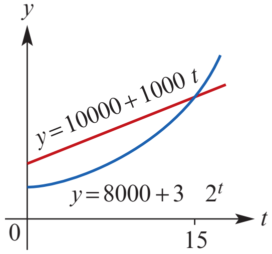
1 a 73.5366°C **b** 59.5946

2 a 770 **b** 1840

3 a $k = 22\,497,\ \lambda = 0.22$ **b** \$11 523

4 a $A = 65\,000,\ p = 0.064$ **b** \$47 300

5 a



b i (12.210, 22 209.62)
ii $t = 12.21$ **iii** 22 210

c ii (12.21, 12.21)

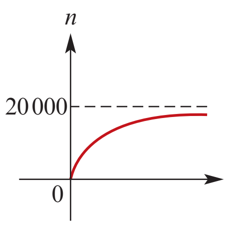
d $c = 0.52$

6 a iii $a = \frac{1}{2}$ or $a = 1$

iv If $a = 1$, then $e^{-2B} = 1$, and so $B = 0$;
If $a = \frac{1}{2}$, then $B = \frac{1}{2}\ln 2$
But if $B = 0$, $n = A(1 - 1) = 0$ for all t.
Therefore $B = \frac{1}{2}\ln 2$.

v $A = 20\,000$

b



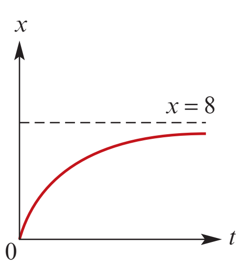
c $\dfrac{\ln 0.1}{\frac{1}{2}\ln(\frac{1}{2})} = \dfrac{2\ln 10}{\ln 2} \approx 6.644$

After 6.65 hours, the population is 18 000

7 a 75 **b** 2.37 **c** 0.646

8 $k = -0.5,\ A_0 = 100$

9 a



b i 0 grams **ii** 2.64 grams **iii** 6.92 grams

c 10.4 minutes

10 a $k = 0.235$ **b** 22.7°C **c** 7.17 minutes

Chapter 3

Short-answer questions

1 a Domain = $\{x \in \mathbb{R} : x \neq 0\}$;
Range = $\{y \in \mathbb{R} : y \neq 2\}$

b Domain = $\left[\frac{2}{3}, \infty\right)$; Range = $(-\infty, 3]$

c Domain = $\{x \in \mathbb{R} : x \neq 2\}$; Range = $(3, \infty)$

d Domain = $\{x \in \mathbb{R} : x \neq 2\}$;
Range = $\{y \in \mathbb{R} : y \neq 4\}$

e Domain = $[2, \infty)$; Range = $[-5, \infty)$

2 1

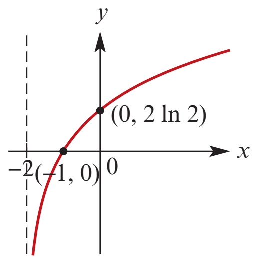
3 $x = \sqrt[3]{12a}$

4 $x = -9$

5 $x = \ln\left(\frac{3}{2}\right)$

6 a $a = -1$ and $b = 2\ln 2$

b



7 $x = 0$ or $x = 1$

8 $A = \frac{8}{5}$ and $k = \ln\left(\frac{5}{2}\right)$

Extended-response questions

1 a $k = 0.0292$ **b** 150×10^6

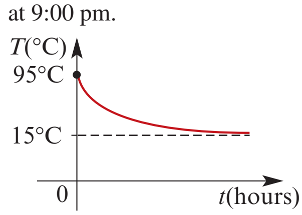
c 6.4494×10^8 **d** 23.762 years

2 a $A = 80,\ k = 0.3466$

b 17.5°C

c The temperature would reach 24°C 6 hours 18 minutes and 14 seconds after 2:00 p.m., i.e. 8:18:14. Therefore Jenny would first record this temperature, on the hour, at 9:00 pm.

d



3 a Area = $0.02(0.92)^{\frac{x}{10}}$ **b** $0.0197\ \text{mm}^2$

c Load = $0.02(0.92)^{10-2.9x}$ **d** $x < 2.59$ m

4 a Carriage A: $(0.83)^n I$
Carriage B: $0.66(0.89)^n I$

b 6 stations

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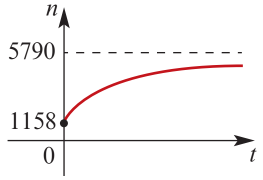
5 a i $n = 5790$ **ii** 1158

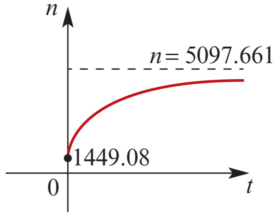
iii



iv $t = \dfrac{-100}{3}\ln\left(\dfrac{179}{1600}\right)$

b i $a = 2.518,\ b = 0.049,\ c = 5097.661$

ii



Chapter 4

Exercise 4A

1 -1

2 -1

3 a $h + 9$ **b** 9

4 a $x + 1$ **b** $2x^3 + 1$ **c** 40 **d** 0
e 5 **f** 1 **g** $2x + 1$ **h** $3x$
i $3x^3 + x$ **j** $6x$

5 a $2 + 3h + h^2$ **b** 2

6 $2x + h,\ 2x$

7 $h + 6,\ 6$

8 a $10x$ **b** 3 **c** 0 **d** $6x + 4$
e $15x^2$ **f** $10x - 6$

Exercise 4B

1 a $5x^4$ **b** $28x^6$ **c** 6 **d** $10x - 4$
e $12x^2 + 12x + 2$ **f** $20x^3 + 9x^2$
g $-4x + 4$ **h** $18x^2 - 4x + 4$

2 a -4 **b** -8 **c** -2 **d** -4

3 a -4 **b** -36

4 a $3t^2$ **b** $3t^2 - 2t$ **c** $x^3 + 9x^2$

5 a -2 **b** 0 **c** $15x^2 - 6x + 2$
d $\dfrac{6x^2 - 8}{5}$ **e** $4x - 5$ **f** $12x - 12$
g $50x^4$ **h** $27x^2 + 3$

6 a $4x - 15x^2$ **b** $-4z - 6$ **c** $18z^2 - 8z$
d $-2 - 15x^2$ **e** $-4z - 6$ **f** $-3z^2 - 8z$

7 a $(-\frac{1}{2}, 3\frac{1}{2})$ **b** $(2, 32), (-2, -32)$
c $(2, 6)$ **d** $(0, 0), (2, -4)$

8 a $(1, 7)$ **b** $\left(\dfrac{5}{4}, \dfrac{59}{8}\right)$

9 a $x = 1$ **b** $x = 0$ **c** $x = \dfrac{1 + \sqrt{3}}{2}$
d $x = \dfrac{3 + \sqrt{3}}{6}$ **e** $x = \dfrac{1 - \sqrt{3}}{2}$

10 a $78.69°$ **b** $0°$ **c** $45°$ **d** $135°$
e $63.43°$ **f** $116.57°$

11 a $8x - 4$ **b** $2x + 2$ **c** $6x^2 - 12x + 18$
d $x^2 - 2x + 1$

12 a $(3, 16)$, gradient $= 8$
b $(0, -1)$, gradient $= -1$
c $(-1, 6)$, gradient $= -8$
d $(4, 594)$, gradient $= 393$
e $(1, -28)$, gradient $= -92$
f $(2\frac{1}{2}, 0)$, gradient $= 0$

13 a $x = 1$ **b** $x = 1$ **c** $x > 1$ **d** $x < 1$
e $x = 2\frac{2}{3}$ **f** $x = 4$ or $x = -2$

14 a $(-\infty, -1) \cup (1, \infty)$ **b** $(-1, 1)$ **c** $\{1, -1\}$

15 a $(-1, 0.5) \cup (2, \infty)$ **b** $(-\infty, -1) \cup (0.5, 2)$
c $\{-1, 0.5, 2\}$

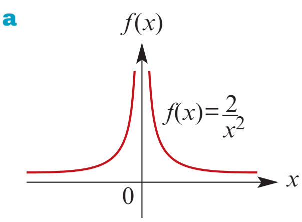
16 a $(-\frac{1}{4}, 2) \cup (2, \infty)$ **b** $(-\infty, -\frac{1}{4})$ **c** $\{-\frac{1}{4}, 2\}$

17 a $(2, -12)$ **b** $(3, -11)$ **c** $\left(\dfrac{5}{4}, -\dfrac{183}{16}\right)$

21 a $(-\infty, -1]$ **b** $[2, \infty)$ **c** $(-\infty, 0]$
d $\left[\dfrac{3}{2}, \infty\right)$

Exercise 4C

1 a



b $\dfrac{-2(2 + h)}{(1 + h)^2}$ **c** -4

2 a $\dfrac{-1}{(x - 3)^2}$ **b** $\dfrac{-1}{(x + 2)^2}$

3 $-4x^{-5}$

4 a $-6x^{-3} - 5x^{-2}$ **b** $12x - \dfrac{15}{x^4}$
c $\dfrac{15}{x^4} - \dfrac{8}{x^3}$ **d** $-18x^{-4} - 6x^{-3}$ **e** $-\dfrac{2}{x^2}$

5 a $\dfrac{4}{z^2}$ **b** $\dfrac{-18 - 2z}{z^4}$ **c** $3z^{-4}$
d $\dfrac{-2z^3 + z^2 - 4}{z^2}$ **e** $\dfrac{6 - 12z}{z^4}$
f $-6x - \dfrac{6}{x^2}$

6 a $11\frac{3}{4}$ **b** $\dfrac{1}{32}$ **c** -1 **d** 5

7 $f'(x) = 10x^{-6} > 0$ for all $\{ x \in \mathbb{R} : x \neq 0 \}$

8 $\pm\dfrac{1}{2}$ **9** $a = -1,\ b = 4$

10 $\dfrac{1}{2}$ **11** $a = -9,\ b = 1$

12 $k = 0$ or $k = \dfrac{3}{2}$

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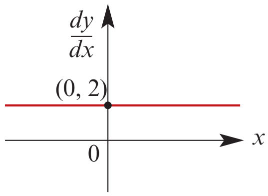
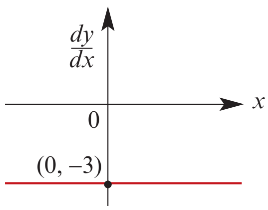
Exercise 4D

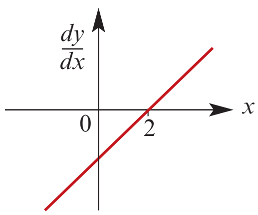
1 a



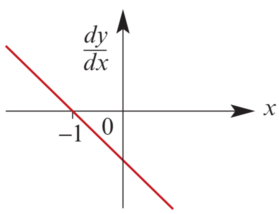
b



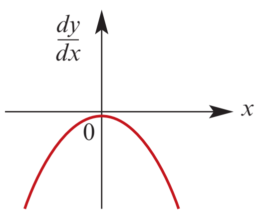
c



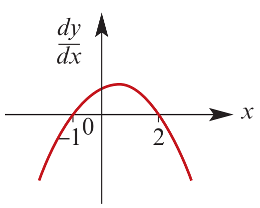
d



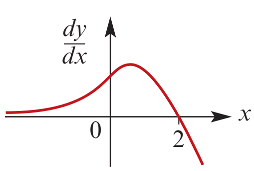
e



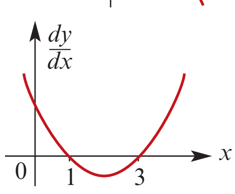
f



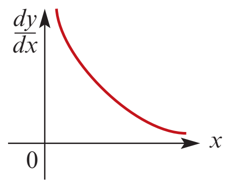
g



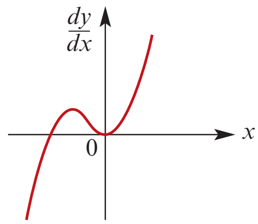
h



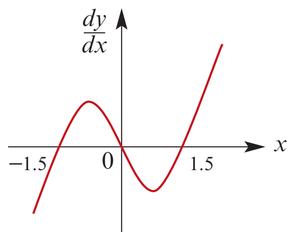
i



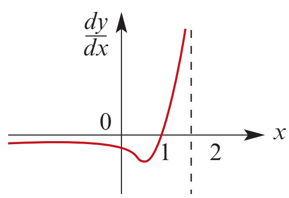
2 a



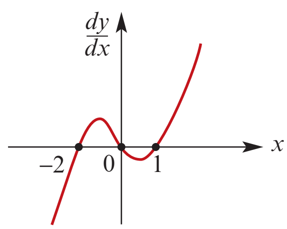
b



c

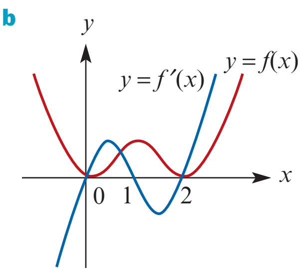


d



3 a B **b** C **c** A **d** D

4 a b



c i 0 **ii** 0 **iii** 0 **iv** 96

d i 1 **ii** 0.423

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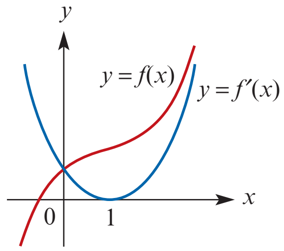
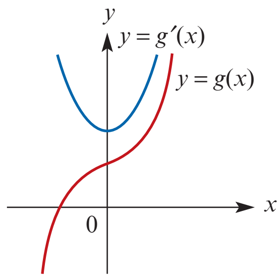
5



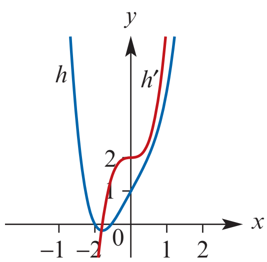
Gradient is 0 at $\left(1, \frac{4}{3}\right)$;
Gradient is positive for $\mathbb{R} \setminus \{1\}$

6



Gradient is always positive;
Minimum gradient where $x = 0$

7 a



b i $x = -1.4945$ or $x = 0.7976$
ii $x = 0.6300$

Exercise 4E

1 a $8x(x^2+1)^3$ **b** $20x(2x^2-3)^4$
c $24(6x+1)^3$ **d** $an(ax+b)^{n-1}$
e $2anx(ax^2+b)^{n-1}$ **f** $\dfrac{6x}{(1-x^2)^4}$
g $-3\left(x^2-\dfrac{1}{x^2}\right)^{-4}\left(2x+\dfrac{2}{x^3}\right)$ **h** $(1-x)^{-2}$

2 a $6(x+1)^5$ **b** $4x^3(3x+1)(x+1)^7$
c $4\left(6x^3+\dfrac{2}{x}\right)^3\left(18x^2-\dfrac{2}{x^2}\right)$ **d** $-4(x+1)^{-5}$

3 -10 **4** $-\dfrac{1}{2}$ and $\dfrac{1}{2}$ **5** $2x\sqrt{3x^2+1}$

6 a $n[f(x)]^{n-1}f'(x)$ **b** $\dfrac{-f'(x)}{[f(x)]^2}$

Exercise 4F

1 $x^{-\frac{1}{2}}$

2 a $\dfrac{x^{-\frac{4}{5}}}{5}$ **b** $\dfrac{5}{2}x^{\frac{3}{2}}$
c $\dfrac{5}{2}x^{\frac{3}{2}}-\dfrac{3}{2}x^{\frac{1}{2}}$ **d** $\dfrac{3}{2}x^{-\frac{1}{2}}-\dfrac{20}{3}x^{\frac{2}{3}}$
e $-\dfrac{6}{7}x^{-\frac{13}{7}}$ **f** $-\dfrac{1}{4}x^{-\frac{5}{4}}+2x^{-\frac{1}{2}}$

3 a $\dfrac{1}{27}$ **b** $\dfrac{1}{12}$ **c** $\dfrac{2}{9}$ **d** $\dfrac{5}{2}$

4 a $\dfrac{1}{\sqrt{2x+1}}$ **b** $\dfrac{-3}{2\sqrt{4-3x}}$
c $\dfrac{x}{\sqrt{x^2+2}}$ **d** $\dfrac{-1}{\sqrt[3]{(4-3x)^2}}$
e $\dfrac{3}{2}\sqrt{x}-\dfrac{1}{\sqrt{x^3}}$ **f** $3\sqrt{x}\left(\dfrac{5x+6}{2}\right)$

7 a $\dfrac{x}{\sqrt{x^2+2}}$ **b** $\dfrac{2x-5}{3\sqrt[3]{(x^2-5x)^2}}$
c $\dfrac{2x+2}{5\sqrt[5]{(x^2+2x)^4}}$

Exercise 4G

1 a $5e^{5x}$ **b** $-21e^{-3x}$
c $-12e^{-4x}+e^x-2x$ **d** e^x-e^{-x}
e $e^{-2x}(e^x-1)$ **f** $2e^{2x}-2e^{-2x}$

2 a $-6x^2e^{-2x^3}$ **b** $2xe^{x^2}+3$
c $(2x-4)e^{x^2-4x}+3$ **d** $(2x-2)e^{x^2-2x+3}-1$
e $-\dfrac{1}{x^2}e^{\frac{1}{x}}$ **f** $\dfrac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

3 a $\dfrac{9}{2}$ **b** $\dfrac{1}{2}e^{\frac{1}{2}}+4$

4 a 5 **b** $5e^4+2$

5 a $2f'(x)e^{2f(x)}$ **b** $2e^{2x}f'(e^{2x})$

6 a $8e^{2x}(e^{2x}-1)^3$ **b** $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ **c** $\dfrac{e^x}{2\sqrt{e^x-1}}$
d $\dfrac{2}{3}e^{x^{\frac{2}{3}}}x^{-\frac{1}{3}}$ **e** $(2x-3)e^{(x-1)(x-2)}$
f e^{e^x+x}

Exercise 4H

1 a $\dfrac{2}{x}$ **b** $\dfrac{2}{x}$ **c** $2x+\dfrac{3}{x}$
d $\dfrac{3x-1}{x^2}$ **e** $\dfrac{3+x}{x}$ **f** $\dfrac{1}{x+1}$
g $\dfrac{1}{x+2}$ **h** $\dfrac{3}{3x-1}$ **i** $\dfrac{6}{6x-1}$

2 a $\dfrac{3}{x}$ **b** $\dfrac{3(\ln x)^2}{x}$ **c** $\dfrac{2x+1}{x^2+x-1}$
d $\dfrac{3x^2+2x}{x^3+x^2}$ **e** $\dfrac{4}{2x+3}$ **f** $\dfrac{4}{2x-3}$

3 a $\dfrac{2x}{x^2+1}$ **b** 1

4 a $(e, 1),\ m = \dfrac{1}{e}$
b $(e, \ln(e^2+1)),\ m = \dfrac{2e}{e^2+1}$
c $(-e, 1),\ m = -\dfrac{1}{e}$ **d** $(1, 1),\ m = 2$

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e $(1, 0),\ m = 0$ **f** $\left(\frac{3}{2}, \ln 2\right),\ m = 1$

5 $\frac{1}{2}$ **6** $\frac{1+2x}{1+x+x^2}$ **7** $\frac{3}{5}$ **8** 2

Exercise 4I

1 a $5\cos(5x)$ **b** $-5\sin(5x)$
c $5\sec^2(5x)$ **d** $2\sin x\cos x$
e $3\sec^2(3x+1)$ **f** $-2x\sin(x^2+1)$
g $2\sin\left(x-\frac{\pi}{4}\right)\cos\left(x-\frac{\pi}{4}\right)$
h $-2\cos\left(x-\frac{\pi}{3}\right)\sin\left(x-\frac{\pi}{3}\right)$
i $6\sin^2\left(2x+\frac{\pi}{6}\right)\cos\left(2x+\frac{\pi}{6}\right)$
j $-6\cos^2\left(2x+\frac{\pi}{4}\right)\sin\left(2x+\frac{\pi}{4}\right)$

2 a $\frac{1}{\sqrt{2}},\ \sqrt{2}$ **b** $1,\ 0$ **c** $2,\ 0$
d $0,\ 0$ **e** $1,\ 0$ **f** $1,\ 4$

3 a $-5\sin(x) - 6\cos(3x)$
b $-\sin x + \cos x$ **c** $\cos x + \sec^2 x$
d $2\tan x\sec^2 x$

4 a $-\frac{\pi}{90}\sin x^\circ$ **b** $\frac{\pi}{60}\cos x^\circ$
c $\frac{\pi}{60}\sec^2(3x)^\circ$

5 a $\tan x$ **b** $\frac{-1}{\sin x\cos x}$

6 a $2\cos(x)\,e^{2\sin x}$ **b** $-2\sin(2x)\,e^{\cos(2x)}$

Exercise 4J

1 a $20x^4 + 36x^2 + 4x$ **b** $9x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$
c $3(2x-1)^2(8x-1)$ **d** $8x(2x^2+1)(6x^2+1)$
e $5(3x+1)^{\frac{1}{2}}(3x+4)$ **f** $\frac{5x^2-8x+1}{\sqrt{2x-4}}$
g $x^2(3x^2+4x+3)(3x^2+2x+1)^{-2}$
h $2x^3(5x^2-2)(2x^2-1)^{-\frac{1}{2}}$
i $2x\sqrt[3]{x^2+2x} + \frac{2x^2(x+1)}{3\sqrt[3]{(x^2+2x)^2}}$
j $\frac{4(5x^2-4)^2(5x^2+2)}{x^3}$
k $\frac{3(x^6-16)}{x^4}$ **l** $\frac{2x^3(9x^2-8)}{5\sqrt[5]{(x^3-x)^4}}$

2 a $e^x(x^2+2x+1)$
b $e^{2x}(2x^3+3x^2+6x+5)$
c $2e^{4x+1}(x+1)(2x+3)$ **d** $\frac{-8x-7}{2e^{4x}\sqrt{x+1}}$

3 a $1+\ln x$ **b** $2x+4x\ln x$ **c** $e^x\ln x + \frac{e^x}{x}$
d $1+\ln(-x)$

4 a $\frac{2x^3(2-x)}{e^{2x}}$ **b** $2e^{2x+3}$
c $\frac{3}{2}(2e^{2x}+1)(e^{2x}+x)^{\frac{1}{2}}$ **d** $\frac{e^x(x-1)}{x^2}$
e $xe^{\frac{1}{2}x^2}$ **f** $-x^2e^{-x}$

5 a $e^x(f'(x)+f(x))$ **b** $\frac{e^x(f(x)-f'(x))}{[f(x)]^2}$
c $f'(x)e^{f(x)}$ **d** $2e^xf'(x)f(x) + [f(x)]^2e^x$

6 a $3x^2\cos(x) - x^3\sin(x)$
b $2x\cos x - (1+x^2)\sin x$
c $-e^{-x}\sin x + e^{-x}\cos x$
d $6\cos x - 6x\sin x$
e $3\cos(3x)\cos(4x) - 4\sin(4x)\sin(3x)$
f $2\sin(2x) + 2\tan(2x)\sec(2x)$
g $12\sin x + 12x\cos x$
h $2xe^{\sin x} + x^2\cos x\,e^{\sin x}$
i $2x\cos^2 x - 2x^2\cos x\sin x$
j $e^x\tan x + e^x\sec^2 x$

7 a $-e^{\pi}$ **b** 0

8 2

Exercise 4K

1 a $\frac{4}{(x+4)^2}$ **b** $\frac{4x}{(x^2+1)^2}$ **c** $\frac{x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{2(1+x)^2}$
d $\frac{(x+2)^2(x-3)(x-1)}{(x^2+1)^2}$ **e** $\frac{2+2x-x^2}{(x^2+2)^2}$
f $\frac{-4x}{(x^2-1)^2}$ **g** $\frac{x^2+4x+1}{(x^2+x+1)^2}$
h $\frac{-2(4x^3+3x^2+1)}{(2x^3+2x)^2}$

2 a $81,\ 378$ **b** $0,\ 0$ **c** $0, 0$
d $\frac{1}{2},\ 0$ **e** $\frac{3}{2},\ -\frac{1}{2}$

3 a $\frac{2x^2+x+1}{\sqrt{x^2+1}}$ **b** $\frac{x(7x^3+3x+4)}{2\sqrt{x^3+1}}$
c $\frac{5}{(x+3)^2}$

4 a $\frac{3e^x-2e^{4x}}{(3+e^{3x})^2}$ **b** $-\left(\frac{(x+1)\sin(x)+\cos(x)}{(x+1)^2}\right)$
c $\frac{x - x\ln(x)+1}{x(x+1)^2}$

5 a $\frac{1-\ln x}{x^2}$ **b** $\frac{1+x^2-2x^2\ln x}{x(1+x^2)^2}$

6 a $\frac{9e^{3x}}{(3+e^{3x})^2}$ **b** $\frac{-2e^x}{(e^x-1)^2}$ **c** $\frac{-8e^{2x}}{(e^{2x}-2)^2}$

7 a -2 **b** -6π **c** $-e^{\pi}$ **d** $-\frac{1}{\pi}$

Exercise 4L

1 a 0 **b** $56x^6$ **c** $\frac{-1}{4\sqrt{x^3}}$
d $48(2x+1)^2$ **e** $-\sin x$ **f** $-\cos x$
g e^x **h** $\frac{-1}{x^2}$ **i** $\frac{2}{(x+1)^3}$
j $-4\sin\left(2x+\frac{\pi}{4}\right)$

2 a $\frac{15\sqrt{x}}{4}$ **b** $8(x^2+3)^2(7x^2+3)$
c $-\frac{1}{4}\sin\left(\frac{x}{2}\right)$ **d** $-48\cos(4x+1)$

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e $2e^{2x+1}$ **f** $\dfrac{-4}{(2x+1)^2}$ **g** $12x^2 + 6$

h $(x^3 + 6x^2 + 6x)e^x$ **i** $\dfrac{1}{x}$

3 a $f''(x) = 24e^{3-2x}$

b $f''(x) = 8e^{-0.5x^2}(1 - x^2)$ **c** $f''(x) = 0$

d $f''(x) = \dfrac{-2(x^2 + 2x + 2)}{(x^2 + 2x)^2}$

e $f''(x) = 360(1 - 3x)^3$

f $f''(x) = (4x^2 + 2)e^{x^2}$

g $f''(x) = \dfrac{-4}{(x+1)^3}$ **h** $f''(x) = \dfrac{3}{4\sqrt{(1-x)^5}}$

i $f''(x) = -5\sin(3 - x)$

j $f''(x) = -9\cos(1 - 3x)$

k $f''(x) = -\dfrac{1}{9}\sin\left(\dfrac{x}{3}\right)$ **l** $f''(x) = -\dfrac{1}{16}\cos\left(\dfrac{x}{4}\right)$

4 a 1 **b** −1 **c** −1 **d** 0

5 a $x = -\dfrac{2}{3}$ **b** $x = \dfrac{1}{15}$ **c** $x = \pm\dfrac{1}{\sqrt{2}}$

6 a $x = -2$ **b** $x = 0$ or $x = -3$

c $x = 0$, $x = -3 + \sqrt{3}$ or $x = -3 - \sqrt{3}$

Chapter 4 review

Short-answer questions

1 a 8 **b** −8

2 a $1 - \dfrac{x}{\sqrt{1 - x^2}}$ **b** $\dfrac{-4x^2 - 2x + 12}{(x^2 + 3)^2}$

c $\dfrac{3}{2\sqrt{1 + 3x}}$ **d** $\dfrac{-2}{x^2} - \dfrac{1}{2}x^{-\frac{3}{2}}$

e $\dfrac{3x - 15}{2\sqrt{x - 3}}$ **f** $\dfrac{1 + 2x^2}{\sqrt{1 + x^2}}$ **g** $\dfrac{4x}{(x^2 + 1)^2}$

h $\dfrac{-x^2 + 1}{(x^2 + 1)^2}$ **i** $\dfrac{10x}{3}(2 + 5x^2)^{-\frac{2}{3}}$

j $\dfrac{-2x^2 - 2x + 4}{(x^2 + 2)^2}$ **k** $4x(3x^2 + 2)^{-\frac{1}{3}}$

3 a −6 **b** 1 **c** 5 **d** $\dfrac{1}{6}$

4 a $\dfrac{1}{x + 2}$ **b** $3\cos(3x + 2)$

c $-\dfrac{1}{2}\sin\left(\dfrac{x}{2}\right)$ **d** $(2x - 2)e^{x^2 - 2x}$ **e** $\dfrac{1}{x - 3}$

f $2\pi\cos(2\pi x)$

g $6\sin(3x + 1)\cos(3x + 1)$ **h** $\dfrac{1}{2x\sqrt{\ln x}}$

i $\dfrac{2 - 2\ln(2x)}{x^2}$

j $2x\sin(2\pi x) + 2\pi x^2\cos(2\pi x)$

5 a $e^x\sin(2x) + 2e^x\cos(2x)$

b $4x\ln x + 2x$ **c** $\dfrac{1 - 3\ln x}{x^4}$

d $2\cos(2x)\cos(3x) - 3\sin(2x)\sin(3x)$

e $\dfrac{2}{\cos^2(2x)} = 2\sec^2(2x)$

f $-9\cos^2(3x + 2)\sin(3x + 2)$

g $2x\sin^2(3x) + 6x^2\cos(3x)\sin(3x)$

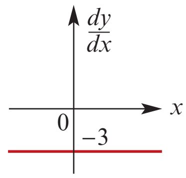
6 a $2e^2 \approx 14.78$ **b** 0

c $15e^3 + 2 \approx 303.28$ **d** 1

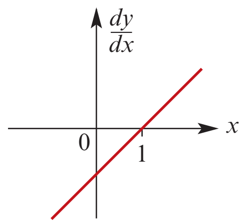
7 a ae^{ax} **b** ae^{ax+b} **c** $-be^{a-bx}$

d $abe^{ax} - abe^{bx}$ **e** $(a - b)e^{(a-b)x}$

8 a



b



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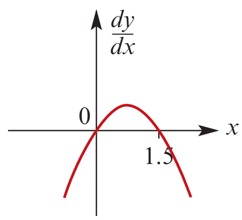

9 $2\left(4 - \dfrac{9}{x^2}\right)\left(4x + \dfrac{9}{x}\right)$, $x = \pm\dfrac{3}{2}$

10 b $\left(\dfrac{3}{2}, \infty\right) \cap (-1, 4) = \left(\dfrac{3}{2}, 4\right)$

11 a $24(2x - 1)$ **b** $2e^{x^2+1}(2x^2 + 1)$

c $2\cos x - x\sin x$

Extended-response questions

1 a i −4 **ii** −6 **iii** −18 **iv** −18 **v** 6 **vi** $-\dfrac{1}{6}$

b $a = \dfrac{5}{2}$, $b = 1$, $c = -\dfrac{7}{2}$, $d = 6$

2 a i −1 and 3 **ii** $x > 3$ and $x < -1$

b (3, 6) and (7, 1) **c** $\left(\dfrac{1}{2}, 6\right)$ and $\left(\dfrac{5}{2}, 1\right)$

d (2, 6) and (10, 1) **e** (2, 18) and (10, 3)

Chapter 5

Exercise 5A

1 $y = 4x - 5$

2 $y = -\dfrac{1}{3}x - 1$

3 $y = x - 2$ and $y = -x + 3$

4 $y = 18x + 1$, $y = -\dfrac{1}{18}x + 1$

5 $\left(\dfrac{3}{2}, -\dfrac{11}{4}\right)$, $c = -\dfrac{29}{4}$

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6 a i $y = 2x - 3$ **ii** $y = -\frac{1}{2}x - \frac{1}{2}$

b i $y = -3x - 1$ **ii** $y = \frac{1}{3}x - 1$

c i $y = -x - 2$ **ii** $y = x$

d i $y = 8x + 2$ **ii** $y = -\frac{1}{8}x - \frac{49}{8}$

e i $y = \frac{3}{2}x + 1$ **ii** $y = -\frac{2}{3}x + 1$

f i $y = \frac{1}{2}x + \frac{1}{2}$ **ii** $y = -2x + 3$

g i $y = \frac{2}{3}x + \frac{4}{3}$ **ii** $y = -\frac{3}{2}x + \frac{7}{2}$

h i $y = 4x - 16$ **ii** $y = -\frac{1}{4}x - \frac{15}{2}$

i i $y = -2$ **ii** $x = 2$

j i $y = 4x - 4$ **ii** $y = -\frac{1}{4}x + \frac{1}{4}$

7 $y = 56x - 160$

8 a $y = -1$ **b** $y = \frac{3}{2}x + \frac{1}{2}$

c $y = -2x - 1$ **d** $y = -4x + 5$

9 a $y = 2x$ **b** $y = -1$ **c** $y = 2x - \frac{\pi - 2}{2}$

d $y = 2x$ **e** $y = x$ **f** $y = -x + \frac{\pi - 2}{2}$

10 a $y = 2$ **b** $y = x$

c $y = 4e^2x - 3e^2$ **d** $y = \frac{e}{2}(x + 1)$

e $y = 3ex - 2e$ **f** $y = 4e^{-2}$

11 a $y = x - 1, y = -x + 1$ **b** $y = 2x - 1$

c $y = kx - 1$

12 a $x = 0$ **b** $x = 0$ **c** $x = 4$

d $x = -5$ **e** $x = -\frac{1}{2}$ **f** $x = -5$

13 $\frac{\pi - 2}{2}$ **14** $a = 1$ **15** $a = e$

16 $a = 0$ **17** $a = 0$ or $a = \frac{3}{2}$

Exercise 5B

1 a 21 **b** $3h + 18$ **c** 18

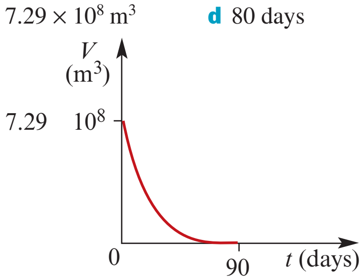
2 a $\frac{dV}{dt}$ **b** $\frac{dS}{dr}$ **c** $\frac{dV}{dx}$ **d** $\frac{dA}{dt}$ **e** $\frac{dV}{dh}$

3 Wanes by 0.006 units per day

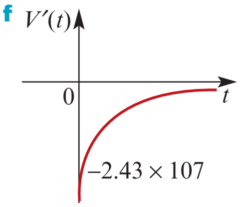
4 a $-3 \times 10^3(90 - t)^2$ **b** 90 days

c 7.29×10^8 m^3 **d** 80 days

e

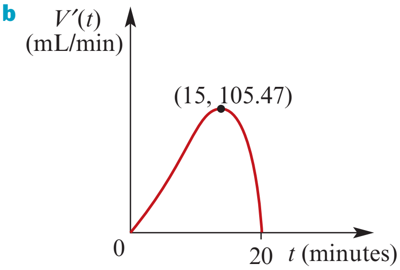


f



5 a $V'(t) = \frac{t^3}{160}(20 - t)$

b



c $t = 15$

6 a $t \approx 100$, $t \approx 250$, $t \approx 500$

b $\approx 430\,000$ m^3/day **c** $\approx 270\,000$ m^3/day

d $(100, 250) \cup (500, 600)$

7 a $\lambda = 0.1373$, $P_0 = 30$

b 9.625 hours

c i 4.120 units/hour **ii** 1.373 units/hour

8 a $-0.3(T - 15)$

b i -22.5°C/minute **ii** -13.5°C/minute

iii -4.5°C/minute

9 $\frac{dy}{dx} = 3 - 2\sin x$, gradient always positive

10 a 4.197 **b** -0.4

11 a $v = 6t^2 - 18t + 12$

b $t = 1, x = 5$; $t = 2, x = 4$

c -6 m/s^2, 6 m/s^2 **d** -1.5 m/s

12 a 8 cm from O **b** 2 cm/s

c 9 cm from O after 1 s **d** -2 cm/s^2

13 a $v = \frac{\sqrt{2}t}{\sqrt{t^2 + 1}}$ **b** $a = \frac{\sqrt{2}}{(t^2 + 1)^{\frac{3}{2}}}$

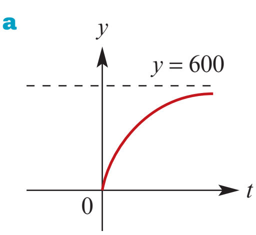
c 1 cm/s, $\frac{1}{2}$ cm/s^2

14 When $t = 0$, $v = 0.4$ m/s;

When $t = 1$, $v = 0.4e \approx 1.0874$ m/s;

When $t = 2$, $v = 0.4e^2 \approx 2.956$ m/s

15 a



b 3.33

16 a $-2y$ **b** ky

17 a 0.18 kg **b** 3.47 hours

c i 6.93 hours **ii** 10.4 hours

d $0.2\,m$

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Exercise 5C

1 $-28h$

2 a $\frac{dy}{dx} = 12x^2 - 16x$ **b** 16 **c** 0.32

3 $\frac{dy}{dx} = 5 - \frac{8}{x^2}$, $\delta y \approx 3h$

4 a $\delta y \approx 3h$ **b** $\delta y \approx \frac{h}{2\sqrt{a}}$
c $\delta y \approx \frac{-6h}{(3a+1)^2}$ **d** $\delta y \approx 8h(2a+1)^3$
e $\delta y \approx 36ah(6a^2 - 1)^2$
f $\delta y \approx \frac{-2h(a^2 + 5a - 1)}{(a^2+1)^2}$
g $\delta y \approx \frac{2ah}{3\sqrt[3]{(a^2+10)^2}}$ **h** $\delta y \approx \frac{5h}{(a+1)^2}$

5 -0.48

6 a $\frac{dT}{d\ell} = \frac{\pi}{\sqrt{g\ell}}$ **b** $\frac{\pi}{40}\sqrt{\frac{10}{g}}$ seconds

7 $\frac{1}{20}$ cm

8 4%

9 a $46h$ **b** $\frac{4600h}{111}\%$

10 a i $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$ **ii** $f'(100) = -\frac{1}{2000}$
b $-\frac{3}{2000}$
c 0.0985
d $\frac{1}{\sqrt{a+h}} \approx \frac{1}{\sqrt{a}} - \frac{h}{2(\sqrt{a})^3}$

11 $\frac{31}{30}$

12 $10\pi h$ cm^2

13 4%

14 a i $f'(x) = \frac{1}{(1-x)^2}$ **ii** $f'(0) = 1$

15 a i $f'(x) = \frac{1}{2\sqrt{1+x}}$ **ii** $f'(0) = \frac{1}{2}$

17 a $\frac{8\pi}{5}$ cm^2; 2.5% **b** $\frac{16\pi}{5}$ cm^3; 3.75%

18 $2h$

19 a $f'(x) = ae^{ax}$ **b** $f(h) \approx 1 + ah$
c $f(b+h) \approx f(b) \cdot (1 + ah)$

20 a $\delta y \approx he^{\frac{a}{2}}$ **b** $\delta y \approx -2he^a$
c $\delta y \approx (1+a)he^a$ **d** $\delta y \approx (1-a)he^{-a}$

21 0

22 $\frac{h}{2}$

23 0.01

24 a $\delta y \approx -2h\sin(2a)$ **b** $\delta y \approx \frac{h}{2}\cos\left(\frac{a}{2}\right)$
c $\delta y \approx 2h\sec^2(2a)$ **d** $\delta y \approx -\frac{h}{2}\sec^2\left(\frac{a}{2}\right)$
e $\delta y \approx h\sin\left(\frac{\pi}{4} - a\right)$ **f** $\delta y \approx -\frac{h}{2}\cos\left(\frac{a}{2}\right)$

25 a i $h\sec^2\theta$ **ii** $2h$
b $\frac{\pi}{90} + 1$

26 a $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
b i $\frac{1-h}{\sqrt{2}}$ **ii** 0.6967

Exercise 5D

1 a $(2, -16), (-2, 16)$ **b** $(1, -2)$
c $(0, 0), (1, 1)$ **d** $(4, 48)$
e $(0, 0), \left(\frac{2}{\sqrt{3}}, \frac{16}{3}\right), \left(\frac{-2}{\sqrt{3}}, \frac{16}{3}\right)$ **f** $\left(\frac{1}{3}, \frac{14}{3}\right)$
g $(3, 2)$ **h** $(0, -10), (2, 6)$

2 a $(0, 1)$ **b** $\left(\frac{1}{3e}, -\frac{1}{3e}\right)$
c $(0, 1), (-\pi, 1), \left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right), (\pi, 1)$
d $(-1, -e^{-1})$ **e** $(0, 0), (2, 4e^{-2})$
f $(e^{-1}, -2e^{-1})$

3 a $a = 6$ **b** $b = 3$

4 $b = -2,\ c = 1,\ d = 3$

5 $a = 2,\ b = -4,\ c = -1$

6 $a = \frac{2}{3},\ b = -2\frac{1}{2},\ c = -3,\ d = 7\frac{1}{2}$

7 a $a = 2$ and $b = 9$ **b** $(-1, -5)$

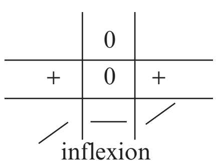
8 $x = \frac{1}{2}$ or $x = \frac{1-4n}{2n+2}$

9 $x = \pm 1$ or $x = 0$

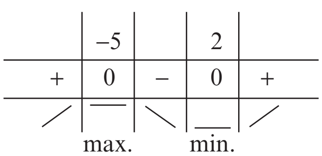
10 $(1, \frac{1}{2}),\ (-1, -\frac{1}{2})$

Exercise 5E

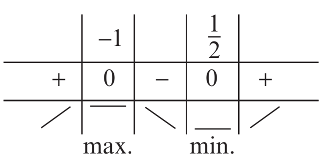
1 a $x = 0$



b $x = 2, x = -5$



c $x = -1, x = \frac{1}{2}$



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d $x = -3, x = 4$

	-3		4	
$-$	0	$+$	0	$-$
\	—	/	—	\
	min.		max.	

e $x = -3, x = 4$

	-3		4	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

f $x = 0, x = \frac{27}{5}$

	0		$\frac{27}{5}$	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

g $x = 1, x = 3$

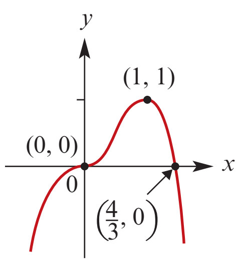
	1		3	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

h $x = 1, x = 3$

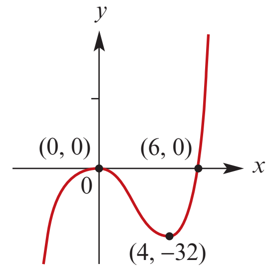
	1		3	
$-$	0	$+$	0	$-$
\	—	/	—	\
	min.		max.	

2 a $x = -2$ (max), $x = 2$ (min)
b $x = 0$ (min), $x = 2$ (max)
c $x = \frac{1}{3}$ (max), $x = 3$ (min)
d $x = 0$ (inflection)
e $x = -2$ (inflection), $x = 0$ (min)
f $x = -\frac{1}{\sqrt{3}}$ (max), $x = \frac{1}{\sqrt{3}}$ (min)

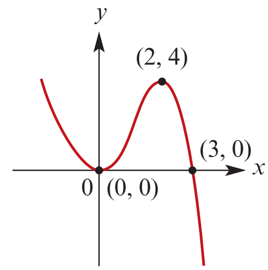
3 a i $(0, 0)$, $(\frac{4}{3}, 0)$
ii $(0, 0)$ inflection, $(1, 1)$ max



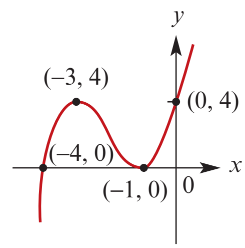
b i $(0, 0)$, $(6, 0)$
ii $(0, 0)$ max, $(4, -32)$ min



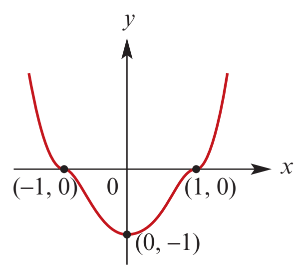
c i $(0, 0)$, $(3, 0)$
ii $(0, 0)$ min, $(2, 4)$ max



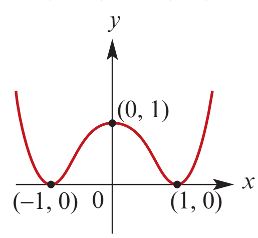
d i $(-4, 0)$, $(-1, 0)$, $(0, 4)$
ii $(-3, 4)$ max, $(-1, 0)$ min



e i $(-1, 0)$, $(0, -1)$, $(1, 0)$
ii $(-1, 0)$ infl, $(0, -1)$ min, $(1, 0)$ infl



f i $(-1, 0)$, $(0, 1)$, $(1, 0)$
ii $(-1, 0)$ min, $(0, 1)$ max, $(1, 0)$ min



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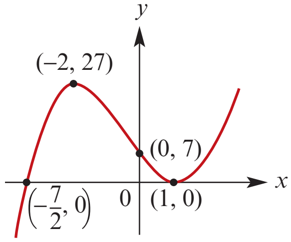
4 a $(-2, 27)$ max, $(1, 0)$ min

b $(1, 0)$ is a turning point

c $\left(-\frac{7}{2}, 0\right)$, $(0, 7)$

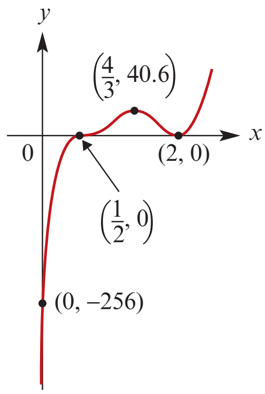
d



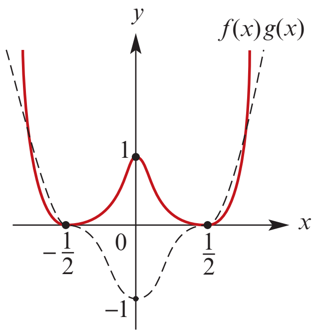
5 b $a = 3$, $b = 2$, $(0, 2)$ min, $(-2, 6)$ max

6 a $(0, -256)$, $(\frac{1}{2}, 0)$, $(2, 0)$

b $\left(\frac{1}{2}, 0\right)$ inflection, $\left(\frac{4}{3}, 40.6\right)$ max, $(2, 0)$ min



7 a



b i $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

ii $\left(\frac{-\sqrt{66}}{12}, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left(\frac{\sqrt{66}}{12}, \infty\right)$

8 a $(-2, 0)$ max, $\left(\frac{4}{3}, -18\frac{14}{27}\right)$ min

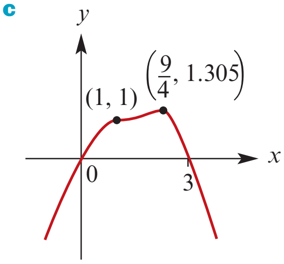
b No stationary points

9 a $(0, -1)$ stationary point of inflection, $(-1, -1)$ minimum

b $(0, -1)$ stationary point of inflection, $(-1.5, -2.6875)$ minimum

c No stationary points, gradient is always positive

10 b $x \le \frac{9}{4}$ **c**



11 a $x = -1$ (infl), $x = 1$ (min), $x = 5$ (max)

b $x = 0$ (max), $x = 2$ (min)

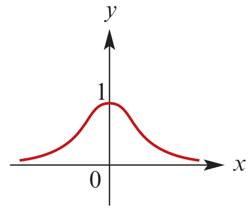
c $x = -4$ (min), $x = 0$ (max)

d $x = -3$ (min), $x = 2$ (infl)

12 a $(0, 0)$ local max; $(2\sqrt{2}, -64)$ and $(-2\sqrt{2}, -64)$ local min

b $(0, 0)$ local max; $\left(\pm 4\sqrt{\frac{m-1}{m}}, -\frac{16^m(m-1)^{m-1}}{m^m}\right)$ local min

13



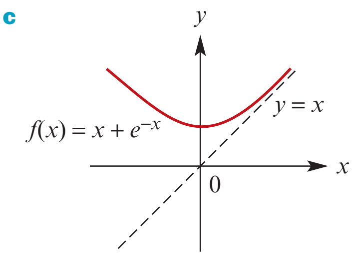
14 $\{x : -2 < x < 0\}$

15 $x < 1$; Max value $= \frac{100}{e^4} \approx 1.83$

16 a Min value $= f(0) = 0$

17 a $(0, 1)$ min **b** $y = x$

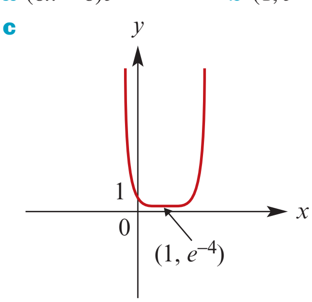
c



18 $p = 1, q = -6, r = 9$

19 a $(8x - 8)e^{4x^2 - 8x}$ **b** $(1, e^{-4})$ min

c



d $y = -\frac{1}{8}x + \frac{5}{4}$

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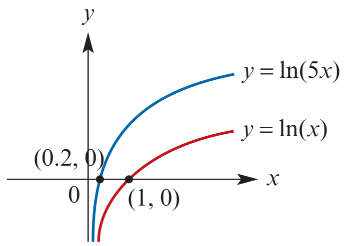
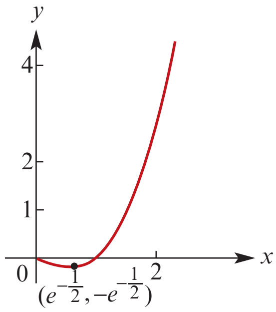
20 Tangents are parallel for any given value of x.



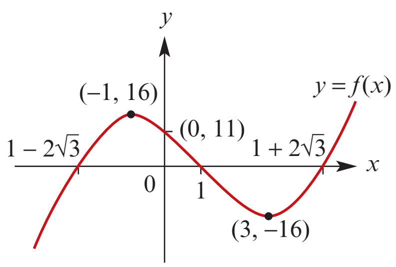
21 a $2x\ln(x) + x$ **b** $x = 1$

c $x = e^{-\frac{1}{2}}$

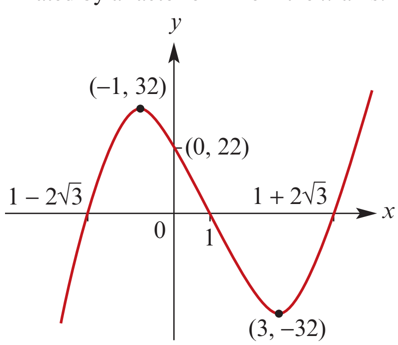
d



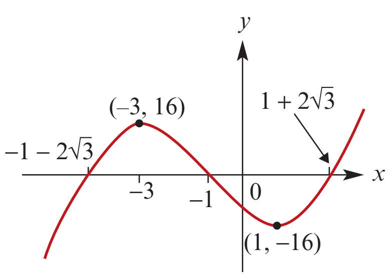
22 a



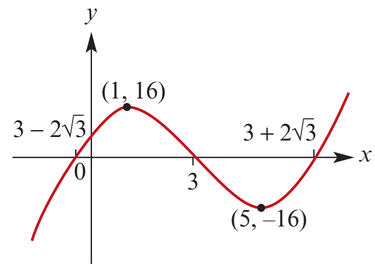
b Dilated by a factor of 2 from the x-axis:



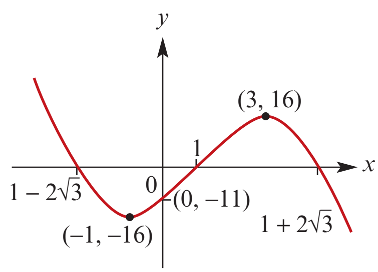
c Translated 2 units to the left:



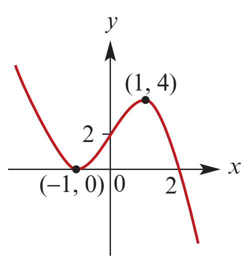
d Translated 2 units to the right:



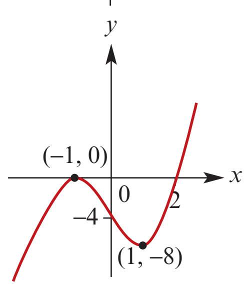
e Reflected in the x-axis:



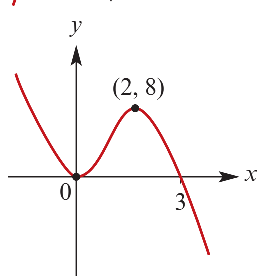
23 a



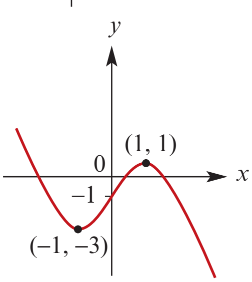
b



c



d



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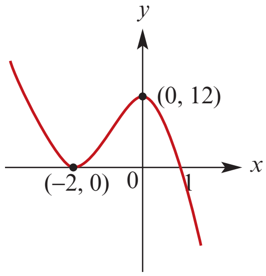
e



24 a $(a+\ell, 0)$, $(b+\ell, 0)$ **b** $(h+\ell, kp)$

25 a Max $x = \frac{\pi}{3}, \frac{5\pi}{3}$; Min $x = 0, \pi, 2\pi$

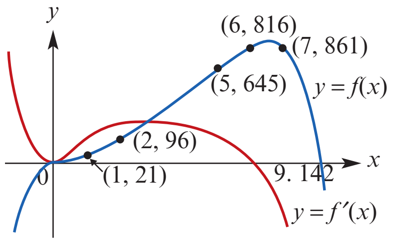
b Max $x = \frac{\pi}{6}$; Min $x = \frac{5\pi}{6}$; Infl $x = \frac{3\pi}{2}$

c Max $x = \frac{\pi}{2}, \frac{3\pi}{2}$; Min $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

d Max $x = \frac{\pi}{3}$; Infl $x = \pi$; Min $x = \frac{5\pi}{3}$

26 a $y = -x^4 + 8x^3 + 10x^2 + 4x$

b



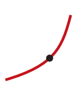
Local max at (6.761, 867.07); No stationary point of inflection: $\frac{dy}{dx} = 4$ when $x = 0$

c −960

d $x = 4.317$ or $x = 8.404$

Exercise 5F

1 a



b

c

d

2 a Point of inflection (0, 0); Concave up on $(0, \infty)$

b Point of inflection $\left(\frac{1}{3}, -\frac{2}{27}\right)$; Concave up on $\left(\frac{1}{3}, \infty\right)$

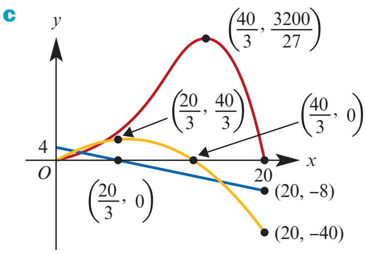
c Point of inflection $\left(\frac{1}{3}, \frac{2}{27}\right)$; Concave up on $\left(-\infty, \frac{1}{3}\right)$

d Points of inflection (0, 0), $\left(\frac{1}{2}, -\frac{1}{16}\right)$; Concave up on $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$

3 a Local min (0, 0); local max $\left(\frac{40}{3}, \frac{3200}{27}\right)$

b $\left(\frac{20}{3}, \frac{1600}{27}\right)$; gradient $= \frac{40}{3}$

c



4 a i $6x^2 + 12x$ **ii** $12x + 12$

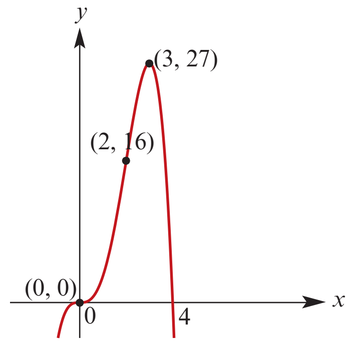
b Local min (0, −12); local max (−2, −4)

c (−1, −8)

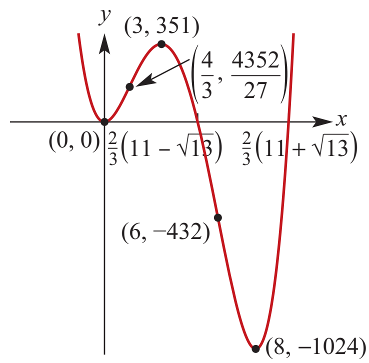
5 a $f'(x) = \cos x$; $f''(x) = -\sin x$; max $\left(\frac{\pi}{2}, 1\right)$; min $\left(\frac{3\pi}{2}, -1\right)$; infl $(\pi, 0)$

b $f'(x) = e^x(x+1)$; $f''(x) = e^x(x+2)$; min $(-1, -e^{-1})$; infl $(-2, -2e^{-2})$

6

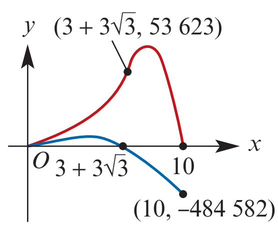


7



8 a $f'(x) = e^x(10 + 8x - x^2)$, $f''(x) = e^x(18 + 6x - x^2)$

b



c $3 + 3\sqrt{3}$, $(3 + 3\sqrt{3}, 53\,623)$

9 (0, 0), (π, π), (2π, 2π), (3π, 3π), (4π, 4π)

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10 a $0, \pi, 2\pi$ **b** $0, \pi, 2\pi$

c $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

11 $\frac{d^2y}{dx^2} = 2a \neq 0$

12 a $\left(\frac{3}{2}, 2\right)$ **b** $\left(1, \frac{3}{2}\right)$

13 a $(0, 0), -6$ **b** $(-1, -1), 8;\ \ (1, -1), -8$

c $(0, 3), 0$ **d** No points of inflection

e No points of inflection

f $\left(-\sqrt{3}, \frac{-\sqrt{3}}{2}\right), \frac{-1}{4};\ (0, 0), 2;\ \left(\sqrt{3}, \frac{\sqrt{3}}{2}\right), \frac{-1}{4}$

14 a $\frac{-7\pi}{4}, \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

16 a $f'(x) = 2x(1 + 2\ln x)$

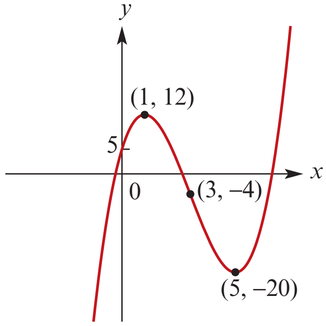
b $f''(x) = 2(3 + 2\ln x)$

c Stationary point at $(e^{-\frac{1}{2}}, -e^{-1})$;

point of inflection at $(e^{-\frac{3}{2}}, -3e^{-3})$

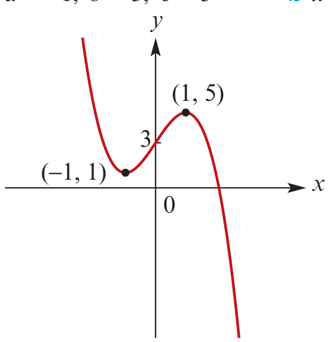
17 a $a = -9,\ b = 15,\ c = 5$

b



18 a $a = -1,\ b = 3,\ c = 3$ **b** $x = -1$

c



Exercise 5G

1 Absolute max = 2; Absolute min = −70

2 Absolute max = 15; Absolute min = −30

3 Absolute max = 0; Absolute min = −20.25

4 Absolute max = 2304; Absolute min = −8

5 b $\frac{dV}{dx} = 30x - 36x^2$

c Local max at $\left(\frac{5}{6}, \frac{125}{36}\right)$

d Absolute max value is 3.456 when $x = 0.8$

e Absolute max value is $\frac{125}{36}$ when $x = \frac{5}{6}$

6 a $25 \leq y \leq 28$

b Absolute max = 125; Absolute min = 56

7 a $\frac{1}{(x-4)^2} - \frac{1}{(x-1)^2}$ **b** $\left(\frac{5}{2}, \frac{4}{3}\right)$

c Absolute max = $\frac{3}{2}$; Absolute min = $\frac{4}{3}$

8 b $\frac{dA}{dx} = \frac{1}{4}(x - 5)$ **c** $x = 5$ **d** $\frac{61}{8}$ m^2

9 Absolute max = 12.1; Absolute min = 4

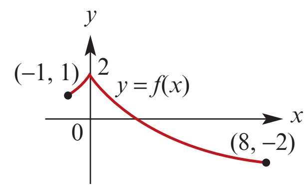
10 a $\frac{1}{(x-4)^2} - \frac{1}{(x+1)^2}$ **b** $\left(\frac{3}{2}, \frac{4}{5}\right)$

c Absolute max = $\frac{5}{4}$; Absolute min = $\frac{4}{5}$

11 Absolute max = $\frac{\sqrt{2}}{2}$; Absolute min = −1

12 Absolute max = 1; Absolute min = $\frac{\sqrt{2}}{2}$

13 Absolute max = 2; Absolute min = −2



14 Absolute max = $\frac{1}{e^2} + 2e^2$;

Absolute min = $2\sqrt{2}$

15 Absolute max = $-\ln 10$;

Absolute min = $-\frac{10}{e}$

Exercise 5H

1 625 m^2

2 First = $\frac{4}{3}$; Second = $\frac{8}{3}$

3 Max value of P is 2500

4 Max area is 2 km × 1 km = 2 km^2

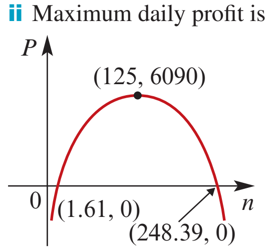
5 $p = \frac{3}{2},\ q = \frac{8}{3}$

6 b $V = \frac{75x - x^3}{2}$ **c** 125 cm^3

7 a i $n = 125$

ii Maximum daily profit is \$6090

b



c $2 \leq n \leq 248$

d $n = 20$

8 12°C

9 8 mm for maximum; $\frac{4}{3}$ mm for minimum

10 a $8\cos\theta$

b Area = $16(1 + \cos\theta)\sin\theta$;

Max area = $12\sqrt{3}$ square units

11 $(1, 1)$

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12 a $\dfrac{75}{\cos\theta}$ seconds

b $220 - 60\tan\theta$ seconds

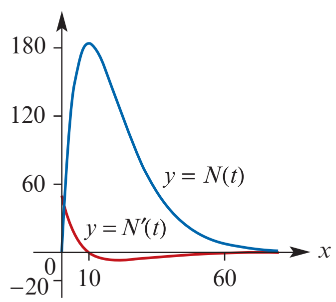
d $\dfrac{dT}{d\theta} = \dfrac{75\sin\theta - 60}{\cos^2\theta}$

e $\theta = \sin^{-1}\left(\dfrac{4}{5}\right) \approx 53.13°$

f Min time $T = 265$ seconds occurs when distance BP is 400 metres

13 Max population $\dfrac{500}{e}$ occurs when $t = 10$

14 a



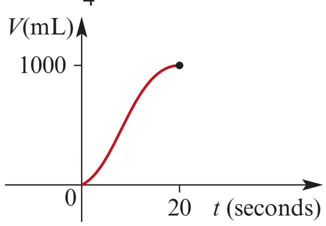
b Max rate of increase is 50, occurs at $t = 0$;

Max rate of decrease is $\dfrac{50}{e^2}$, occurs at $t = 20$

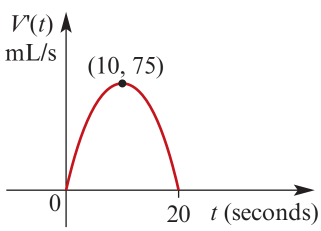
15 a i $V(0) = 0$ mL **ii** $V(20) = 1000$ mL

b $V'(t) = \dfrac{3}{4}(20t - t^2)$

c



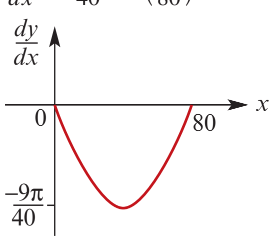
d



e $t = 10$ s, 75 mL/s

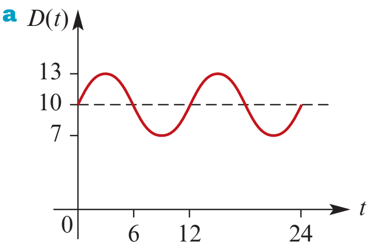
16 a $\dfrac{dy}{dx} = \dfrac{-9\pi}{40}\sin\left(\dfrac{\pi x}{80}\right)$

b



c $(40, 12)$

17 a



b $\{t : D(t) \geq 8.5\} = [0, 7] \cup [11, 19] \cup [23, 24]$

c i 0 m/h **ii** $-\dfrac{\pi}{2}$ m/h **iii** $\dfrac{\pi}{2}$ m/h

d i $t = 0, 12, 24$ **ii** $t = 6, 18$

Exercise 5I

1 a $f'(x) = (x-1)(3x - 2b - 1)$

b $\left(\dfrac{2b+1}{3}, \dfrac{-4(b-1)^3}{27}\right)$ and $(1, 0)$

c $f''(1) = 2 - 2b < 0$, since $b > 1$ **d** $b = \dfrac{11}{2}$

2 a $(0, 0)$, $(\sqrt{2}, -4)$, $(-\sqrt{2}, -4)$

b (a, b), $(\sqrt{2} + a, -4 + b)$, $(\sqrt{2} + a, -4 + b)$

3 a $a = c - 20$, $b = 30 - 2c$ **b** $c = \dfrac{150}{7}$

4 a Increasing $\left[0, \dfrac{2}{3a}\right]$; Decreasing $\left[\dfrac{2}{3a}, \infty\right)$

b $y = \dfrac{-1}{a}x + \dfrac{1}{a^2}$ **c** $y = ax - 1$

d $\left(-\infty, \dfrac{4}{27a^2}\right]$

5 a i $2(a-3)$ **ii** $m = 2(a-3)$

b $(a, (a-3)^2)$

c $y = 2(a-3)x - a^2 + 9$ **d** $\dfrac{3+a}{2}$

6 a $h = 1$ or $h = -3$ **b** $a = 2$

c $a = -48$, $b = -64$

7 a $(a, 0)$, $\left(\dfrac{a+2}{3}, \dfrac{-4(a-1)^3}{27}\right)$

b $(a, 0)$ local minimum;

$\left(\dfrac{a+2}{3}, \dfrac{4(a-1)^3}{27}\right)$ local maximum

c i $y = (a-1)^2 x - (a-1)^2$ **ii** $y = 0$

iii $y = -\dfrac{(a-1)^2}{4}(x - a)$

8 a $2(x-1)(x-b)(2x - b - 1)$

b $(1, 0)$, $(b, 0)$, $\left(\dfrac{b+1}{2}, \dfrac{(b-1)^4}{16}\right)$ **c** $b = 3$

9 $a = \dfrac{1}{486}$, $b = 0$, $c = \dfrac{-1}{161}$, $d = \dfrac{1459}{243}$

10 a $a = \dfrac{4d-1}{4}$, $b = \dfrac{-(2d+3)}{2}$, $c = \dfrac{11-4d}{4}$

b $d = \dfrac{38}{67}$

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Chapter 5 review

Short-answer questions

1 a $y = -x$ **b** $(0, 0)$

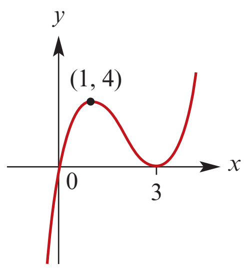
2 $y = 6ax - 3a^2$, $P(0, -3a^2)$

3 a $y = 3x - 3$ **b** $x = \frac{11}{3}$

4 a 5π square units/unit
b 6π square units/unit

5 a $(1, 1)$ max; $(0, 0)$ inflection
b $(-1, 0)$ max; $(1, -4)$ min
c $(-\sqrt{3}, 6\sqrt{3} + 1)$ max; $(\sqrt{3}, -6\sqrt{3} + 1)$ min

6



7 $x = 1$ (inflection); $x = 2$ (minimum)

8 $y = -9x + 7$

9 $(0, 2)$

10 $x = \frac{-1}{2}$

11 a $f'(x) = \frac{4}{5}(x - 1)^{-\frac{1}{5}}, \{x \in \mathbb{R} : x \neq 1\}$
b $y = \frac{4}{5}x - \frac{3}{5}$ and $y = -\frac{4}{5}x + 1$ **c** $\left(1, \frac{1}{5}\right)$

12 a $64\pi\ \text{cm}^3/\text{cm}$ **b** $64\pi\ \text{cm}^3/\text{s}$

13 0.25 m/s, $0.25e$ m/s, $0.25e^2$ m/s, $0.25e^4$ m/s

14 a $(25e^{100t})°\text{C/s}$ **b** $(25e^5)°\text{C/s}$

15 $y = ex$

16 b 20 cm/year

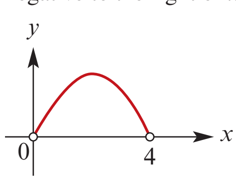
17 2

18 a $y = \frac{1}{e}x$ **b** $y = \frac{x}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} + \sqrt{2}$
c $y = x - \frac{3\pi}{2}$ **d** $y = \frac{-2}{\sqrt{e}}x - 1$

19 a $\left(\frac{8}{3}, \frac{-1024}{27}\right)$ **b** $\left(\frac{\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right)$
c $\left(2, \frac{1}{2} + \ln 2\right)$

Extended-response questions

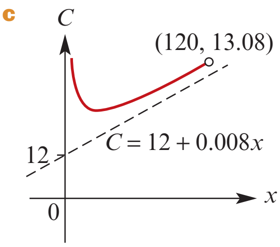
1 a $y = 4x - x^2$ **b** $0 < x < 4$ **c** $y = 4, x = 2$
d Gradient is positive to the left of $x = 2$, and negative to the right of $x = 2$
e



f $0 < y < 4$

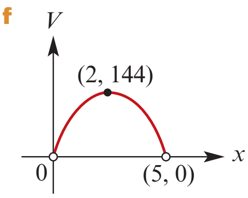
2 a $A = 4xy$ **b** $y = -\frac{2}{3}x + 8$
c $A = 32x - \frac{8}{3}x^2$ **d** $x = 6, y = 4$
e $96\ \text{m}^2$

3 a i \$12.68 **ii** \$12.74
b $C = 12 + 0.008x + \frac{14.40}{x}$
c



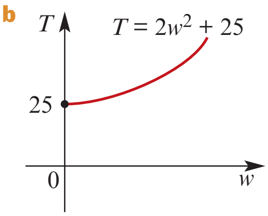
d 42.43 km/h

4 a $V = 4(x^3 - 13x^2 + 40x)$
b $0 < x < 5$ **c** $x = 2$
d 2 cm, 12 cm, 6 cm **e** $144\ \text{cm}^3$
f



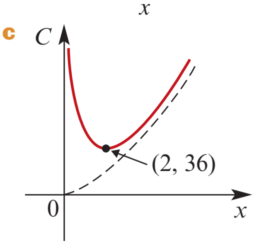
5 32

6 a $T = 2w^2 + 25$
b



c $A = \frac{25}{w} + 2w$
d i $\frac{5\sqrt{2}}{2} \approx 3.54$ kg **ii** $10\sqrt{2} \approx 14.14$ s

7 b $C = 3x^2 + \frac{48}{x}$
c



d i $x = 2, h = 3$; i.e. 2 m, 2 m, 3 m **ii** $36\ \text{m}^2$

8 10 m, 10 m, 5 m; Area $400\ \text{m}^2$

9 a $A = \frac{1}{2}a^2\theta$ **b** $A = \frac{1}{2}\left(\frac{100}{\theta + 2}\right)^2\theta$
c $\theta = 2$ **d** $625\ \text{cm}^2$

10 b i $r = \frac{L}{4}$ **ii** $\theta = 2$ **iii** Maximum

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11 b $\dfrac{dT}{dx} = \dfrac{x}{\sqrt{x^2 + 900}} - \dfrac{3}{5}$

c i $x = 22.5$ **ii** 71 seconds

d 63 seconds

12 a $y = ex$ **b** $y = 2ex$ **c** $y = kex$

e i $k = \dfrac{1}{e}$ or $k \leq 0$ **ii** $k > \dfrac{1}{e}$

13 b $T = \dfrac{20 + 16\sqrt{2}}{15} \approx 2.84$ hours

14 $t = 1.16$, 1.2 km apart

15 b $0 < x < 1$ **c** $x = \dfrac{1}{\sqrt{2}}$, $y = \pm 1$

d $A = 2\sqrt{2}$

16 c ii $\dfrac{dA}{dx} = -3x^2 - 2ax + a^2$

17 $t = 5$, $N(5) = \dfrac{120}{e}$

18 a $b = 5$, $c = 6$

b i 6 weeks **ii** 3.852 weeks

c $190\frac{2}{3}$ cm^2

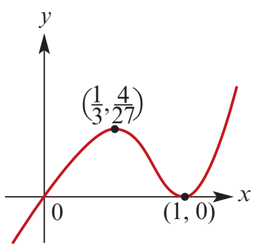
19 a $(1, -6)$ **b** $3(x - 1)^2 + 3$

c $3(x - 1)^2 + 3 > 3$ for all $x \in \mathbb{R} \setminus \{1\}$

20 a $a = 1$, $c = 1$, $b = -2$, $d = 0$

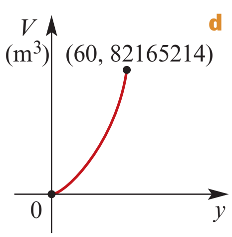
b $\left\{x : \dfrac{1}{3} < x < 1\right\}$

c



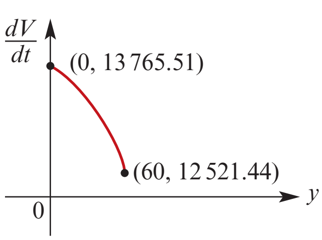
21 a 53 109 671 m^3 **b** $\dfrac{dV}{dy} = \pi(y + 630)^2$

c



d 82 165 214 m^3

e



22 a i $r = \dfrac{2\pi - \theta}{2\pi}$

ii $h = \sqrt{1 - \left(\dfrac{2\pi - \theta}{2\pi}\right)^2}$

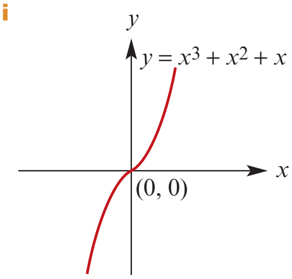
b $\dfrac{49\sqrt{15}\pi}{1536}$

c 0.3281, 2.5271

d i $\theta = 1.153$ **ii** $V_{max} = 0.403$ m^3

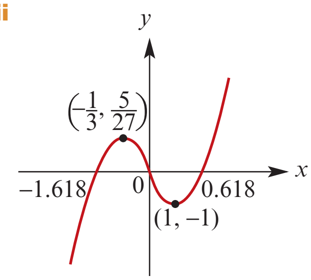
e 0.403 m^3

23 a i



No stationary points

ii



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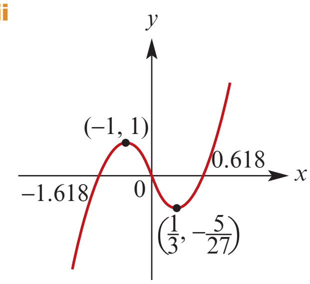

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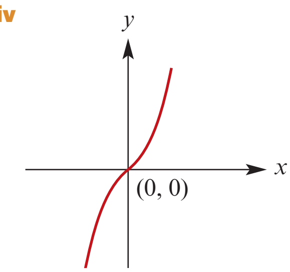

No stationary points

b i $f'(x) = 3x^2 + 2ax + b$

ii $x = \dfrac{-a \pm \sqrt{a^2 - 3b}}{3}$

c ii $a = -3$ or $a = 3$; $(-1, -1)$, $(1, 1)$ stationary points of inflection

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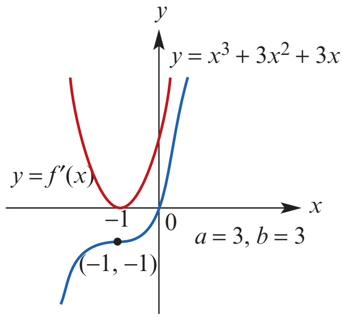
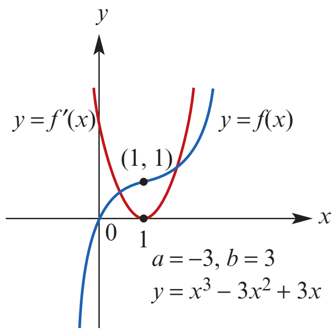
iii



iv

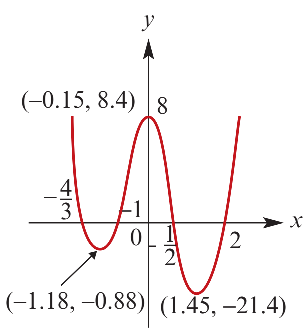


d $a^2 < 3b$

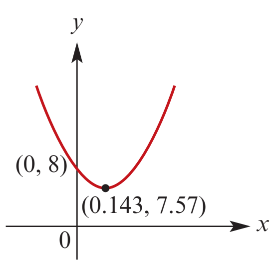
24 $x = e$

25 a i $a = -21$

ii



b i



ii Min at $(0.143, 7.57)$

iii $g'(x) = 24x^3 - 3x^2 + 42x - 6$

iv 0.1427

v $g'(0) = -6,\ g'(10) = 24\,114$

vi $g''(x) = 72x^2 - 6x + 42$

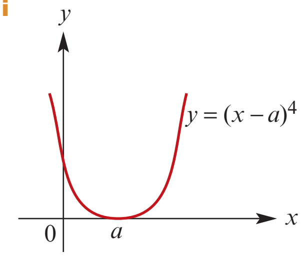
vii $g''(x) > 0$ for all x; thus $y = g'(x)$ has no turning points and crosses the x-axis only once

26 b i $x = a$ or $x = b$ or $x = \dfrac{b+a}{2}$

ii $x = a$ or $x = b$

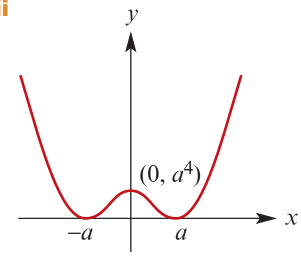
c $(a, 0),\ (b, 0),\ \left(\dfrac{a+b}{2}, \dfrac{(a-b)^4}{16}\right)$

e i



ii $(a, 0),\ (-a, 0),\ (0, a^4)$

iii



27 b i $x = a$ or $x = \dfrac{3b+a}{4}$

ii $x = a$ or $x = b$

c Local min at $\left(\dfrac{3b+a}{4}, \dfrac{-27}{256}(b-a)^4\right)$;
Stationary point of inflection at $(a, 0)$

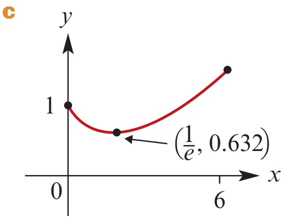
e $\left(-\dfrac{a}{2}, \dfrac{-27a^4}{16}\right)$ and $(a, 0)$

f i $b = -\dfrac{a}{3}$

28 a $f'(x) = \ln x + 1$

b $x = \dfrac{1}{e} \approx 0.37$, i.e. during the first month

c

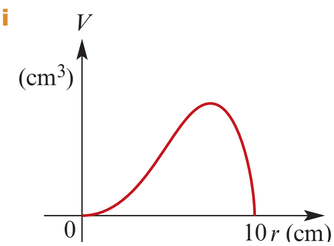


d When $x = 6$

29 a i $y = \sqrt{100 - r^2},\ \ h = 2\sqrt{100 - r^2}$

ii $V = 2\pi r^2\sqrt{100 - r^2}$

b i



ii $V = 2418.4,\ \ r = 8.165,\ \ h = 11.55$

iii $r = 6.456$ or $r = 9.297$

c i $\dfrac{dV}{dr} = \dfrac{400\pi r - 6\pi r^3}{\sqrt{100 - r^2}}$

ii $V_{max} = \dfrac{4000\pi\sqrt{3}}{9}$ when $r = \dfrac{10\sqrt{6}}{3}$

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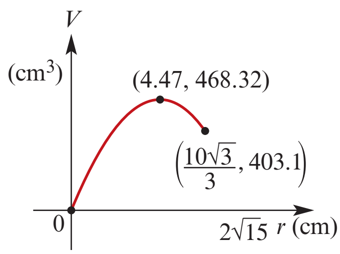
d **ii** $\dfrac{dV}{dr} > 0$ for $r \in \left(0, \dfrac{20\sqrt{6}}{6}\right)$

iii $\dfrac{dV}{dr}$ is increasing for $r \in (0, 5.21)$

30 **a** **i** $h = \dfrac{100 - 3r^2}{2r}$ **ii** $V = \dfrac{\pi r}{6}(300 - 5r^2)$

iii $0 < r < \dfrac{10\sqrt{3}}{3}$ **iv** $\dfrac{dV}{dr} = \dfrac{\pi}{6}(300 - 15r^2)$

v



31 **a** $a = \dfrac{1}{3}\ln\left(\dfrac{10}{3}\right)$

b **i** $x = 0$ and $x = \dfrac{5}{2}$

ii $x = \dfrac{-4 + 5a \pm \sqrt{25a^2 + 16}}{4a}$

32 **a** $f'(x) = \dfrac{3\,000\,000e^{-0.3x}}{(1 + 100e^{-0.3x})^2}$

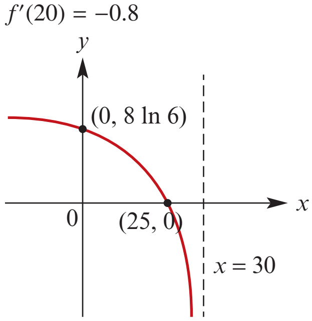
b **i** 294 kangaroos per year

ii 933 kangaroos per year

33 **a** $a = 30$ **b** $(0, 8\ln 6)$, $(25, 0)$

c $f'(20) = -0.8$

d



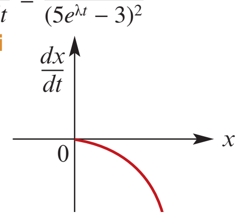
34 **b** $\left(\dfrac{\pi}{2}, e\right), \left(\dfrac{3\pi}{2}, \dfrac{1}{e}\right)$ **c** $\left[\dfrac{1}{e}, e\right]$

d Period 2π, since $g(x + 2\pi) = g(x)$

36 **a** **i** 30 g **ii** 12.28 g

b $\dfrac{dx}{dt} = \dfrac{-300\lambda e^{\lambda t}}{(5e^{\lambda t} - 3)^2}$

c **ii**



37 **b** $MP = \dfrac{2}{\tan\theta}$ **c** $NQ = 8\tan\theta$

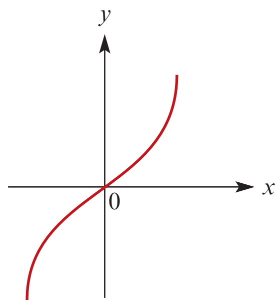
d $x = \dfrac{2}{\tan\theta} + 8\tan\theta + 10$

e $\dfrac{dx}{d\theta} = -2\operatorname{cosec}^2\theta + 8\sec^2\theta$

f $x = 18$, $\theta = 26.6°$

38 **a** $f'(x) = e^x + e^{-x}$ **b** $\{0\}$

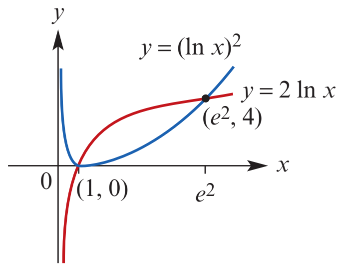
d



39 **a** $x = 1$ or $x = e^2$

b When $x = 1$, gradient of $y = 2\ln x$ is 2 and gradient of $y = (\ln x)^2$ is 0

c



d $\{x : 2\ln x > (\ln x)^2\} = (1, e^2)$

40 **a** $h = a(1 + \cos\theta)$ **b** $r = a\sin\theta$

d $\dfrac{dV}{d\theta} = \dfrac{\pi a^3}{3}[2\sin\theta\cos\theta(1 + \cos\theta) - \sin^3\theta]$

$\theta = \cos^{-1}\left(\dfrac{1}{3}\right) \approx 70.53°$

e $V = \dfrac{32\pi a^3}{81}$ cm^3

41 **b** $\dfrac{dy}{dt} = \dfrac{bAe^{bt}}{(1 + Ae^{bt})^2}$

e After 7 hours (to the nearest hour)

42 **a** $f'(x) = \dfrac{xe^x - e^x}{x^2}$ **b** $x = 1$

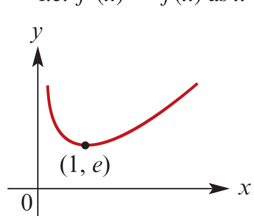
c $(1, e)$ minimum

d **i** $\dfrac{f'(x)}{f(x)} = \dfrac{x - 1}{x}$

ii $\lim\limits_{x \to \infty} \dfrac{f'(x)}{f(x)} = 1$

i.e. $f'(x) \to f(x)$ as $x \to \infty$

e



f $t = \dfrac{1}{k} \approx 45.27$ years, i.e. during 1945

43 **a** $A = 1000$, $k = \dfrac{1}{5}\ln 10 \approx 0.46$

b $\dfrac{dN}{dt} = kAe^{kt}$ **c** $\dfrac{dN}{dt} = kN$

d **i** $\dfrac{dN}{dt} \approx 2905.7$ **ii** $\dfrac{dN}{dt} \approx 4.61 \times 10^{12}$

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44 a $t \approx 34.66$ years **b** $t \approx 9.12$ years

45 a Max height 0.7 m first occurs at $t = \frac{1}{6}$ s

b $\frac{2}{3}$ **c** 0.6π m/s, 0.6π m/s, 0 m/s

46 a i $r = \frac{1}{6}$ **ii** $p = 12,\ q = 8$

b $T'(3) = -\frac{4\pi}{3}$, i.e. length of night decreasing by $\frac{4\pi}{3}$ hours/month; $T'(9) = \frac{4\pi}{3}$, i.e. length of night increasing by $\frac{4\pi}{3}$ hours/month

c $-\frac{8}{3}$ hours/month

d $t = 9$, i.e. after 9 months

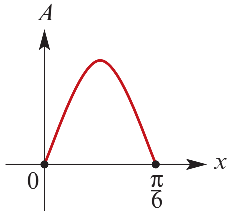
47 a $A = 2x\cos(3x)$

b i $\frac{dA}{dx} = 2\cos(3x) - 6x\sin(3x)$

ii When $x = 0$, $\frac{dA}{dx} = 2$;

When $x = \frac{\pi}{6}$, $\frac{dA}{dx} = -\pi$

c i



ii $x = 0.105$ or $x = 0.449$

iii Max area 0.374, occurs when $x = 0.287$

d ii (0.287, 1.162)

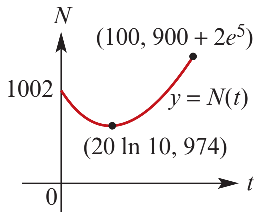
48 a i $N'(t) = -1 + \frac{1}{10}e^{\frac{t}{20}}$

ii Minimum population is 974, occurs when $t = 20\ln 10$

iii $N(0) = 1002$

iv $N(100) = 900 + 2e^5$

v



b i $N_2(0) = 1002$

ii $N_2(100) = 990 + 2e^{\frac{1}{2}}$

iv Minimum population is 974, occurs when $t = (20\ln 10)^2$

c ii Minimum population is 297, occurs when $t = 100.24$

d i $N_3'(t) = -\frac{3}{2}t^{\frac{1}{2}} + \frac{1}{10}e^{\frac{t}{20}}$

49 c $\frac{\pi}{180}\sec^2(\alpha°)$ metres

Chapter 6

Exercise 6A

1 a 3.81 square units **b** 1.34 square units **c** $\frac{35}{2}$ square units

2 a 13.2 square units **b** 10.2 square units

3 a 10 square units **b** 10.64 square units

4 a 0.72 square units **b** 2.88, decrease strip width

5 a 36.8 square units **b** 36.7 square units

6 11.9 square units

7 a ≈ 48 square units **b** Distance travelled

8 a $\frac{9}{2}$ **b** 9 **c** 4

Exercise 6B

1 a $\frac{x^4}{8} + c$ **b** $\frac{5}{4}x^4 - x^2 + c$

c $\frac{x^4}{5} - x^3 + c$ **d** $2z + \frac{5}{2}z^2 - z^3 + c$

2 a $y = -\frac{1}{2x^2} + c$ **b** $y = 3x^{\frac{4}{3}} + c$

c $y = \frac{4}{5}x^{\frac{5}{4}} + \frac{5}{2}x^{\frac{2}{5}} + c$

3 a $-\frac{3}{x} + c$ **b** $-\frac{2}{3x^3} + 3x^2 + c$

c $-\frac{2}{x} - \frac{3}{x^2} + c$ **d** $\frac{9}{4}x^{\frac{4}{3}} - \frac{20}{9}x^{\frac{9}{4}} + c$

e $\frac{12}{7}x^{\frac{7}{4}} - \frac{14}{3}x^{\frac{3}{2}} + c$ **f** $\frac{5}{2}x^{\frac{8}{5}} + \frac{9}{2}x^{\frac{8}{3}} + c$

4 a $y = x^2 - 3x + 3$ **b** $y = \frac{x^4}{4} + 6$

c $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 - \frac{22}{3}$

5 a $\frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$ **b** $\frac{3z^3 - 4}{2z} + c$

c $\frac{5}{3}x^3 + x^2 + c$ **d** $\frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + c$

e $\frac{2x^3}{3} + \frac{3x^5}{5} + c$ **f** $\frac{3}{7}x^{\frac{7}{3}} + \frac{3}{16}x^{\frac{16}{3}} + c$

6 $f(x) = x^3 + \frac{1}{x} - \frac{17}{2}$

7 $s = \frac{3}{2}t^2 + \frac{8}{t} - 8$

8 a $k = -32$ **b** $f(7) = 201$

Exercise 6C

1 a $\frac{1}{6}(2x - 1)^3 + c$ **b** $-\frac{1}{4}(t - 2)^4 + c$

c $\frac{1}{20}(5x - 2)^4 + c$ **d** $\frac{1}{24 - 16x} + c$

e $\frac{1}{8(6 - 4x)^2} + c$ **f** $\frac{-1}{8(3 + 4x)^2} + c$

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g $\frac{2}{9}(3x+6)^{\frac{3}{2}}+c$ **h** $\frac{2}{3}(3x+6)^{\frac{1}{2}}+c$

i $\frac{1}{9}(2x-4)^{\frac{9}{2}}+c$ **j** $\frac{1}{7}(3x+11)^{\frac{7}{3}}+c$

k $-\frac{2}{9}(2-3x)^{\frac{3}{2}}+c$ **l** $-\frac{1}{10}(5-2x)^5+c$

2 a $\frac{1}{2}\ln(x)+c$ **b** $\frac{1}{3}\ln(3x+2)+c$

c $\ln(1+4x)+c$ **d** $\frac{5}{3}\ln(3x-2)+c$

e $-\frac{3}{4}\ln(1-4x)+c$ **f** $-6\ln(x-4)+c$

3 a $y=\frac{1}{2}\ln(x)+1,\ x>0$

b $y=10-\ln(5-2x),\ x<\frac{5}{2}$

4 $y=10\ln(x-5)$

5 $y=3\ln\left(\frac{2-x}{2}\right)+10$

6 $y=\frac{5}{4}\ln\left(\frac{5}{1-2x}\right)+10$

7 $y=\frac{5}{4}\ln\left(\frac{1}{2x-1}\right)+10$

Exercise 6D

1 a $\frac{1}{6}e^{6x}+c$ **b** $\frac{1}{2}e^{2x}+\frac{3}{2}x^2+c$

c $-\frac{1}{3}e^{-3x}+x^2+c$ **d** $-\frac{1}{2}e^{-2x}+\frac{1}{2}e^{2x}+c$

2 a $\frac{1}{2}e^{2x}-2e^{\frac{x}{2}}+c$ **b** $e^x-e^{-x}+c$

c $\frac{2}{3}e^{3x}+e^{-x}+c$ **d** $15e^{\frac{x}{3}}-10e^{\frac{x}{5}}+c$

e $\frac{9}{2}e^{\frac{2x}{3}}-\frac{15}{7}e^{\frac{7x}{5}}+c$ **f** $\frac{15}{4}e^{\frac{4x}{3}}-\frac{9}{2}e^{\frac{2x}{3}}+c$

3 a $y=\frac{1}{2}(e^{2x}-x^2+9)$ **b** $y=-\frac{3}{e^x}-e^x+8$

4 $y=9-2e^{-2}$

5 a $k=2$ **b** $y=\frac{1}{2}e^{2x}+\frac{1}{2}e^2$

6 a $k=3$ **b** $y=-\frac{1}{3}e^{3x}-\frac{2}{3}e^3$

Exercise 6E

1 a $\frac{7}{3}$ **b** 20 **c** $-\frac{1}{4}$ **d** 9

e $\frac{1}{2}$ **f** $\frac{140}{3}$ **g** $15\frac{1}{3}$ **h** $343\frac{11}{20}$

2 a 10 **b** 1 **c** $\frac{13}{3}$ **d** $\frac{1}{3}$

e $\frac{10}{441}$ **f** 34 **g** $\frac{2}{3}\left(2^{\frac{3}{2}}-1\right)$

h $2-2^{\frac{1}{2}}$ **i** $\frac{1}{15}$

3 a $\frac{1}{2}(e^2-1)$ **b** $\frac{1}{2}(3-e^{-2})$

c $6e^{\frac{1}{3}}-4$ **d** e^2-e^{-2}

4 a 10 **b** 17 **c** -5 **d** 9 **e** -3

5 a $\ln\left(\frac{1}{3}\right)$ **b** $\frac{1}{2}\ln 5$ **c** $\frac{3}{2}\ln\left(\frac{19}{17}\right)$

Exercise 6F

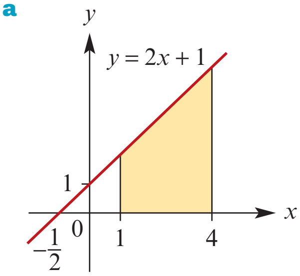
1 a 3

b 44

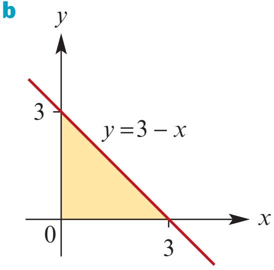
c i 8 **ii** 10

2 a $\frac{4}{3}$ **b** $\frac{1}{6}$ **c** $121\frac{1}{2}$ **d** $\frac{1}{6}$ **e** $4\sqrt{3}$ **f** 108

3 a



b



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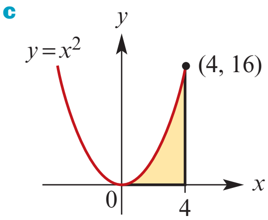

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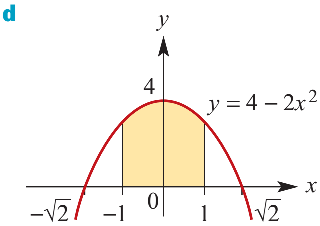

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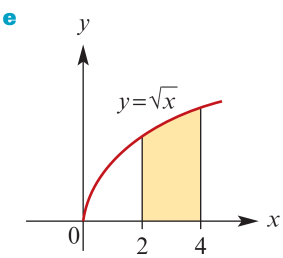

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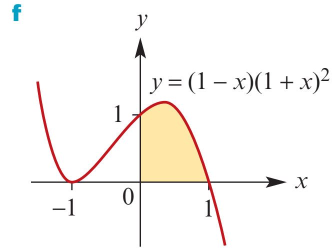
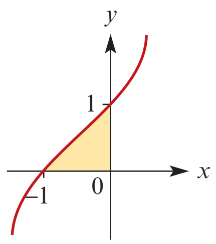
f



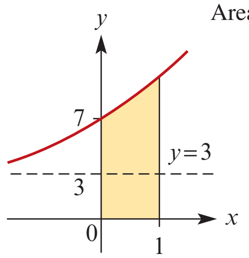
4 $\frac{321}{10}$ square units

5 Area $= \frac{3}{4}$ square units

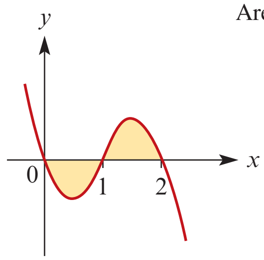


6 Area $= 2e^2 + 1$

≈ 15.78 square units



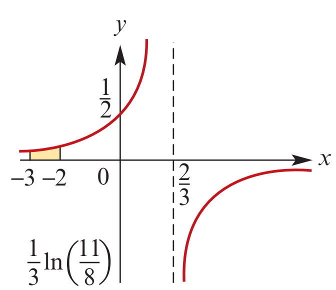
7 Area = 0.5 square units



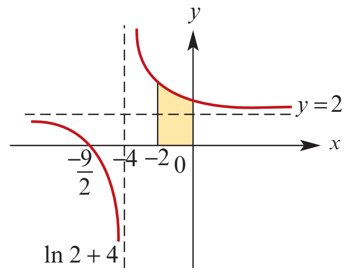
8 a $\frac{5}{6}$ square units **b** $8\frac{1}{6}$ square units

9 a $A(0, 3)$, $B(1, 0)$ **b** 2 square units

10



11



12 b Derivative: $(\ln a)e^{x \ln a} = (\ln a) \cdot a^x$

Antiderivative: $\frac{e^{x \ln a}}{\ln a} = \frac{a^x}{\ln a}$

Exercise 6G

1 a $\frac{1}{3}\sin(3x)$ **b** $-2\cos\left(\frac{1}{2}x\right)$ **c** $\sin(3x)$

d $-4\cos\left(\frac{1}{2}x\right)$ **e** $-\frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right)$

f $\frac{1}{3}\sin(3x) - \frac{1}{2}\cos(2x)$

g $\frac{1}{4}\sin(4x) + \frac{1}{4}\cos(4x)$

h $\frac{1}{4}\cos(2x) + \frac{1}{3}\sin(3x)$

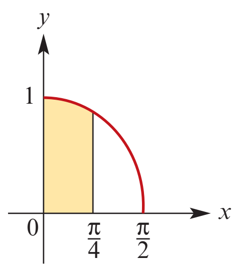
i $-\frac{1}{4}\sin\left(2x + \frac{\pi}{3}\right)$ **j** $-\frac{1}{\pi}\cos(\pi x)$

2 a $1 - \frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $1 + \frac{1}{\sqrt{2}}$ **d** 2 **e** 1

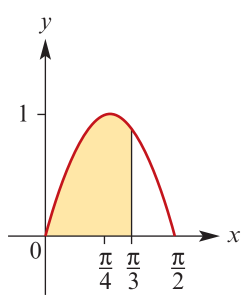
f $\frac{2}{3}$ **g** $-\frac{1}{2}$ **h** 4 **i** $\frac{1 - \sqrt{3}}{4}$ **j** −2

3 $-\sqrt{2} + 2$ square units

4 a $\int_0^{\frac{\pi}{4}} \cos x \, dx = \frac{1}{\sqrt{2}}$



b $\int_0^{\frac{\pi}{3}} \sin(2x) \, dx = \frac{3}{4}$



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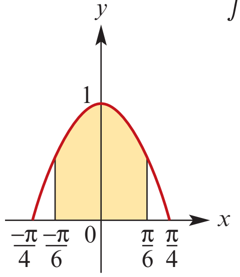
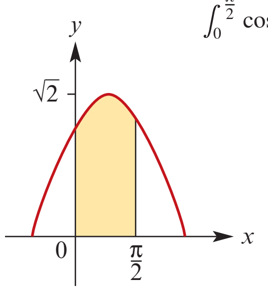
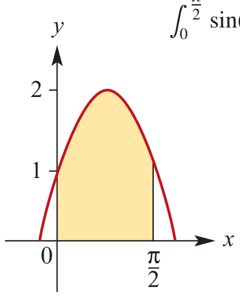
c $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos(2x)\, dx = \frac{\sqrt{3}}{2}$



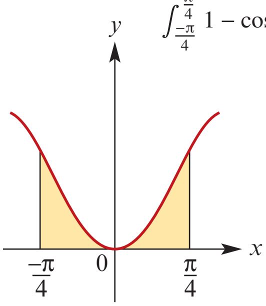
d $\int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta\, d\theta = 2$



e $\int_0^{\frac{\pi}{2}} \sin(2\theta) + 1\, d\theta = 1 + \frac{\pi}{2}$

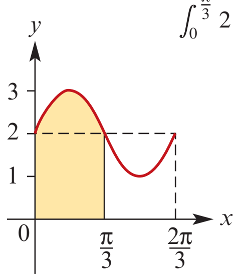


f $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1 - \cos(2\theta)\, d\theta = \frac{\pi}{2} - 1$



5 a $\frac{\sqrt{2}}{2}$ **b** $-\frac{1}{3}$ **c** $-\frac{\sqrt{3}}{3}$ **d** $-\frac{1}{\sqrt{2}}$

6 $\int_0^{\frac{\pi}{3}} 2 + \sin(3x)\, dx = \frac{2\pi}{3} + \frac{2}{3}$



Exercise 6H

1 a $4\frac{2}{3}$ **b** $2\frac{2}{3}$ **c** 12 **d** $\frac{5\sqrt{3}}{4} - 2$

e $\frac{e^4}{2} + 4\ln 2 - \frac{e^2}{2}$ **f** 4 **g** $\frac{5\pi^2}{8} + 1$

h $8\ln 2 + \frac{51}{4}$ **i** $\frac{1}{12}$

2 0.5 square units

3 $\frac{6x}{3x^2 + 7}$, $\frac{1}{6}\ln\left(\frac{19}{7}\right)$

4 a $\frac{1}{3}\ln(x^3 + 3)$ **b** $\frac{1}{2}\ln(x^2 + 4x)$

c $\frac{1}{2}\ln(3 + e^{2x})$ **d** $\frac{1}{3}\ln(x^3 + 3x)$

e $\frac{5}{3}\ln(3x - 2)$

5 a $\frac{1}{\cos^2 x}$, $\tan x$ **b** $-\frac{2}{\sin^2(2x)}$, $-\frac{\cos(2x)}{2\sin(2x)}$

c $\sin(x) + x\cos(x)$, $-1 + \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$

6 a $1 + \ln(2x)$, $-x + \ln(2x)$

b $x + 2x\ln(2x)$, $\frac{1}{2}x^2\ln(2x) - \frac{x^2}{4}$

c $1 + \frac{x}{\sqrt{1 + x^2}}$, $\ln(1 + \sqrt{2})$

7 $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$, $2e^{\sqrt{2}} - 2e$

8 $6\sin^2(2x)\cos(2x)$, $\frac{1}{6}$

9 a 139.68 **b** 18.50 **c** −0.66

10 b $5\ln 3 + 4$

11 b $5 + 6\ln 2$

12 $f(x) = 1 - 2\cos\left(\frac{1}{2}x\right)$

13 a $f(x) = \frac{1}{2}\sin(2x) + 1$

b $f(x) = 3\ln x + 6$ **c** $f(x) = 2e^{\frac{x}{2}} - 1$

14 $\sin(3x) + 3x\cos(3x)$

Hence $\int_0^{\frac{\pi}{6}} x\cos(3x)\, dx = \frac{\pi}{18} - \frac{1}{9}$

15 $a = 1$, $b = -2$; Area $= \frac{12\sqrt{3} - \pi - 12}{3\pi}$ square units

16 a 1.450 square units **b** 1.716 square units

17 0.1345

18 $f(x) = \frac{1}{2}\left(x^2 - \cos(2x) + 3\right)$

19 a $(x^2 + 1)^3 + c$ **b** $\sin(x^2) + c$

c $(x^2 + 1)^3 + \sin(x^2) + c$

d $-(x^2 + 1)^3 + c$ **e** $(x^2 + 1)^3 - 4x + c$

f $3\sin(x^2) + c$

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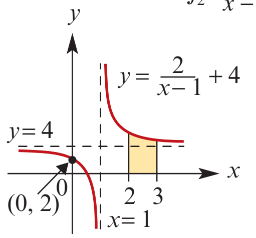
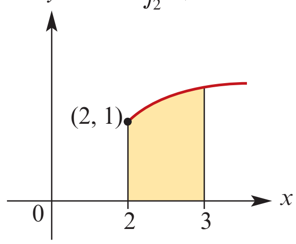
20 $\int_2^3 \frac{2}{x-1} + 4\,dx = 2\ln 2 + 4$



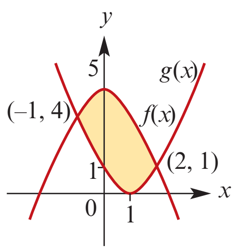
21 $\int_2^3 \sqrt{2x-4} + 1\,dx = \frac{1}{3} \times 2^{\frac{3}{2}} + 1$



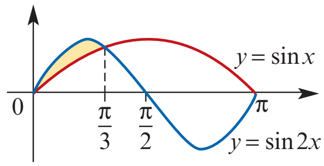
22 a $\frac{4\sqrt{2}}{3} - \frac{2}{3}$ **b** $\frac{2^{\frac{5}{2}}}{3}$ **c** $\frac{1}{3}\ln 4$
d $\frac{1}{2}\ln 3 + 3$ **e** $-2\frac{2}{3}$ **f** $2\sqrt{2} - 2$

Exercise 6I

1 36 square units
2 Area = 9 square units



3 a 36 square units **b** $20\frac{5}{6}$ square units
c 4 square units **d** $4\frac{1}{2}$ square units
e $4\frac{1}{2}$ square units
4 a 2 square units
b $e + e^{-1} - 2 \approx 1.086$ square units
5 3.699 square units
6 Area = $\frac{1}{4}$ square unit



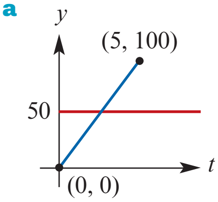
7 $\frac{1}{2}$ square unit
8 $P(\ln 3, 3)$; Area ≈ 2.197 square units

Exercise 6J

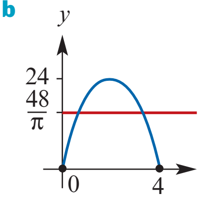
1 a $\frac{2}{3}$ **b** $\frac{2}{\pi}$ **c** $\frac{2}{\pi}$ **d** 0 **e** $\frac{1}{2}(e^2 - e^{-2})$

2 $10(e^5 - 1)e^{-5} \approx 9.93$°C

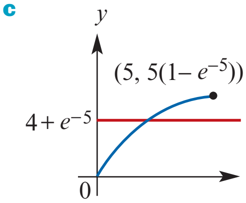
3 a



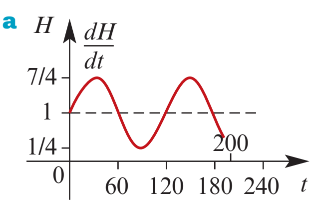
b



c



4 $\frac{147}{10}$ m/s
5 $\frac{a^2}{6}$
6 a $3000(2 - 2^{0.9})$ N/m^2
b $1000(4^{0.1} - 1)$ N/m^2
7 a $x = t^2 - 3t$ **b** $x = 0$ **c** 0 m/s
d $\frac{9}{2}$ m **e** $\frac{3}{2}$ m/s
8 a $x = \frac{2t^3}{3} - 4t^2 + 6t + 4$, $a = 4t - 8$
b When $t = 1$, $x = \frac{20}{3}$ m;
When $t = 3$, $x = 4$ m
c When $t = 1$, $a = -4$ m/s^2;
When $t = 3$, $a = 4$ m/s^2
9 Initial position is 3 m to the left of O
10 Velocity = 73 m/s; Position = $\frac{646}{3}$ m
11 a Velocity = $-10t + 25$
b Height = $-5t^2 + 25t$ **c** $\frac{5}{2}$ s
d $\frac{125}{4}$ m **e** 5 s
12 a



b $t \in (10, 50) \cup (130, 170)$
c $t = 30$ or $t = 150$
d **i** 120 kilojoules
ii 221.48 kilojoules
13 a When $t = 0$, 1000 million litres per hour;
When $t = 2$, 896 million litres per hour
b **i** $t = 0$ and $t = 15$
ii 1000 million litres per hour

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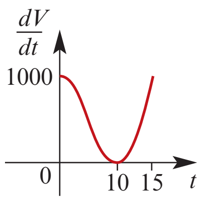
c

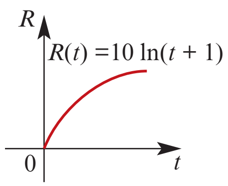


d **i** 5000

ii 5000 million litres flowed out in the first 10 hours

14 a When $t = 5$, ≈ 17.9 penguins per year;
When $t = 10$, ≈ 23.97 penguins per year;
When $t = 100$, ≈ 46.15 penguins per year

b



c 3661; the growth in the penguin population over 100 years

15 $71\,466\frac{2}{3}$ m^3

16 1.26 m

17 a 6 metres

b 18π m^2

c **i** $y - 3 + 3\cos\left(\frac{a}{3}\right) = \dfrac{-1}{\sin\left(\frac{a}{3}\right)}(x - a)$

ii 5.409

18 a i 9 **ii** $\dfrac{3(\sqrt{2}+2)}{2}$ **iii** 12

b Max value is 12; Min value is 0.834

c $\dfrac{48(\pi + 1)}{\pi}$ litres

Exercise 6K

1 a $\ln x$ **b** $-\ln x$ **c** $e^x - 1$ **d** $1 - \cos x$

Chapter 6 review

Short-answer questions

1 a $\frac{65}{4}$ **b** 0 **c** $\frac{-5a^{\frac{3}{2}}}{3}$ **d** $-\frac{55}{3}$

e $\frac{1}{2}$ **f** 1 **g** 0 **h** 0

2 $\frac{23}{2}$ **3** 3 **4** 4 **5** 820

6 $\frac{85}{4}$ **7** $\frac{5}{3}$ **8** $\frac{5}{3}$

9 $\int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx + \int_c^d f(x) - g(x)\,dx$

10 a $P(3, 9)$, $Q(7.5, 0)$ **b** 29.25 square units

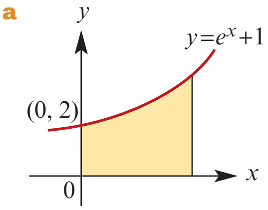
11 a 5 **b** $p = \frac{20}{7}$

12 3.45 square units

13 a $A(0, 6)$, $B(5, 5)$ **b** $15\frac{1}{6}$ square units

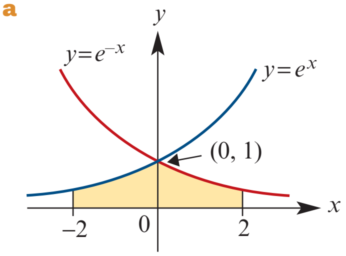
c $\frac{125}{6}$ square units

14 a



b $e^2 + 1 \approx 8.39$

15 a



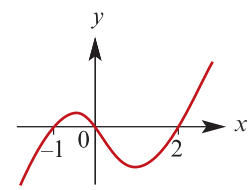
b $2 - 2e^{-2}$

16 a $e - 1 \approx 1.72$

b $2(e - 1) \approx 3.44$ square units

17 $2 + e^2 \approx 9.39$ square units

18 $3\frac{1}{12}$ square units



19 a $3 - e^{-2}$ **b** $\ln\left(\frac{2}{3}\right) - \frac{3}{2}$ **c** $\frac{\pi^2}{8} + 1$

d $\frac{1}{2}\left(\ln\left(\frac{5}{6}\right) - \frac{2}{e^4} + \frac{2}{e^5}\right)$

Extended-response questions

1 a $4y - 5x = -3$ **b** $\left(\frac{3}{5}, 0\right)$

c $(1, 0)$ **d** $\frac{9}{40}$ **e** $9 : 49$

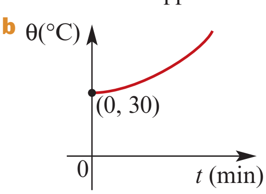
3 a $\frac{1}{3}$ square units

d $1 - \frac{n-1}{n+1} = \frac{2}{n+1}$ square units

e $\frac{9}{11}, \frac{99}{101}, \frac{999}{1001}$

f Area between the curves approaches 1

4 a 968.3° **b**



c 2.7 minutes **d** 64.5°C/min

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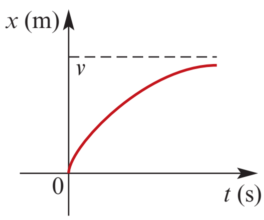
5 a 5×10^4 m/s

b Magnitude of velocity becomes very small

c $5 \times 10^4(1 - e^{-20})$ m **d** $x = v(1 - e^{-t})$

e



6 a $\frac{d}{dx}\left(e^{-3x}\sin(2x)\right) =$
$-3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)$

$\frac{d}{dx}\left(e^{-3x}\cos(2x)\right) =$
$-3e^{-3x}\cos(2x) - 2e^{-3x}\sin(2x)$

c $\int e^{-3x}\sin(2x)\,dx =$
$\frac{-1}{13}\left(3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)\right)$

7 a i $\tan a = \frac{4}{3}$ **ii** $\sin a = \frac{4}{5}$, $\cos a = \frac{3}{5}$

b 2 square units

8 a $\frac{dy}{dx} = -\frac{x}{10}e^{\frac{x}{10}}$, $\frac{dy}{dx} = -x(100 - x^2)^{-\frac{1}{2}}$

b When $x = 0$, $\frac{dy}{dx} = 0$ for both functions

c $-e$

d 6.71 square units

e 8.55%

f $(25\pi - 50)$ square units

g i $10(10e - 20)$

ii $(25\pi - 100e + 200)$ square units

9 a $\frac{dy}{dx} = \ln x + 1$, $\int_1^e \ln x\,dx = 1$

b $\frac{dy}{dx} = (\ln x)^n + n(\ln x)^{n-1}$

d $\int_1^e (\ln x)^3\,dx = 6 - 2e$

10 $s = \sqrt[3]{a^2b}$, $r = \sqrt[3]{ab^2}$

11 a i $R(0) = 0$ **ii** $R(3) = 0$

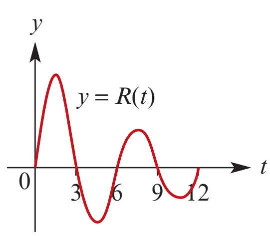
b $R'(t) = e^{-\frac{t}{10}}\left(\frac{10\pi}{3}\cos\left(\frac{\pi t}{3}\right) - \sin\left(\frac{\pi t}{3}\right)\right)$

c i 1.41, 4.41, 7.41, 10.41

ii Local max: (1.41, 8.65), (7.41, 4.75);
Local min: (4.41, −6.41) (10.41, −3.52)

d $t = 0, 3, 6, 9$ or 12

e



f i 16.47 litres **ii** 12.20 litres **iii** 8.27 litres

g 12.99 litres

12 b $1 - \frac{\pi}{4}$

Chapter 7

Short-answer questions

1 a $\frac{-2x^5 + 4x^3 - 2x}{(x^4 - 1)^2}$ **b** $\{0\}$

2 $4(6x - 4)(3x^2 - 4x)^3$ **3** $2x\ln(2x) + x$

4 a $b = \frac{1}{2}$ **b** $k = (2b - 1)e^{2b+1}$

5 $m = \frac{1}{12}$, $a = -\frac{22}{3}$, $c = -\frac{28}{3}$

6 $\frac{1}{6}\ln 7$

7 a $\frac{3}{5}\ln(5x - 2)$ **b** $\frac{3}{10 - 25x}$

8 a -7 **b** -14 **c** -20

9 a $5\frac{1}{2}$ **b** $\frac{1}{8}$

10 $2x\sqrt{3x^2 + 1}$

11 a $4x - 3$ **b** -3 **c** $\{1\}$

12 $\frac{f'(x)}{f(x)}$ **13** $a = \frac{145}{144}$

14 $m = \frac{1}{4}(-3 + \sqrt{105})$

15 a $(0, -4)$ and $(-2, 0)$ **b** 0 **c** 4 **d** $9\frac{1}{2}$

16 $f'(2) = -\frac{3}{25}$, $f''(2) = \frac{18}{125}$ **18** 24π

19 $\left(\frac{5}{18}, \frac{5}{36}\right)$ **20** $\frac{2}{3}$ **21** $-\frac{8x}{(x^2 - 2)^2}$

22 6 **23** $3528(5 - 7x)^7$

24 $\frac{1}{9}$ **25** $\frac{2}{3}$ **26** -70

27 0 **28** -1

29 a $-\frac{2}{(2x + 1)^2}$ **b** -2

30 a $x = 0$ or $x = -2$ **b** $x > 0$ or $x < -2$

c $-2 < x < 0$

31 $x > 1$ **32** $x = \pm\frac{2}{\sqrt{3}}$

33 a $\frac{1}{(1 - x)^2}$ **b** $(y + 1)^2$

34 $-3x(x^2 + 1)^{-\frac{5}{2}}$

36 $f'(x) = 10x^4 > 0$ for all $\{x \in \mathbb{R} : x \neq 0\}$. If $b > 0$, then $b^5 > 0$, and if $b < 0$, then $b^5 < 0$.

37 a $4 - 2\sqrt{2}$ **b** $2(e^{\frac{3}{4}} - 1)$ **c** $\frac{1}{2}\ln 2$

d $-\frac{1}{2}\ln 2$ **e** $\frac{1}{4}$ **f** $\frac{1}{20}$

39 a $\frac{1}{m}\left(2 - \ln\left(\frac{4}{m}\right)\right)$ **b** $0 < m < 4e^{-2}$

40 $\delta V \approx 30\text{ cm}^3$

41 2%

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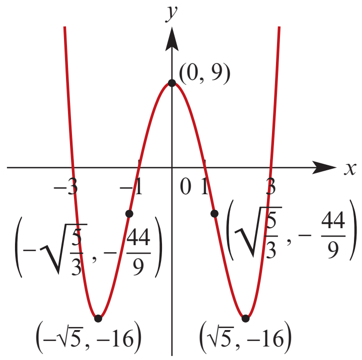
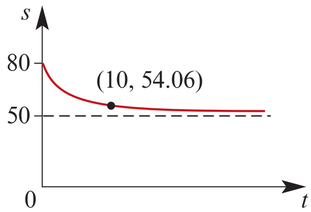
42



Extended-response questions

1 a 54.06 g **b**

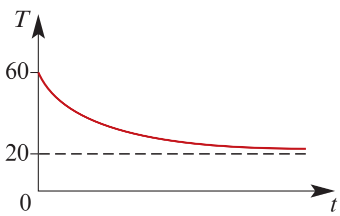


c $\frac{ds}{dt} = -6e^{-\frac{1}{5}t}$ **d** $\frac{ds}{dt} = -\frac{1}{5}(s - 50)$

e 0.8 g/L **f** 17 seconds

2 a 60°C

b

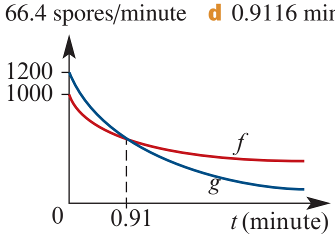


c $\frac{dT}{dt} = -14.4e^{-0.36t}$ **d** $\frac{dT}{dt} = -0.36(T - 20)$

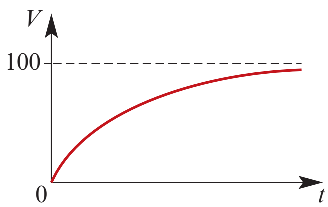
3 a 1.386 minutes **b** 2200, 5.38%

c 66.4 spores/minute **d** 0.9116 minutes

e



4 a



b i $20e^{-0.2t}$ m/s^2 **ii** $20 - \frac{V}{5}$ m/s^2

c 8.05 seconds

5 100

6 b $k = 0.028$ **c** 0.846°C/min

7 a i 0.1155 **ii** 0.2

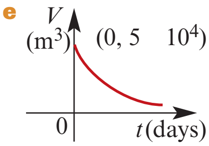
b 13.86 days

8 $600

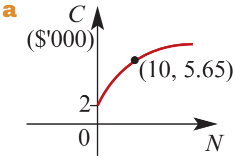
9 a 5×10^4 m^3 **b** $-12\,500$ m^3/day

c -3500 m^3/day **d** After 222.61 days

e



10 a

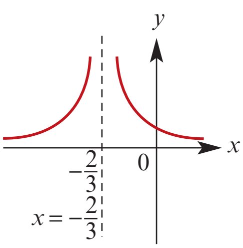


b $\frac{3N^2}{4(N^3 + 16)^{\frac{3}{4}}}$

c Rate of change of cost in $1000s with respect to the increase in the number of bottle tops produced

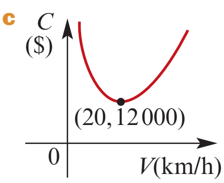
11 $p = 4$; Number of items sold = 50

12 $a = \pm 3$, $b = \pm 2$



13 a $17 000 **b** $C = \frac{160\,000}{V} + 10V^2$

c



d $V = 20$, $C = 12\,000$ **e** $12 560

14 a $\frac{60}{\sqrt{39}} \approx 9.61$ km **b** $\frac{60}{\sqrt{39}} \approx 9.61$ km

15 $4 - \sqrt{\frac{12}{7}} \approx 2.7$ m

16 a $\{x \in \mathbb{R} : x > 1\}$ **b** $\{x \in \mathbb{R} : 0 < x < 2\}$

c $\{x \in \mathbb{R} : x > 1\}$, $\left\{x \in \mathbb{R} : 0 < x < \frac{3}{2}\right\}$

d $\{x \in \mathbb{R} : x > 1\}$, $\left\{x \in \mathbb{R} : 0 < x < \frac{n+1}{n}\right\}$

17 a $(1, 1)$, $(-1, -1)$ **b** $\left(\pm 2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\right)$

c $\left(\pm n^{\frac{1}{2n+2}}, n^{-\frac{n}{2n+2}}\right)$

18 a $A = 48 + 16x + \frac{108}{x}$

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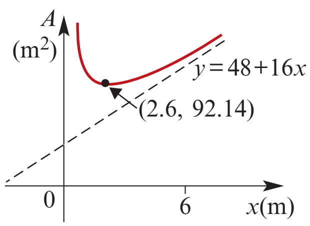
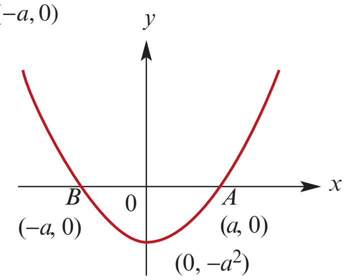
b



c Height = $\frac{3\sqrt{3}}{2}$ m, width = $\frac{4\sqrt{3}}{3}$ m

d $172\ \text{m}^2$

19 a $(a, 0), (-a, 0)$



b $\frac{4a^3}{3}$ **c** $\frac{2a^2}{3}$ **d** 2 : 3

20 a $-5\frac{1}{3}$ **c** $a = 1$ or $a = -2$

21 a **i** $50e^{-1}$ litres/minute
ii $t = 5$
iii 2 minutes 18 seconds
iv 3 minutes 48 seconds
b 14.74 litres
c 53 seconds

Chapter 8

Exercise 8A

1 1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T

2 HH1, HH2, HH3, HH4, HH5, HH6,
HT1, HT2, HT3, HT4, HT5, HT6,
TH1, TH2, TH3, TH4, TH5, TH6,
TT1, TT2, TT3, TT4, TT5, TT6

3 a $\frac{1}{13}$ **b** $\frac{3}{4}$ **c** $\frac{4}{13}$ **d** $\frac{2}{13}$

4 a $\frac{1}{2}$ **b** $\frac{2}{3}$

5 0.8

6 0.65

7 a 0.067 **b** 0.047

8 5%

9 $\frac{6}{7}$

10 a $\frac{17}{500}$ **b** $\frac{9}{250}$ **c** $\frac{41}{125}$ **d** $\frac{41}{500}$

11 a $\frac{13}{20}$ **b** $\frac{7}{20}$

12 a $\frac{57}{100}$ **b** $\frac{2}{19}$ **c** $\frac{27}{100}$ **d** $\frac{1}{250}$

13 $\frac{9}{25}$

14 a $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $\frac{5}{6}$

15 a 0.13 **b** 0.32

16 a 0.40 **b** 0.67 **c** 0.18

17 a 0.35 **b** 0.18 **c** 0.12 **d** 0.17

Exercise 8B

1 a 0.2 **b** 0.675 **c** 0.275

2 a $\frac{1}{6}$ **b** $\frac{1}{3}$

3 a 0.06 **b** $\frac{1}{5}$

4 $\frac{3}{5}$ **5** $\frac{24}{59}$

6 a $\frac{1}{2}$ **b** $\frac{77}{200}$ **c** $\frac{40}{77}$ **d** 0.4

7 a $\frac{65}{224}$ **b** $\frac{115}{448}$ **c** $\frac{21}{65}$ **d** $\frac{61}{246}$

8 a 0.24 **b** 0.86

9 a Yes **b** No **c** No

10 a 0.5 **b** 0.2 **c** 0.7

11 0.39

12 $\frac{1}{7}$

13 0.0479

14 a 0.486 **b** 0.012 **c** 0.138

15 a $\frac{2}{5}$ **b** $\frac{1}{15}$ **c** $\frac{8}{15}$

16 a $\frac{2}{5}$ **b** $\frac{7}{40}$ **c** $\frac{7}{16}$ **d** $\frac{7}{15}$

17 a $\frac{5}{14}$ **b** $\frac{3}{5}$

18 a 0.735 **b** 0.453

19 $\frac{3}{44}$

Exercise 8C

1 a Discrete **b** Not discrete
c Discrete **d** Discrete

2 a Not discrete **b** Discrete
c Not discrete **d** Discrete

3 a $\{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$

b

x	Outcomes
0	TTT
1	HTT, THT, TTH
2	THH, HTH, HHT
3	HHH

c $\frac{1}{2}$

4 a Yes, as the sum of the probabilities is 1 and $p(x) \geq 0$ for all x
b 0.8

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5 a

x	0	1	2	3
$p(x)$	$\frac{125}{729}$	$\frac{300}{729}$	$\frac{240}{729}$	$\frac{64}{729}$

b $\frac{604}{729}$ **c** $\frac{304}{729}$

6 a $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b $Y = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

2nd die

1st die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

c i $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{5}$ **iv** $\frac{7}{10}$ **v** $\frac{1}{5}$ **vi** $\frac{2}{7}$

7 a

2nd die

1st die	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

b 1, 2, 3, 4, 5, 6 **c** 0.19

8 a 0.288 **b** 0.064 **c** 0.352 **d** 0.182

9 a $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b $\mathrm{P}(A) = \frac{1}{6}$, $\mathrm{P}(B) = \frac{1}{6}$, $\mathrm{P}(C) = \frac{5}{12}$, $\mathrm{P}(D) = \frac{1}{6}$

c $\mathrm{P}(A \mid B) = \frac{1}{6}$, $\mathrm{P}(A \mid C) = \frac{1}{5}$, $\mathrm{P}(A \mid D) = \frac{1}{6}$

d i Independent **ii** Not independent

iii Independent

10 a Yes **b** 0.5

11 a and **c**

12 a

x	0	1	2	3
$p(x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

b

x	0	1	2	3
$p(x)$	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

13

x	0	1	2
$p(x)$	0.36	0.48	0.16

14 a

x	1	2	3	4	5
$p(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

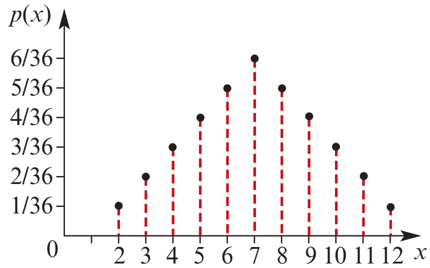
b $\frac{3}{5}$ **c** $\frac{1}{3}$

15 a $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c



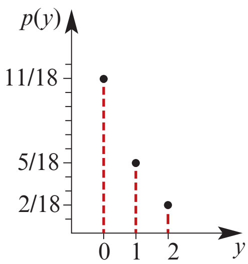
d $\frac{5}{18}$ **e** $\frac{7}{10}$

16 a $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b

y	0	1	2
$p(y)$	$\frac{11}{18}$	$\frac{5}{18}$	$\frac{2}{18}$

c



17 a

x	0	1	2
$p(x)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

b $\frac{7}{15}$

18 a

x	10	20	100
$p(x)$	$\frac{3}{4}$	$\frac{6}{25}$	$\frac{1}{100}$

b

y	20	30	40	110	120	200
$p(y)$	$\frac{9}{16}$	$\frac{9}{25}$	$\frac{36}{625}$	$\frac{3}{200}$	$\frac{3}{625}$	$\frac{1}{10\,000}$

19 a $\frac{1}{4}$

b $\{EENE, ENEE, ENNN, NEEE, NENN, NNEN\}$, $\mathrm{P}(X = 4) = \frac{3}{8}$

c $\mathrm{P}(X = 5) = \frac{3}{8}$

Exercise 8D

1 \$60

2 a $\mathrm{E}(X) = 4.6$ **b** $\mathrm{E}(X) = 0.5$

c $\mathrm{E}(X) = 2.38$ **d** $\mathrm{E}(X) = 0.569$

e $\mathrm{E}(X) = 7$ **f** $\mathrm{E}(X) = 0$

3 Expected profit = \$3000

4 A loss of 17c

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5 a

x	0	1	2	3
$P(X = x)$	0.12	0.36	0.38	0.14

b $E(X) = 1.54$

6

x	1	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

$E(X) = \frac{49}{12}$

7 a $E(X) = 4.11$ **b** $E(X^3) = 78.57$ **c** $E(5X - 4) = 16.55$ **d** $E\left(\frac{1}{X}\right) = 0.255$

8 \$5940

9 a $p = \frac{1}{16}$ **b** $E(X) = 2$ **c** $\text{Var}(X) = 3.5$

10 a $k = \frac{1}{21}$ **b** $E(X) = \frac{91}{21}$ **c** $\text{Var}(X) = \frac{20}{9}$

11 a

x	1	2	3	4	6	8	9	12	16
$p(x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

b i $\frac{1}{4}$ **ii** $\frac{25}{4}$ **iii** $\frac{275}{16}$

12 a $\frac{21}{4}$ **b** $\frac{7}{12}$ **c** $\frac{497}{48}$

13 a $\text{Var}(2X) = 64$ **b** $\text{Var}(X + 2) = 16$ **c** $\text{Var}(1 - X) = 16$ **d** $\text{sd}(3X) = 12$

14 a $c = 0.35$ **b** $E(X) = 2.3$ **c** $\text{Var}(X) = 1.61$, $\text{sd}(X) = 1.27$ **d** 0.95

15 a $k = \frac{1}{15}$ **b** $E(X) = 3.667$ **c** $\text{Var}(X) = 1.556$ **d** 0.9333

16 a 7 **b** 5.83 **c** 0.944

17 a 3 **b** 1.5 **c** 0.9688

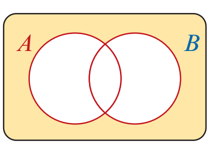
18 $c_1 = 40$, $c_2 = 60$

Chapter 8 review

Short-answer questions

1 Yes, as $P(A \cap B) = 0$

2



$P(A' \cap B') = P(A \cup B)'$

3 a $\frac{40}{81}$ **b** $\frac{5}{9}$

4 0.4

6 a 0.1 **b** 1.3 **c** 2.01

7 a 21.5 **b** $\frac{53}{256}$ **c** 630.75 **d** $\frac{29\sqrt{3}}{2}$

8 a

p	$x - 2$	-2
$P(P = p)$	$\frac{4}{5}$	$\frac{1}{5}$

b $\frac{4}{5}x - 2$

c $x >$ \$2.50

9 a 0.47 **b** $\frac{47}{70}$

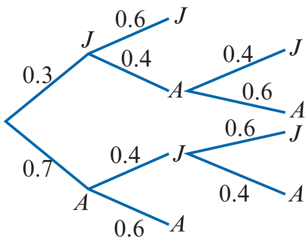
10 21.5%

11 a $\frac{1}{24}$ **b** $\frac{17}{24}$ **c** $\frac{5}{6}$ **d** $\frac{11}{18}$

Extended-response questions

1 a 0.1 **b** 0.2 **c** 4

2 a



b i 0.396 **ii** 0.604

c i

x	2	3
$P(X = x)$	0.6	0.4

ii 2.4

d 0.46

3 \$14

4 a 0.5 **b** 0.05 **c** 0.033 **d** $\frac{25}{33}$

5 a i 1.21
ii $\text{Var}(P) = 1.6659$, $\text{sd}(P) = 1.2907$
iii 0.94

b i

t	1	0.40	0
$p(t)$	0.39	0.27	0.34

ii $E(T) = 0.498 \approx 0.50$ **iii** 1

6 a $E(Y) = 2.002$
b $\text{Var}(Y) = 2.014$, $\text{sd}(Y) = 1.419$
c i

b	0	100	200
$p(b)$	0.677	0.270	0.053

ii $E(B) =$ \$37.60

7 a $\mu = 13.5\%$, $\sigma = 16.2\%$ **b** 0.95
c $E(G) = 6.9\%$, $\text{sd}(G) = 9.726\%$

8 Yes

9 \$1.00

10 a i 0.65 **ii** 0.2275 **iii** 0.079 625
iv 0.042 875
b Expected cost: \$8 439 375
c Expected profit: \$10 703 125

11 b $x = \frac{1}{2}, \frac{49}{288}$

12 a i $\frac{1}{81}$ **ii** $\frac{8}{81}$ **iii** $\frac{4}{81}$ **iv** $\frac{4}{27}$ **v** $\frac{56}{81}$
b 4.4197 cents (as the lowest value coin is 5c, he can settle for that)

13 a i $E(X) = \frac{7}{2}$ **ii** $\text{Var}(X) = \frac{35}{12}$
b i $E(X) = \frac{n+1}{2}$ **ii** $\text{Var}(X) = \frac{n^2 - 1}{12}$
c i $\frac{1}{10}$ **ii** $\frac{2}{5}$ **iii** $\frac{11}{2}$ **iv** $\frac{33}{4}$

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Chapter 9

Exercise 9A

1 **a** and **b**

2 0.2734

3 **a** 0.0256 **b** 0.0016

4 **a** 0.0778 **b** 0.2304 **c** 0.010 24

5 **a** $P(X = x) = \binom{3}{x}(0.5)^x(0.5)^{3-x}$, $x = 0, 1, 2, 3$
b 0.375

6 **a** $P(X = x) = \binom{6}{x}(0.48)^x(0.52)^{6-x}$, $x = 0, 1, 2, 3, \ldots, 6$
b 0.2527

7 **a** 0.0536 **b** 0.0087 **c** 0.2632

8 **a** $P(X = x) = \binom{10}{x}(0.1)^x(0.9)^{10-x}$, $x = 0, 1, 2, 3, \ldots, 10$
b **i** 0.3487 **ii** 0.6513

9 **a** $P(X = x) = \binom{11}{x}(0.2)^x(0.8)^{11-x}$, $x = 0, 1, 2, 3, \ldots, 11$
b **i** 0.2953 **ii** 0.0859 **iii** 0.9141

10 **a** $P(X = x) = \binom{7}{x}(0.2)^x(0.8)^{7-x}$, $x = 0, 1, 2, 3, \ldots, 7$
b **i** 0.000 013 **ii** 0.2097 **iii** 0.3899

11 0.624

12 **a** $\left(\frac{x}{100}\right)^6$ **b** $\frac{6x^5(100-x)}{100^6}$
c $\frac{x^6}{100^6} + \frac{6x^5(100-x)}{100^6} + \frac{15x^4(100-x)^2}{100^6}$

13 0.6836

14 **a** 0.1156 **b** 0.7986 **c** 0.3170

15 0.6791

16 **a** 0.1123 **b** 0.5561 **c** 0.000 01
d 0.000 01

17 0.544

18 **a** $\left(\frac{1}{4}\right)^6 \approx 0.000\,24$ **b** 0.1694

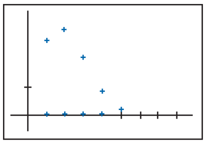
19 **a** 0.0138 **b** 0.2765 **c** 0.8208 **d** 0.3368

20 **a** $(0.8)^8 \approx 0.168$ **b** 0.001 23
c 0.0021

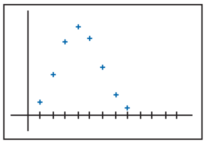
21 **a** $(0.15)^{10} \approx 0.000\,000\,006$
b $1 - (0.85)^{10} \approx 0.8031$ **c** 0.5674

Exercise 9B

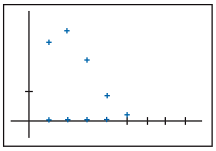
1



2



3 **a** **b**



c The distribution in part b is the reflection of the distribution in part a in the line $X = 5$.

4 **a** Mean = 5; Variance = 4
b Mean = 6; Variance = 2.4
c Mean = $\frac{500}{3}$; Variance = $\frac{1000}{9}$
d Mean = 8; Variance = 6.4

5 **a** 1 **b** 0.2632

6 37.5

7 $n = 48$, $p = \frac{1}{4}$, $P(X = 7) = 0.0339$

8 $n = 100$, $p = \frac{3}{10}$, $P(X = 20) = 0.0076$

9 Mean = 10, sd = $\sqrt{5}$; The probability of obtaining between 6 and 14 heads is 0.95

10 Mean = 120, sd = $4\sqrt{3}$;
The probability that between 107 and 133 students attend a state school is 0.95

Exercise 9C

1 **a** **i** $(0.8)^5 \approx 0.3277$ **ii** 0.6723
b 14 **c** 22

2 **a** **i** 0.1937 **ii** $1 - (0.9)^{10} \approx 0.6513$
b 12

3 7 **4** 7 **5** 10 **6** 42 **7** 86

Chapter 9 review

Short-answer questions

1 **a** $\frac{16}{81}$ **b** $\frac{32}{81}$ **c** $\frac{16}{27}$ **d** $\frac{65}{81}$

2 $\frac{54}{125}$

3 0.409 51

4 **a** 2 **b** $\frac{3\sqrt{5}}{5}$

5 **a** $(1-p)^4$
b $4p(1-p)^3$
c $1-(1-p)^4$
d p^4
e $1-(1-p)^4-4p(1-p)^3$

6 120

7 $\frac{5p(1-p)^4}{1-(1-p)^5}$

8 $\frac{5}{16}$

9 $\frac{32}{625}$

Extended-response questions

1 **a** 0.0173
b 0.2131

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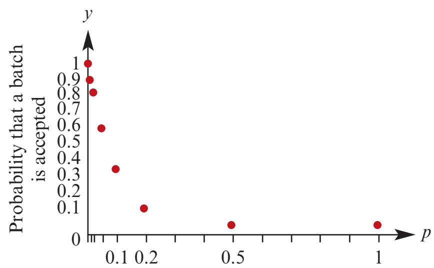
2

p	Probability that a batch is accepted
0	1
0.01	0.9044
0.02	0.8171
0.05	0.5987
0.1	0.3487
0.2	0.1074
0.5	0.000 98
1	0



3 a 0.0582 = 5.82%

b $\mu = 0.4,\ \sigma = 0.6197,\ \mu \pm 2\sigma = 0.4 \pm 1.2394$

c Yes

4 0.0327

5 a i 0.0819 **ii** 0.9011

b i $P = 15p^2(1 - p)^4$

ii $\dfrac{dP}{dp} = 30p(1 - p)^3(1 - 3p)$

6 a 2 **b** $n = 6,\ p = \dfrac{1}{3}$

c

x	0	1	2	3	4	5	6
Freq	17.56	52.68	65.84	43.90	16.46	3.29	0.27

7 a 0.9139 **b** 0.041 45 **c** 10.702

8 a 0.0735 **b** 0.5015 **c** 27

9 $\dfrac{1}{3} \le q \le 1$

Chapter 10

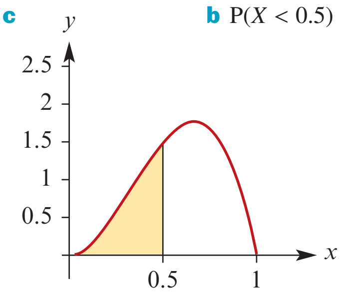
Exercise 10A

1 a 0.16 **b** 0.24

2 a 0.18 **b** 0.5

4 $k = -\dfrac{11}{6}$

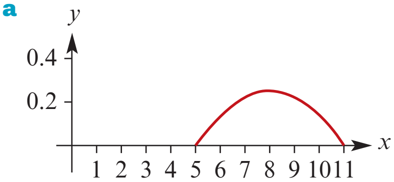
5 a c



b $P(X < 0.5) = \dfrac{5}{16}$

6 a $k = 1$ **b** 0.865

7 a

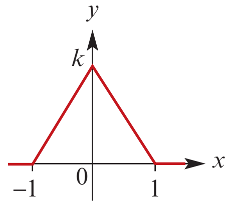


b 0.259 **c** 0.244 **d** 0.28

8 b i 0.024 = 2.4% **ii** 0.155 = 15.5%

9 a $k = 0.005$ **b** 0.007

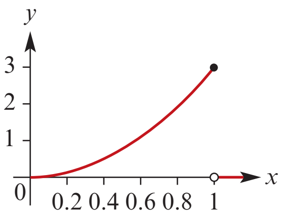
10 a



b $k = 1$

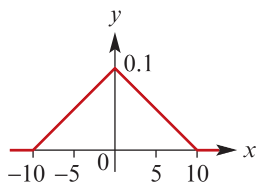
c $\dfrac{3}{4}$

11 a



b 0.406

12 a



b 0.190

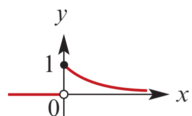
13 a $k = 1000$ **b** 0.5

14 a $\dfrac{2}{3}$ **b** $\dfrac{17}{30}$

15 a 0.202 **b** 0.449

16 a 0.45 **b** 0.711

17 a



b i $1 - e^{-\frac{1}{2}}$ **ii** e^{-1} **iii** $e^{-\frac{1}{2}}$

Exercise 10B

1 a $\dfrac{2}{3}$ **b** $\dfrac{1}{3}$ **c** $\dfrac{1}{2}$ **d** Does not exist

2 a 1 **b** 2.097 **c** 1.132 **d** 0.4444

3 a 0.567 **b** 0.458

4 0

5 $A = \dfrac{2}{9},\ B = 3$

6 a 2 **b** 1.858

7 a 0.632 **b** 0.233 **c** 0.693

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8 a 1 **b** 0.5

9 0.1294 **10** 2.773 minutes

11 a 1 **b** 1

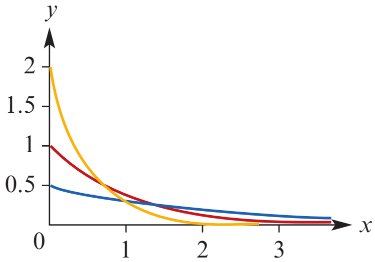
12 a 0.714 **b** 0.736

13 12

14 a 0.4 **b** $\frac{\sqrt{19}-1}{6}$

15 a $ke^{-kx} - k^2xe^{-kx}$, $\frac{-(kx+1)}{k}e^{-kx}$

c



d $y = e^{-x}$ is dilated by factor $\frac{1}{\lambda}$ parallel to the y-axis and by factor λ parallel to the x-axis

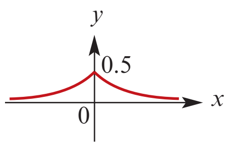
Exercise 10C

1 $\text{Var}(X) = \frac{1}{18}$, $\text{sd}(X) = \frac{\sqrt{2}}{6}$

2 (384, 416)

3 a 0.630 **b** 0.909 **c** 0.279

4 a



b 1.386

5 a $\frac{1}{\ln 9}$ **b** $\text{E}(X) = 3.641$, $\text{Var}(X) = 4.948$

6 a 0.366 **b** $\text{E}(X) = 0.333$, $\text{Var}(X) = 0.056$

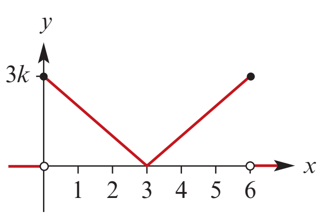
7 0.641

8 a 0.732 **b** $\text{E}(X) = \frac{4}{3}$, $\text{Var}(X) = \frac{2}{9}$

9 a 0.0004 **b** $\frac{16}{3}$ **c** 2.21

10 a $\frac{3}{4a^3}$ **b** $2\sqrt{5}$

11 a



b $k = \frac{1}{9}$ **d** 4.5

Exercise 10D

1 1300

2 $\mu = 1010\text{ g}$, $\sigma = 50\text{ g}$

3 a 0.708, 0.048 **b** \$98.94, \$0.33

4 a 0, 5.4 **b** 3, 0.6 **c** 1, 5.4

5 a $\frac{31}{40}$ **b** $\frac{91}{20}$

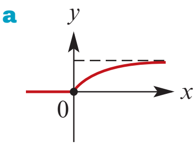
Exercise 10E

1 a $F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{5}x & \text{if } 0 < x < 5 \\ 1 & \text{if } x \ge 5 \end{cases}$ **b** $\frac{3}{5}$

2 a $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4}x & \text{if } 0 \le x < 1 \\ \frac{1}{20}(x^4+4) & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

b $\sqrt[4]{6}$

3 a

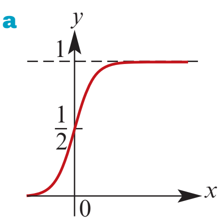


b e^{-4} **c** 0.0182

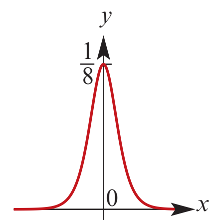
4 a $k = \frac{1}{36}$ **b** $\frac{1}{48}$

5 a $\frac{2}{3}$ **b** 20 **c** $\frac{80}{3}$

6 a



b $f(x) = \frac{0.5e^{-0.5x}}{(1+e^{-0.5x})^2}$



Chapter 10 review

Short-answer questions

1 a 2 **b** 0.21 **c** 0.44

2 $a = \frac{1}{3}$, $b = 2$

3 $\frac{\pi}{2}$

4 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$

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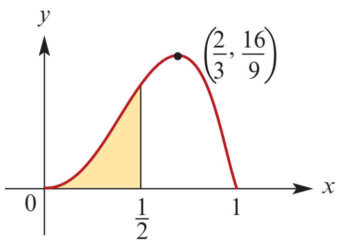
5 a



b $P(X < 0.5) = \frac{5}{16}$

c $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3(4 - 3x) & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

6 a $k = 12$ **b** $P(X < \frac{2}{3}) = \frac{16}{27}$

c $P(X < \frac{1}{3} \mid X < \frac{2}{3}) = \frac{3}{16}$

7 a 0.008 **b** $\frac{8}{27}$

8 $\frac{2}{3}$

9 a $\frac{7}{3}$ **b** $a = 1$

10 a $c = \frac{3}{4}$ **b** 0

12 a 1.649 **b** 0.833

13 (320, 340) **14** (246, 254)

Extended-response questions

1 a $a = \frac{-2}{81}$ **b** 700 hours **c** 736.4 hours

2 a $\frac{1}{4}$ **b** $\frac{5 - \sqrt{5}}{5}$ **c** $\frac{1}{3}, \frac{8}{15}$

3 a Median = 6, IQR = $\frac{10}{3}$

b $E(X) = 6$, $Var(X) = 4.736$

4 a $\frac{7}{25}$ **b** \$22.13

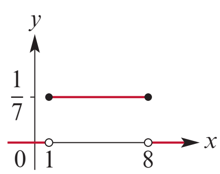
5 $c = \frac{8}{3}$ or $c = 4$

6 a $k = 25$ **b** $\frac{2}{3}$ **d** $\frac{2}{3}$

7 a $k = \frac{1}{4}$ **b** $\mu = 2$, $Var(X) = \frac{2}{3}$ **c** $\frac{3}{4}$

d $\frac{4\sqrt{5}}{5} \approx 1.8$

8 a i



ii $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{7}(x - 1) & \text{if } 1 \le x \le 8 \\ 1 & \text{if } x > 8 \end{cases}$

iii $E(X) = \frac{9}{2}$ **iv** $Var(X) = \frac{49}{12}$

b i $F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x - a}{b - a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$

ii $E(X) = \frac{a + b}{2}$ **iii** $Var(X) = \frac{(b - a)^2}{12}$

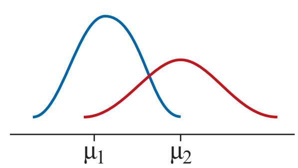
c i $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{15}x & \text{if } 0 \le x \le 15 \\ 1 & \text{if } x > 15 \end{cases}$

ii 7.5 minutes **iii** 18.75

Chapter 11

Exercise 11A

1



2 c

3 a 1

b i $E(X) = \frac{1}{3\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} \, dx$

ii 2

c i $E(X^2) = \frac{1}{3\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} \, dx$

ii 13 **iii** 3

4 a 1

b i $E(X) = \frac{1}{5\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} \, dx$

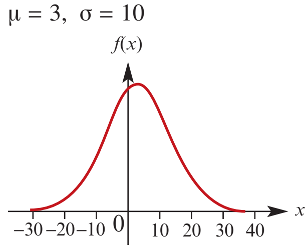
ii −4

c i $E(X^2) = \frac{1}{5\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} \, dx$

ii 41 **iii** 5

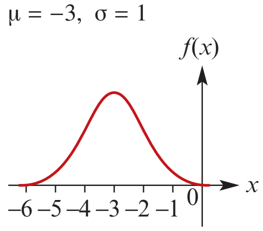
5 a $\mu = 3$, $\sigma = 10$

b



6 a $\mu = -3$, $\sigma = 1$

b



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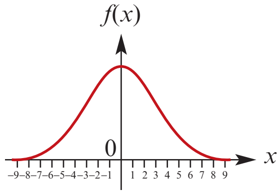
7 a $\mu = 0,\ \sigma = 3$

b



Exercise 11B

1 a 16% **b** 16% **c** 2.5% **d** 2.5%

2 a $\mu = 135,\ \sigma = 5$ **b** $\mu = 10,\ \sigma = \frac{4}{3}$

3 a 68% **b** 16% **c** 0.15%

4 21.1, 33.5

5 one, 95, 99.7, three

6 2.5%

7 a 16% **b** 16%

8 a 68% **b** 16% **c** 2.5%

9 a 95% **b** 16% **c** 50% **d** 99.7%

10 a 0 **b** $-\frac{5}{4}$ **c** 1.5

11 a −1.4 **b** 1.1 **c** 3.5

12 Michael 1.4, Cheryl 1.5; Cheryl

13 Biology 1.73, History 0.90; Biology

14 a

Student	French	English	Maths
Mary	1	0.875	0
Steve	−0.5	−1	1.25
Sue	0	0.7	−0.2

b i Mary **ii** Mary **iii** Steve

c Mary

Exercise 11C

1 a 0.9772 **b** 0.9938 **c** 0.9938 **d** 0.9943
e 0.0228 **f** 0.0668 **g** 0.3669 **h** 0.1562

2 a 0.9772 **b** 0.6915 **c** 0.9938 **d** 0.9003
e 0.0228 **f** 0.0099 **g** 0.0359 **h** 0.1711

3 a 0.6827 **b** 0.9545 **c** 0.9973

4 a 0.0214 **b** 0.9270 **c** 0.0441 **d** 0.1311

5 $c = 1.2816$ **6** $c = 0.6745$

7 $c = 1.96$ **8** $c = -1.6449$

9 $c = -0.8416$ **10** $c = -1.2816$

11 $c = -1.9600$

12 a 0.9522 **b** 0.7977 **c** 0.0478
d 0.154 538

13 a 0.9452 **b** 0.2119 **c** 0.9452 **d** 0.1571

14 a $c = 9.2897$ **b** $k = 8.5631$

15 a $c = 10$ **b** $k = 15.88$

16 a $a = 0.994$ **b** $b = 1.96$ **c** $c = 2.968$

17 a 0.7161 **b** 0.0966 **c** 0.5204
d $c = 33.5143$ **e** $k = 13.029\,13$
f $c_1 = 8.28,\ c_2 = 35.72$

18 a 0.9772 **b** 0.9772 **c** $c = 10.822$
d $k = 9.5792$ **e** $c_1 = 9.02,\ c_2 = 10.98$

Exercise 11D

1 a i 0.2525 **ii** 0.0478 **iii** 0.0901
b 124.7

2 a i 0.7340 **ii** 0.8944 **iii** 0.5530
b 170.25 cm
c 153.267 cm

3 a i 0.0766 **ii** 0.9998 **iii** 0.153
b 57.3

4 a 10.56% **b** 78.51%

5 Mean = 1.55 kg; sd = 0.194 kg

6 a 36.9% **b** $c = 69$

7 a 0.0228 **b** 0.0005 **c** 0.0206

8 a 3.04 **b** 350.27

9 1003 mL

10 a Small 0.1587; Med 0.7745; Large 0.0668
b $348.92

11 a i 0.1169 **ii** 17.7
b 0.0284

12 a A: 0.0228; B: 0.1587 **b** $c = \frac{34}{3}$

Chapter 11 review

Short-answer questions

1 a $1 - p$ **b** $1 - p$ **c** $2p - 1$

2 a $a = -1$ **b** $b = 1$ **c** 0.5

3 $(x, y) \to \left(\frac{x-8}{3}, 3y\right)$

4 a $\frac{q}{p}$ **b** $1 - q$ **c** $\frac{1-p}{1-q}$

5 a $\mathrm{P}(Z < \frac{1}{2})$ **b** $\mathrm{P}(Z < -\frac{1}{2})$ **c** $\mathrm{P}(Z > \frac{1}{2})$
d $\mathrm{P}(-\frac{1}{2} < Z < \frac{1}{2})$ **e** $\mathrm{P}(-\frac{1}{2} < Z < 1)$

6 a 0.84 **b** 0.5 **c** 0.16 **d** 0.68

7 a 0.16 **b** 0.34 **c** 0.32 **d** 0.02

8 a 0.69 **b** 0.19 **c** 0.15 **d** 0.68

9 Best C, worst B

10 a 0.5 **b** $b = -1.5$

Extended-response questions

1

Category	Range
High	> 63
Moderate	[56, 62]
Average	[45, 55]
Little	[37, 44]
Low	< 37

2 $k = 3.92$

3 a i 0.1587 **ii** 0.9747 **iii** 0.0164
b $c = 53\,592$
c 3.7×10^{-11}

4 a 3.17×10^{-5} **b** False
c $c_1 = 13.53,\ c_2 = 16.47$

5 0.0803

6 0.92%

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7 a $\mu = 60.027, \ \sigma = 0.2$ **b** 10%

8 a 0.9044 **b** 5.88 **c** 9.044
d 0.2650 **e** \$17.61, 54.0281

9 a $\mu = 0, \ \sigma = 2.658$ **b** 0.882

10 a 0.1056 **b** 0.0803 **c** 0.5944

Chapter 12

Exercise 12A

1 No; sample will be biased towards the type of movie being shown.

2 a No; biased towards shoppers.
b Randomly select a sample from telephone lists or an electoral roll.

3 No; only interested people will call, and they may call more than once.

4 a No; biased towards older, friendly or sick guinea pigs which may be easier to catch.
b Number guinea pigs and then generate random numbers to select a sample.

5 No; a student from a large school has less chance of being selected than a student from a small school.

7 a Unemployed will be under represented.
b Unemployed or employed may be under represented, depending on time of day.
c Unemployed will be over represented.
Use random sampling based on the whole population (e.g. electoral roll).

8 a Divide platform into a grid of 1 m^2 squares. Select squares using a random number generator to give two digits, one a vertical reference and one a horizontal reference.
b Yes, if crabs are fairly evenly distributed; otherwise, five squares may not be enough.

9 No; a parent's chance of selection depends on how many children they have at the school.

10 Not a random sample; only interested people will call, and they may call more than once.

11 People who go out in the evenings will not be included in the sample.

12 a All students at this school
b $p = 0.35$ **c** $\hat{p} = 0.42$

13 a 0.22 **b** $\hat{p}$

Exercise 12B

1 a 0.5 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$P(\hat{P} = \hat{p})$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

d $\frac{1}{2}$

2 a $\frac{3}{5}$ **b** $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

c

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$P(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384
$\hat{p}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$P(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

d 0.3065 **e** 0.6924

3 a 0.5 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$P(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

d 0.9

4 a 0.4 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$P(\hat{P} = \hat{p})$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

d $\frac{1}{3}$

e $\frac{3}{5}$

5 a 0.5 **b** $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

c

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$P(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

d $\frac{5}{16}$

6 a $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$P(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

c $\frac{3}{16}$ **d** $\frac{25}{26}$

7 a $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

b

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$P(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

c $\frac{17}{369}$

8 $\mu = 0.5, \ \sigma = 0.25$ **9** $\mu = 0.5, \ \sigma = 0.224$

10 $\mu = 0.2, \ \sigma = 0.2$ **11** $\mu = 0.3, \ \sigma = 0.084$

12 $\mu = 0.4, \ \sigma = 0.049$ **13** $\mu = 0.2, \ \sigma = 0.04$

14 a 0.1844 **b** 0.7600 **c** 0.9683

Exercise 12C

1 0.2858 **2** 0.8568 **3** 0.1587
4 0.0092 **5** 0.0614

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6 a 1 **b** 0.5000 **c** 0.0412

7 0.9545

8 a 0.9650 **b** 0.9650

9 a 0.575 **b** 0.0139

10 a 0.848 **b** 0.2817 **c** Yes

Exercise 12D

1 a 0.08 **b** $(0.0268, 0.1332)$

2 a 0.192 **b** $(0.1432, 0.2408)$

3 a 0.2 **b** $(0.1216, 0.2784)$

4 $(0.2839, 0.3761)$

5 a $(0.4761, 0.5739)$ **b** $(0.5095, 0.5405)$

c The second interval is narrower because the sample size is larger

6 a $(0.8035, 0.8925)$ **b** $(0.8339, 0.8621)$

c The second interval is narrower because the sample size is larger

7 1537

8 246

9 a 897 **b** 2017

c Reducing margin of error by 1% requires the sample size to be more than doubled

10 a 2017 **b** 2401

c i $M = 1.8\%$ **ii** $M = 2.2\%$

d 2401, as this ensures that M is 2% or less, whoever is correct

11 90%: $(0.5194, 0.6801)$, 95%: $(0.5034, 0.6940)$, 99%: $(0.4738, 0.7262)$; Interval width increases as confidence level increases

12 90%: $(0.5111, 0.5629)$, 95%: $(0.5061, 0.5679)$, 99%: $(0.4964, 0.5776)$; Interval width increases as confidence level increases

13 d 8 **e** 40

Chapter 12 review

Short-answer questions

1 a All employees of the company **b** $p = 0.35$

c $\hat{p} = 0.40$

2 a No; only people already interested in yoga

b Use electoral roll

3 a $\dfrac{k}{100}$ **b** $\dfrac{k}{100} \pm \dfrac{1.96\sqrt{k(100-k)}}{1000}$

4 a $\hat{p} = 0.9$ **b** $M = \dfrac{0.588}{\sqrt{n}}$

c Margin of error would decrease by a factor of $\sqrt{2}$

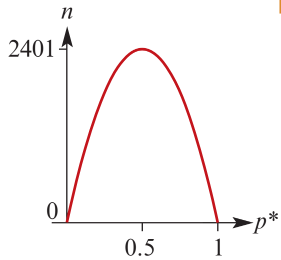
5 a 38 **b** $(0.95)^{40}$

6 a 45 **b** $5.9(0.9)^{49}$

7 a 0.60 **b** 0.10 **c** Increase sample size

Extended-response questions

1 a



b 0.5 **c** 2401

2 a 0.1537 **b** 0.8257

3 a $(0.4730, 0.6670)$

b i 0.7738 **ii** 0.000 000 3

iii 0.2262 **iv** 4.75

c $(0.5795, 0.6645)$

4 a $p = \dfrac{500}{N}$ **b** $\hat{p} = 0.15$

c $N = 3333.33 \approx 3333$ **e** $(2703, 4348)$

Chapter 13

Short-answer questions

1 a $\dfrac{\pi}{2}$ **b** 2 **c** 0.2929 **d** 0.1716

2 a $\dfrac{1}{5}$ **b** $\dfrac{4}{9}$ **c** 1.7 **d** 2.01

3 a $\dfrac{1}{36}$ **b** 0.7407 **c** 3

d 3 **e** 0.5185 **f** 0.65

4 a $\dfrac{3}{28}$ **b** $\dfrac{3}{14}$ **c** $\dfrac{5}{7}$ **d** $\dfrac{5}{14}$ **e** $\dfrac{3}{14}$

5 a $\dfrac{1}{7}$ **b** $\dfrac{1}{3}$

6 a $\dfrac{3}{20}$ **b** $\dfrac{9}{35}$ **c** $\dfrac{7}{12}$

7 a 0.3369 **b** 0.2995

8 a $a = 0.34,\ b = 0.06$ **b** 1.6644

9 a 0.75 **b** 0.28

10 b 2.726

11 $\dfrac{1-q}{2}$

12 a $F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x(2-x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$ **b** $a = \dfrac{1}{2}$

13 a $(1-p)^3$ **b** $p = \dfrac{1}{3}$

14 a $\dfrac{a+b}{2}$ **b** $\dfrac{b-a}{2}$

Extended-response questions

1 b $\dfrac{9}{16}$

2 \$0.76

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3 a $P = \begin{cases} 0.75x - 0.5s, & x \le s \\ 0.5s - 0.25x, & x > s \end{cases}$

b \$5.95

c $E(P) = \sum_{x=24}^{s} (0.75x - 0.5s)p(x) + \sum_{x=s+1}^{30} (0.5s - 0.25x)p(x)$

d 27

5 a i $\frac{1}{6}$ **ii** $\frac{1}{36}$ **iii** $\frac{1}{6}$

b i $\frac{4}{25}$ **ii** $\frac{41}{100}$

c $\frac{121}{600}$

6 a 0.6915 **b** 0.1365

7 a 0.0436 **b** 26.67% **c** 183 **d** 59 271

8 a i 0.1587 **ii** 511.63

b 0.1809

9 a i $\frac{1}{8}$ **ii** A: 0.6915, B: 0.5625

iii $E(X) = 10$, $E(Y) = 10.67$; Machine A

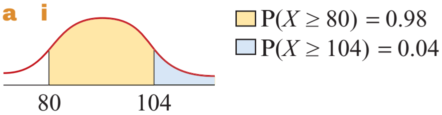
b $\frac{3}{4}$

10 a $c = \frac{20}{49}$ **b** $E(X) = \frac{120}{49}$ **c** $\text{Var}(X) = \frac{6180}{2401}$

11 a i $\frac{1}{2500}$ **ii** $\frac{16}{3}$ **iii** 0.8281 **iv** 0.7677

b 0.9971

12 a i



ii $\mu = 92.956$, $\sigma = 6.3084$

b i 16.73% of sensors **ii** 81°C

13 a i 0.2 **ii** 0.7 **iii** 0.125 **iv** $\frac{3}{160}$

b i 0.360 15 **ii** $\frac{128}{625}$

c 0.163 08

14 a 0.1056 **b** 1027.92 g

15 a i 0.0105 **ii** 0.0455

b 0.4396 **c** $\frac{1149}{1909}$

16 a i $\mu = 4.25$ **ii** $\sigma = 0.9421$ **iii** 0.94 **iv** 0.9

b i Binomial **ii** 18 **iii** 1.342 **iv** 0.3917

17 a i $P(\text{Black}) = \frac{n-3}{n}$ **ii** $P(\text{White}) = \frac{3}{n}$

b $\frac{(n-3)^2}{n^2 - 3n + 3}$

18 a (0.0814, 0.1186)

b (0.0792, 0.1208)

c Larger sample of females

d 900 of each sex

e 0.078 or 0.922

Chapter 14

Short-answer questions

1 $f(g(x)) = (3x + 1)^2 + 6 = 9x^2 + 6x + 7$

2 a $f'(x) = 21x^6(5x^2 - 3)^6(5x^2 - 1)$

b $f'(0) = 2$

3 a $f'\left(\frac{\pi}{2}\right) = \frac{-2}{(\pi + 1)^2}$ **b** $f'\left(\frac{\pi}{3}\right) = 8\pi - 3\sqrt{3}$

4 a $2\ln(2x) + 3$

b $f''(x) = 4e^{\sin(2x)}(\cos^2(2x) - \sin(2x))$

5 a $x = \ln\left(\frac{y+3}{5}\right) + 1$ **b** $x = 3 - e^{2y}$

6 $\frac{1}{4}(e^4 - 1)$ **7** $0.12\pi \text{ cm}^2$

8 $\left(\frac{\sqrt{6}}{3}, \frac{20}{9}\right), \left(-\frac{\sqrt{6}}{3}, \frac{20}{9}\right)$

9 $\frac{-1}{3}\ln(1 - 3x) + c$

10 a 0.5 **b** 0.68 **c** 0.32

11 a $\frac{1}{6}$ **b** $\sqrt{31}$

12 $\frac{-2}{3}, \frac{2}{3}$

13 a $A = 32a - 8a^3$

b Max value $A = \frac{128\sqrt{3}}{9}$ when $a = \frac{2\sqrt{3}}{3}$

14 $b = 3$ **15** 0.36

16 a \$0.65 **b** 0.425

17 0.37

18 a $h = \frac{4000}{x^2}$ **c** $2000(2 + \sqrt{2})$

19 a No **b** Electoral roll

20 a 0.53 **b** (0.4322, 0.6278)

21 a 0.37 **b** $\frac{0.9463}{\sqrt{n}}$

c Increased by a factor of $\sqrt{2}$

Extended-response questions

1 a i $\left(\frac{1}{2}, 8\right)$ **ii** Minimum

b ii $A = \frac{x}{12}(60 - 5x)$ **iii** Max area 15 cm^2

2 a $y = -\frac{1}{2}x + \frac{3}{2}$

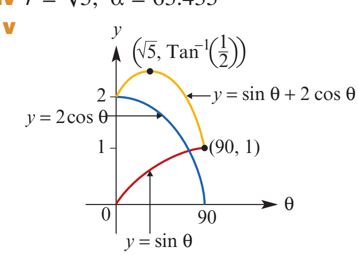
b i $\frac{dy}{d\theta} = \cos\theta - 2\sin\theta$

ii $\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$

iii (26.57, 2.2361); exact: $\left(\tan^{-1}\left(\frac{1}{2}\right), \sqrt{5}\right)$

iv $r = \sqrt{5}$, $\alpha = 63.435°$

v



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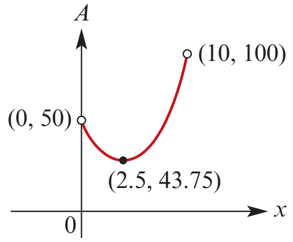
3 a i $A = x^2 - 5x + 50$ **ii** $(0, 10)$

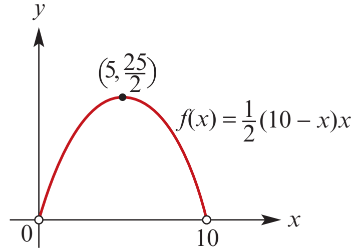
iii



iv Minimum area = 43.75 cm^2

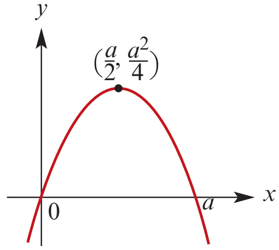
b i $f(x) = \frac{1}{2}(10 - x)x$

ii



c $AYX : OXYZ : ABY : CBYZ = 1 : 2 : 2 : 3$

4 a



b $\frac{a^3}{6}$ square units

c i $y = \frac{2a^2}{9}, y = \frac{2a^2}{9}$ **ii** $\frac{a^3}{162}$ square units

5 a i $f'(t) = -100e^{-\frac{t}{10}}(t^2 - 30t + 144)$

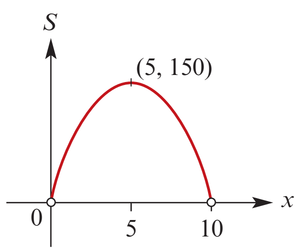
ii $f''(t) = 10e^{-\frac{t}{10}}(t^2 - 50t + 444)$

b i $t \in (6, 24)$ **ii** $t \in (11.546, 35)$;
exact: $t \in (25 - \sqrt{181}, 25 + \sqrt{181})$

iii $t \in (11.546, 24)$

6 a $S = 60x - 6x^2$ **b** $0 < x < 10$

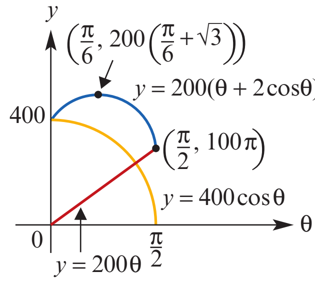
c $x = 5$ **d**



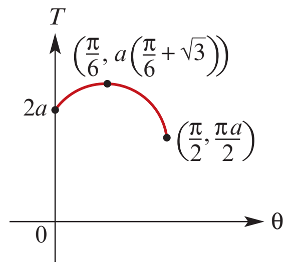
7 a i $OP = \frac{1}{\sin\theta}$ **ii** $BQ = \frac{1 - \cos\theta}{\sin\theta}$

d Min value $S = \frac{\sqrt{3}}{2}$ when $AP = \frac{2\sqrt{3} - 3}{3}$

8 b



c



Minimum value for T is $\frac{\pi a}{2}$

9 a iii $x = 1$ or $x = k - 2$

b i $b = 3 - 2a,\ c = a - 2$

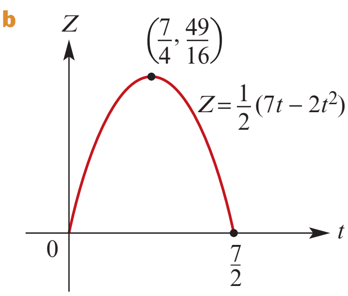
ii $h = a - 2$

iii $a = 0,\ b = 3,\ c = -2$

iv $a = -1,\ b = 5,\ c = -3$

10 a $Z = \frac{1}{2}(7t - 2t^2)$

b



c Max value $Z = \frac{49}{16}$ when $t = \frac{7}{4}$

11 a i $\frac{3}{8}$ **ii** $\frac{4}{15}$

b i $\frac{27}{125}$ **ii** $\frac{8}{125}$ **iii** $\frac{38}{125}$

12 a $k = \frac{b}{a^2}$

b i $y = \frac{b}{2a}x + \frac{b}{2}$

ii $\left(\frac{-a}{2}, \frac{b}{4}\right)$

d $S_1 : S_2 = 27 : 37$

13 a i 0.9332 **ii** 0.0668

iii 0.1151 **iv** 0.1151

b i 33.3% **ii** 866.4

iii 199.4

14 $90 - 8\sqrt{3}$ metres from A towards E

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15 a i $y = -e^{-n}x + e^{-n}(n+1)$

ii $x = n + 1$

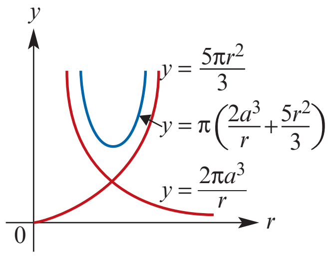
b i $\frac{1}{e^n}\left(1 - \frac{1}{e}\right)$

ii $e : e - 2$

16 b i $h = \frac{3a^3 - 2r^3}{3r^2}$

ii $S = \pi\left(\frac{2a^3}{r} + \frac{5r^2}{3}\right)$

c i



ii Local minimum at $\left((\sqrt[3]{0.6})a,\ \pi a^2\left(\frac{2}{\sqrt[3]{0.6}} + \frac{5}{3}(\sqrt[3]{0.6})^2\right)\right)$

17 a 0.0023

b

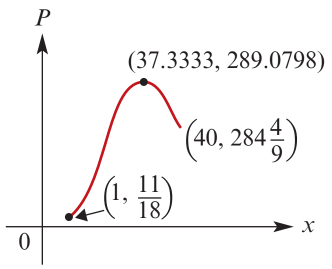
Q	$s-1$	-1
$P(Q = q)$	$\frac{3}{4}$	$\frac{1}{4}$

c $E(Q) = \frac{3}{4}s - 1$, $sd(Q) = \frac{\sqrt{3}}{4}s$

18 a 0.091 21 **b** 0.2660 **c** 0.275

19 a $\frac{dP}{dx} = \frac{1}{90}(112x - 3x^2)$

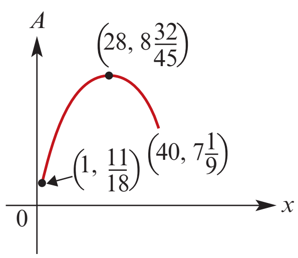
b i



ii Max value of P is 289.0798 tonnes

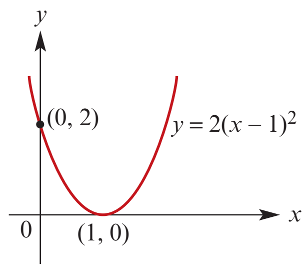
c $A = \frac{x}{90}(56 - x)$

i

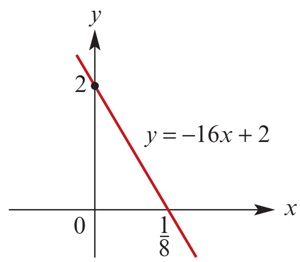


ii Max value of A is $8\frac{32}{45}$ tonnes per miner, occurs when $x = 28$

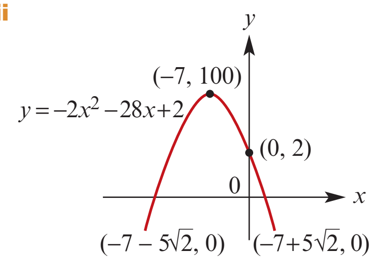
20 a i



ii



iii



b $\left(\frac{2-3k}{k+2}, \frac{-2(2-3k)^2 + 2(k+2)}{k+2}\right)$

i $\left(-2, \frac{2}{3}\right)$ **ii** $\left\{\frac{2}{3}\right\}$

iii $(-\infty, -2) \cup \left(0, \frac{14}{9}\right)$ **iv** $(-2, 0) \cup \left(\frac{14}{9}, \infty\right)$

c $k < -2$

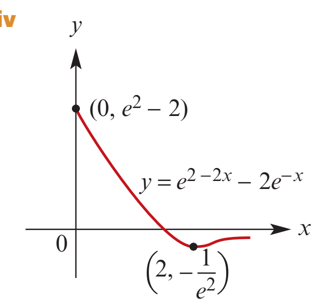
d i $k = 0$ or $k = \frac{14}{9}$ **ii** $0 < k < \frac{14}{9}$

21 a $x = 2 - \ln 2$

b i $\frac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$ **ii** $x = 2$

iii $\left(2, -\frac{1}{e^2}\right)$

iv



c $\left(-\frac{1}{e^2}, 0\right)$

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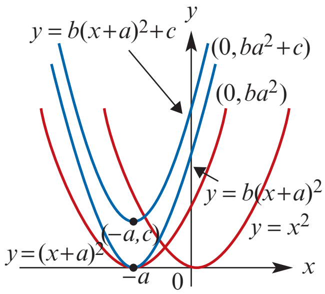
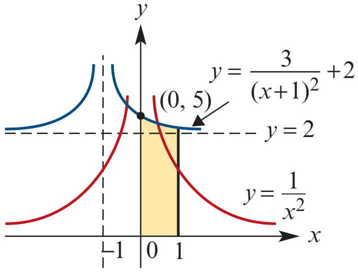
22 a



c Dilation of factor 3 parallel to the y-axis, then translation 1 unit to the left and 2 units up

d $\frac{7}{2}$

e



23 a i $y = 50$ **ii** $y = x - 25$

b $a = -\frac{1}{15}$, $c = -\frac{25}{3}$

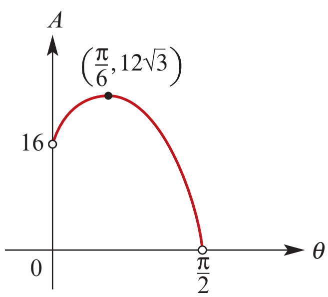
c i 1250 square units

ii $\frac{14\,375}{18}$ square units

iii $\frac{36\,875}{18}$ square units

24 c $\theta = \frac{\pi}{6}$

d



Maximum value of A is $12\sqrt{3}$

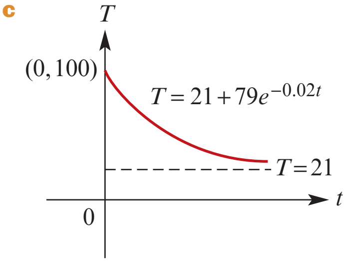
25 a $\mu = 5.0290$, $\sigma = 0.0909$

b \$409.28

26 a $k = \frac{1}{10}\ln\left(\frac{79}{63}\right) \approx 0.02$, $A = 79$

b Approx 2:44 p.m.

c



d Average rate of change $= -1.6$°C/minute

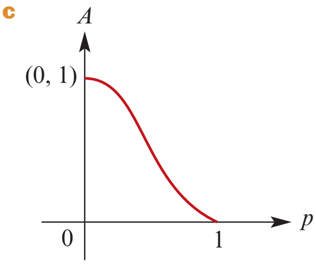
e i 2.0479°C/minute

ii -0.8826°C/minute

27 a $\frac{3}{16}$

b $b = 4$

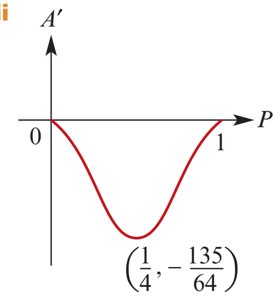
c



d i 0.076 **ii** 0.657

e i $A'(p) = -20p(1-p)^3$

ii



iii $p = \frac{1}{4}$

iv Most rapid rate of change of probability occurs when $p = \frac{1}{4}$

28 a 0.065 36

b i 0.6595 **ii** 0.198 14

c i 23.3% **ii** $c = 0.1075$

29 a i 0.32 **ii** 0.18 **iii** 0.5

b 0.64

c i 0.043 95 **ii** 0.999 **iii** $\frac{7}{128}$

30 a $\frac{527}{1000}$ **b** (0.4961, 0.5580)

c 67% **d** 99.9955%

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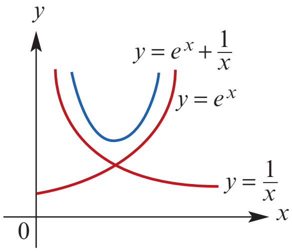
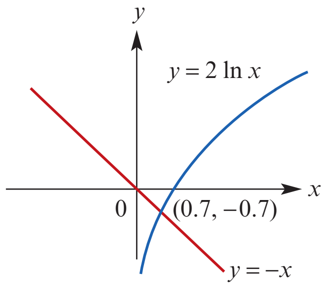
31 a b



c $y = \frac{1}{x} + e^x$, $\frac{dy}{dx} = -\frac{1}{x^2} + e^x$

d ii $2 \ln x < 0$, $x \in (0, 1)$

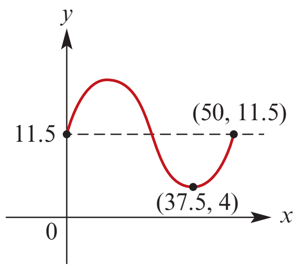
iii



iv (0.7, 3.4)

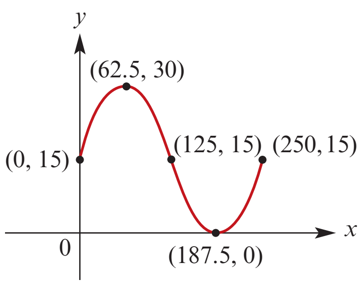
32 a i $m = 12.5$, $n = 15$, $d = 37.5$, $a = 7.5$, $b = 7.5$

ii



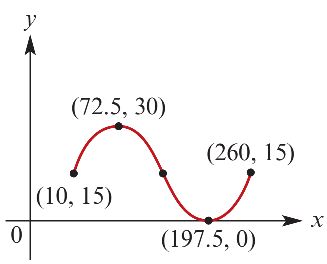
b (2.704, 10), (22.296, 10)

c



d i $h(x) = 15 + 15 \sin\left(\frac{\pi(x - 10)}{125}\right)$

ii



33 a $k = 4$

b i $E(X) = \frac{13}{6}$ **ii** $\frac{10 - \sqrt{2}}{4}$ **iii** $\frac{\sqrt{2}}{12}$

c 0.1857

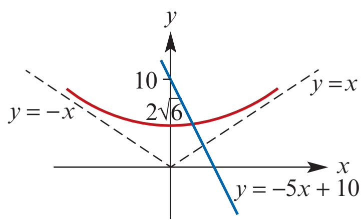
34 a $k = \frac{2}{a^2}$ **b** $E(X) = \frac{a}{3}$, $Var(X) = \frac{a^2}{18}$

c $\frac{6 - 4\sqrt{2}}{9}$ **d** $a = 1000(\sqrt{2} + 2)$

35 a 7 **c** $y = \frac{1}{10}x - \ln 10$ **d ii** 36.852

36 a $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 24}}$ **b** $(0, 2\sqrt{6})$ **c** Even

d



e $y = -5x + 10$

g $12 \ln(\sqrt{7} - 1) - 2\sqrt{7} + \frac{35}{2}$

37 a i 0 **ii** −0.6745 **iii** 0.6745

iv 1.3490 **v** 99.3% **vi** 0.7%

b i μ **ii** μ − 0.645σ

iii μ + 0.645σ **iv** 1.3490σ

v 0.9930% **vi** 0.7%

38 a $k = n + 1$ **b** $E(X) = \frac{n + 1}{n + 2}$

c $\frac{n + 1}{(n + 2)^2(n + 3)}$ **d** Median $= \sqrt[n+1]{\frac{1}{2}}$

39 a i $\frac{1}{1 - b}$ **ii** $1 + \sqrt{b - 1}$

b i 1 **ii** $1 + e^{-7}$

c i $\frac{b(b - 2)}{2(b - 1)}$ **ii** $9 + \sqrt{65}$

d e

40 a i $\frac{-(b + 1)}{b^2}$ **ii** $\left(\frac{2b^2}{b + 1}\right)^{\frac{1}{3}}$

b i $\frac{(b^2 + 1)(b - 1)}{2b^2}$ **ii** $b = 3$

41 a Local maximum $\left(\frac{m}{n}, \left(\frac{m}{n}\right)^m e^{n-m}\right)$

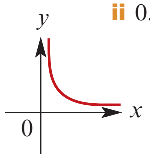
b $\left(\frac{m - 1}{n}, \left(\frac{m - 1}{n}\right)^m e^{n-m+1}\right)$

c i $\frac{4}{e^2}$ **ii** $1 - \frac{5}{e^2}$

42 a i q **ii** $\frac{1}{q}$ **iii** $\frac{1}{q^2}$

b $\frac{1}{3}$

c i **ii** 0.37 **iii** 69.31



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Appendix A

TI-Nspire CAS CX with OS4.0

Keystroke actions and short cuts for the TI-Nspire CAS CX

esc : removes menus and dialogue boxes

ctrl + esc : undo last move

shift + esc : redo last move

tab : move to next entry box (field)

ctrl + tab : switch applications in split screen

Navpad (Touchpad)

ctrl : accesses secondary (blue) commands

ctrl + ▲ : displays page sorter

ctrl + ◀ : displays previous page

ctrl + ▶ : displays next page

on : displays icon page to select applications, mode, My Documents and start a new document

menu : options for each application

ctrl + menu : contextual menus (same as right mouse click)

mouse pointer : mouse pointer (cursor). Selects items.

ctrl + mouse pointer : grab

del : backspace, deletes a character

catalogue : catalogue

2D maths template : 2D maths template

ctrl + ÷ : adds fraction template

enter : completes commands and displays results

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Mode Settings

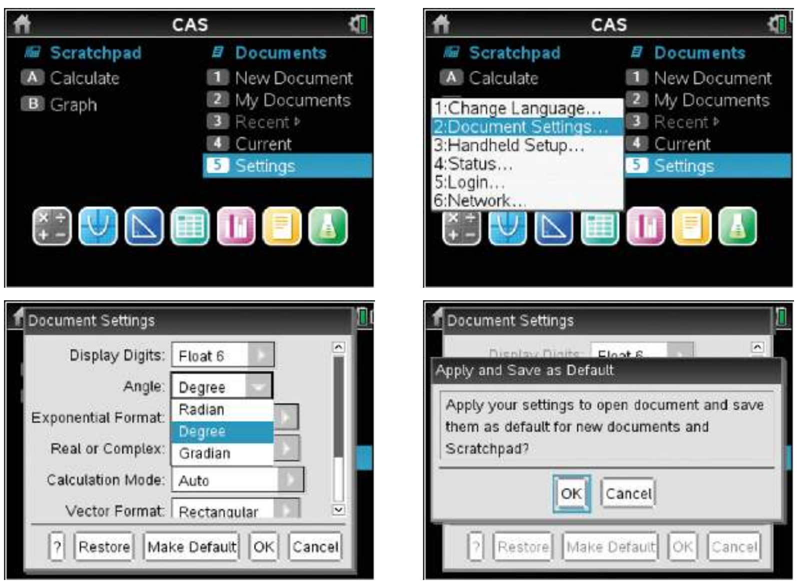
How to set in Degree mode

For this subject it is necessary to set the calculator to **Degree** mode right from the start. This is very important for the Trigonometry topic. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press ⌂on and move to **Settings>Document Settings**, arrow down to the **Angle** field, press ► and select **Degree** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note that there is a separate settings menu for the **Graphs** and **Geometry** pages. These settings are accessed from the relevant pages. For Mathematics it is not necessary for you to change these settings.

Note: When you start your new document you will see **DEG** in the top status line.



How to set in Approximate (Decimal) mode

For this subject it is useful to set the calculator to **Approximate (Decimal)** mode right from the start. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press ⌂on and move to **Settings>Document Settings**, arrow down to the **Calculation Mode** field, press ► and select **Approximate** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note: You can make both the **Degree** and **Approximate Mode** selections at the same time if desired.

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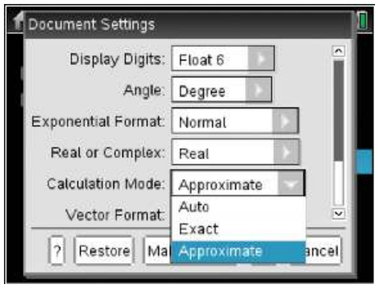

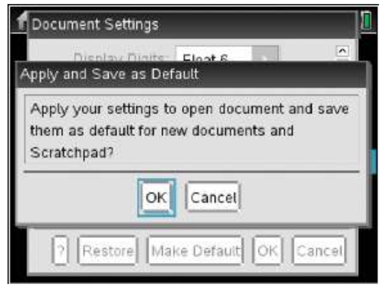

The home screen is divided into two main areas – **Scratchpad** and **Documents**.

All instructions given in the text, and in the Appendix, are based on the **Documents** platform.

Documents

Documents can be used to access all the functionality required for this subject including all calculations, graphing, statistics and geometry.

Starting a new document

1 To start a new document, press [⌂ on] and select **New Document**.

2 If prompted to save an existing document move the cursor to **No** and press [enter].
Note: Pressing [ctrl]+[N] will also start a new document.

A: Calculator page - this is a fully functional CAS calculation platform that can be used for calculations such as arithmetic, algebra, finance, trigonometry and matrices. When you open a new document select **Add Calculator** from the list.

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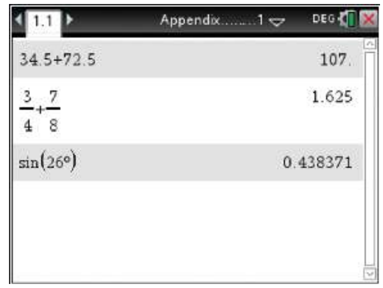
1 You can enter fractions using the fraction template if you prefer. Press ctrl ÷ to paste the fraction template and enter the values. Use the tab key or arrows to move between boxes. Press enter to display the answer. Note that all answers will be either whole numbers or decimals because the mode was set to approximate (decimal).



2 For problems that involve angles (e.g. evaluate sin(26°)) it is good practice to include the degree symbol even if the mode is set to degree (DEG) as recommended.

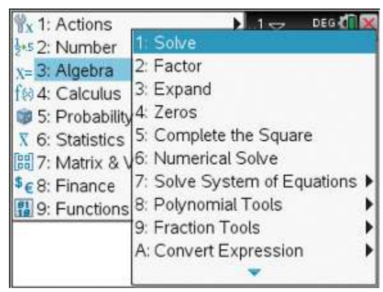
Note: If the calculator is accidentally left in radian (RAD) mode the degree symbol will override this and compute using degree values.

The degree symbol can be accessed using ?!▸. Alternatively select from the **Symbols** palette ctrl.To enter trigonometry functions such as *sin, cos*, press the trig key or just type them in with an opening parenthesis.

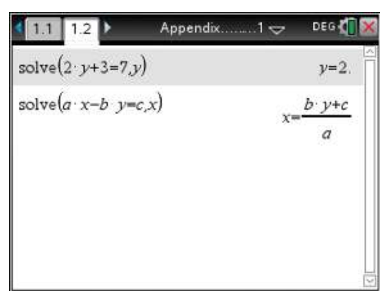
Solving equations

Using the **Solve command** Solve $2y + 3 = 7$ for y.

In a **Calculator** page press menu **>Algebra>Solve** and complete the **Solve** command as shown opposite. You must include the variable you are making the subject at the end of the command line.



Hint: You can also type in **solve(** directly from the keypad but make sure you include the opening bracket.



Literal equations such as $ax - by = c$ can be solved in a similar way.

Note that you must use a multiplication sign between two letters.

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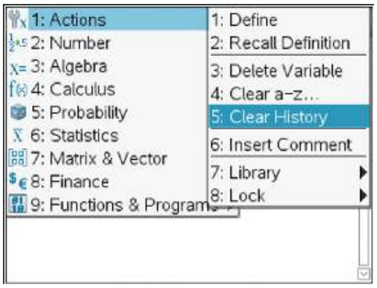
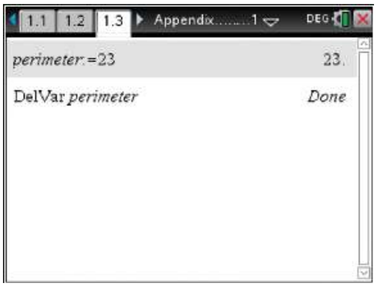
Clearing the history area



Once you have pressed enter the computation becomes part of the **History** area. To clear a line from the History area, press ▲ repeatedly until the expression is highlighted and press enter. To completely clear the History area, press menu **>Actions>Clear History** and press enter again.

Alternatively press ctrl+menu to access the contextual menu.



It is also useful occasionally to clear any previously stored values. Clearing **History** does not clear stored variables.
Pressing menu **>Actions>Clear a-z...** will clear any stored values for single letter variables that have been used.
Use menu **>Actions>Delete Variable** if the variable name is more than one letter. For example, to delete the variable perimeter, then use **DelVar *perimeter***.

Note: When you start a new document any previously stored variables are deleted.

How to construct parallel boxplots from two data lists

Construct parallel boxplots to display the pulse rates of 23 adult females and 23 adult males.

Pulse rate (beats per minute)	
Females	Males
65 73 74 81 59 64 76 83 95 70 73 79 64 77 80 82 77 87 66 89 68 78 74	80 73 73 78 75 65 69 70 70 78 58 77 64 76 67 69 72 71 68 72 67 77 73

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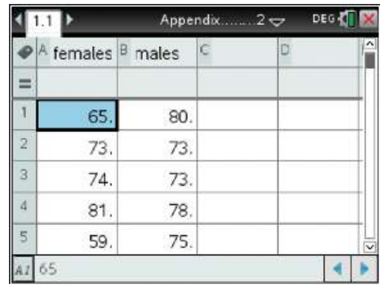
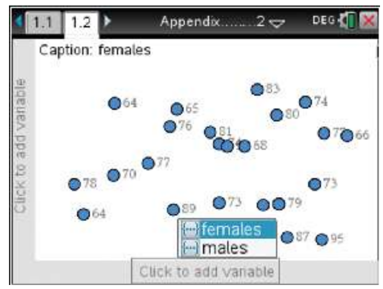
Steps

1 Start a new document: [ctrl]+[N].

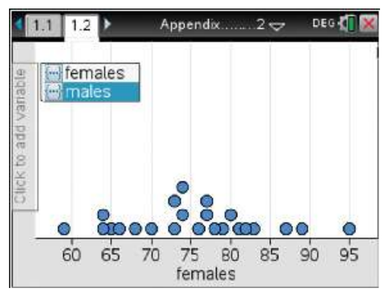
2 Select **Add Lists & Spreadsheet**. Enter the data into lists called ***females*** and ***males*** as shown.



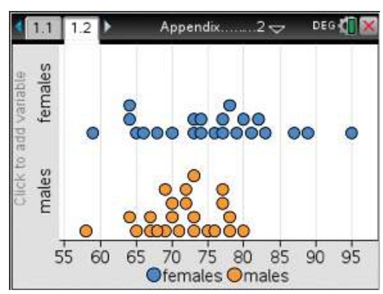
3 Statistical graphing is done through the **Data & Statistics** application.
Press [ctrl]+[I] and select **Add Data & Statistics**
(or press [on] and arrow to and press [enter]).
Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



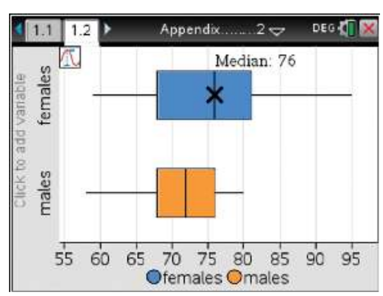
a Press [tab], or navigate and click on the "Click to add variable" box to show the list of variables. Select the variable, ***females***. Press [enter] or [click] to paste the variable to the x-axis. A dot plot is displayed by default as shown.



b To add another variable to the x-axis press [menu]**>Plot Properties>Add X Variable**, then [enter]. Select the variable ***males***. Parallel dot plots are displayed by default.



c To change the plots to box plots press [menu]**>Plot Type>Box Plot**, then press [enter]. Your screen should now look like that shown below.



Use ▼ to trace the other plot.

Press [esc] to exit the **Graph Trace** tool.

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4 Data analysis

Use menu **>Analyze>Graph Trace** and use the cursor arrows to navigate through the key points. Alternatively just move the cursor over the key points. Starting at the far left of the plots, we see that, for females, the

- minimum value is 59: **MinX** = **59**
- first quartile is 68: **Q1**= **68**
- median is 76: **Median** = **76**
- third quartile is 81: **Q3** = **81**
- maximum value is 95: **MaxX** = **95**

and for males, the

- minimum value is 58: **MinX** = **58**
- first quartile is 68: **Q1** = **68**
- median is 72: **Median** = **72**
- third quartile is 76: **Q3** = **76**
- maximum value is 80: **MaxX** = **80**

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Appendix B

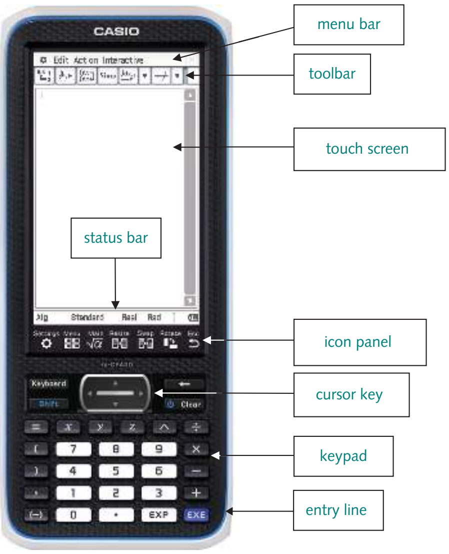
Casio ClassPad II

Operating system

Written for operating system 2.0 or above.

Terminology

Some of the common terms used with the ClassPad are:



The menu bar

The toolbar

The touch screen contains the work area where the input is displayed on the left and the output is displayed on the right. Use your finger or stylus to tap and perform calculations.

The icon panel contains seven permanent icons that access settings, applications and different view settings. Press **escape** to cancel a calculation that causes the calculator to freeze.

The cursor key works in a similar way to a computer cursor keys.

The keypad refers to the hard keyboard.

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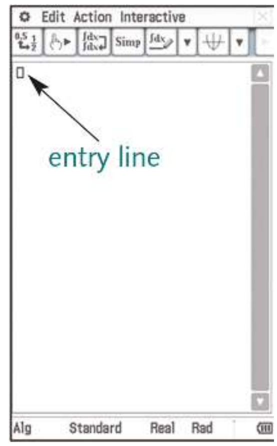
Calculating

Tap from the **icon panel** to display the application menu if it is not already visible.

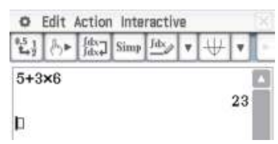
Tap to open the **Main** application.

Note: There are two application menus. Alternate between the two by tapping on the screen selector at the bottom of the screen.

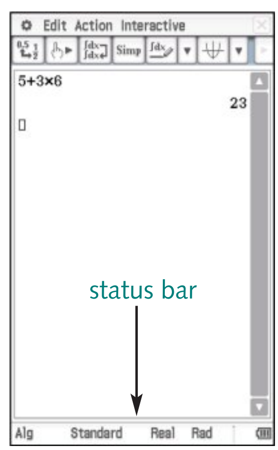
1 The main screen consists of an entry line which is recognised by a flashing vertical line (cursor) inside a small square. The history area, showing previous calculations, is above the entry line.



2 To calculate, enter the required expression in the entry line and press EXE. For example, if we wish to evaluate $5 + 3 \times 6$, type the expression in the entry line and press EXE.
You can move between the entry line and the history area by tapping or using the cursor keys (i.e. ▽, ◁, △, ▷).



3 The ClassPad gives answers in either exact form or as a decimal approximation. Tapping settings in the **status bar** will toggle between the available options.



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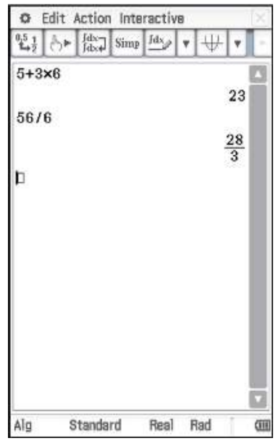
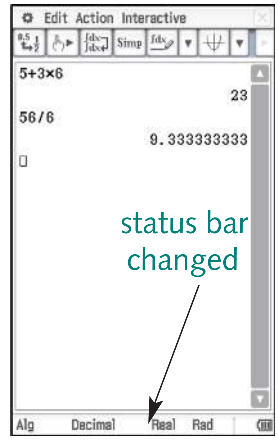
4 For example, if an exact answer is required for the calculation 56 ÷ 6, the **Standard** setting must be selected.

5 If a decimal approximation is required, change the **Standard** setting to **Decimal** by tapping it and press EXE.

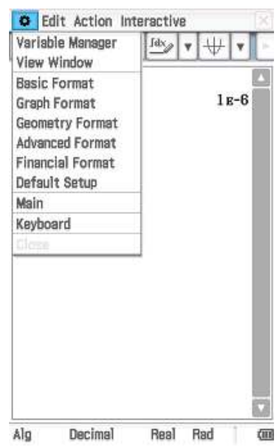




Extremely large and extremely small numbers

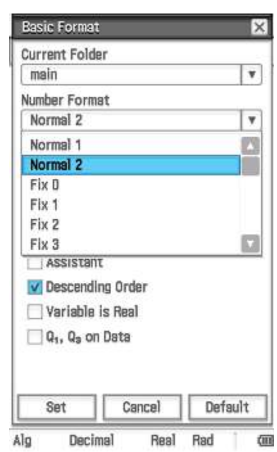
When solving problems that involve large or small numbers the calculator's default setting will give answers in scientific form.

For example, one millionth, or 1/1000000, in scientific form is written as 1×10^{-6} and the calculator will present this as 1E-6.



To change this setting, tap on the settings icon and select **Basic Format.**

Under the Number Format select **Normal 2** and tap SET.



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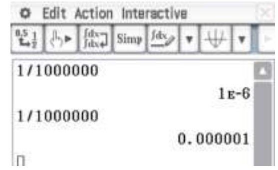
In the Main screen type 1/1000000 and press **EXE**.

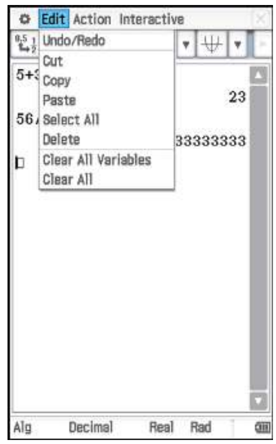
The answer will now be presented in decimal form 0.000001

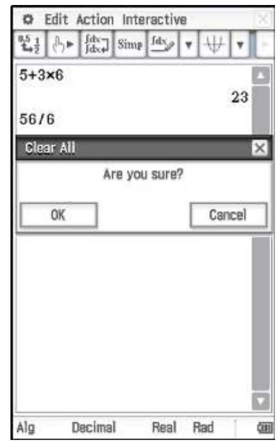
This setting will remain until the calculator is reset.



Clearing the history screen

To clear the **Main** application screen, select **Edit** from the menu bar and then tap **Clear All**. Confirm your selection by tapping **OK**. The entire screen is now cleared. To clear the entry line only, press **Clear** on the front calculator.

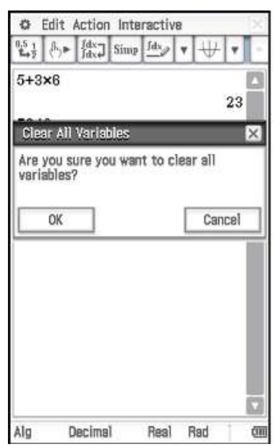


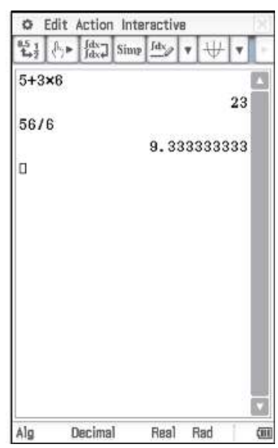


Clearing variables

To clear stored variable values, select **Edit** from the menu bar and then tap **Clear All Variables**. Confirm your selection by tapping OK.

The variables are cleared but the history created on the main screen is kept.





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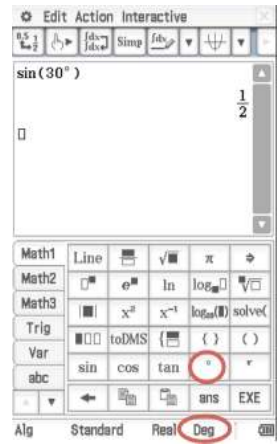
Degree mode

When solving problems in trigonometry, your calculator should be kept in **Degree** mode. In the main screen, the status bar displays the angle mode.

To change the angle mode, tap on the angle unit in the status bar until **Deg** is displayed.



In addition, it is recommended that you always insert the degree symbol after any angle. This overrides any mode changes and reminds you that you should be entering an angle, not a length.

The degree symbol is found in the **Math1** keyboard.

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