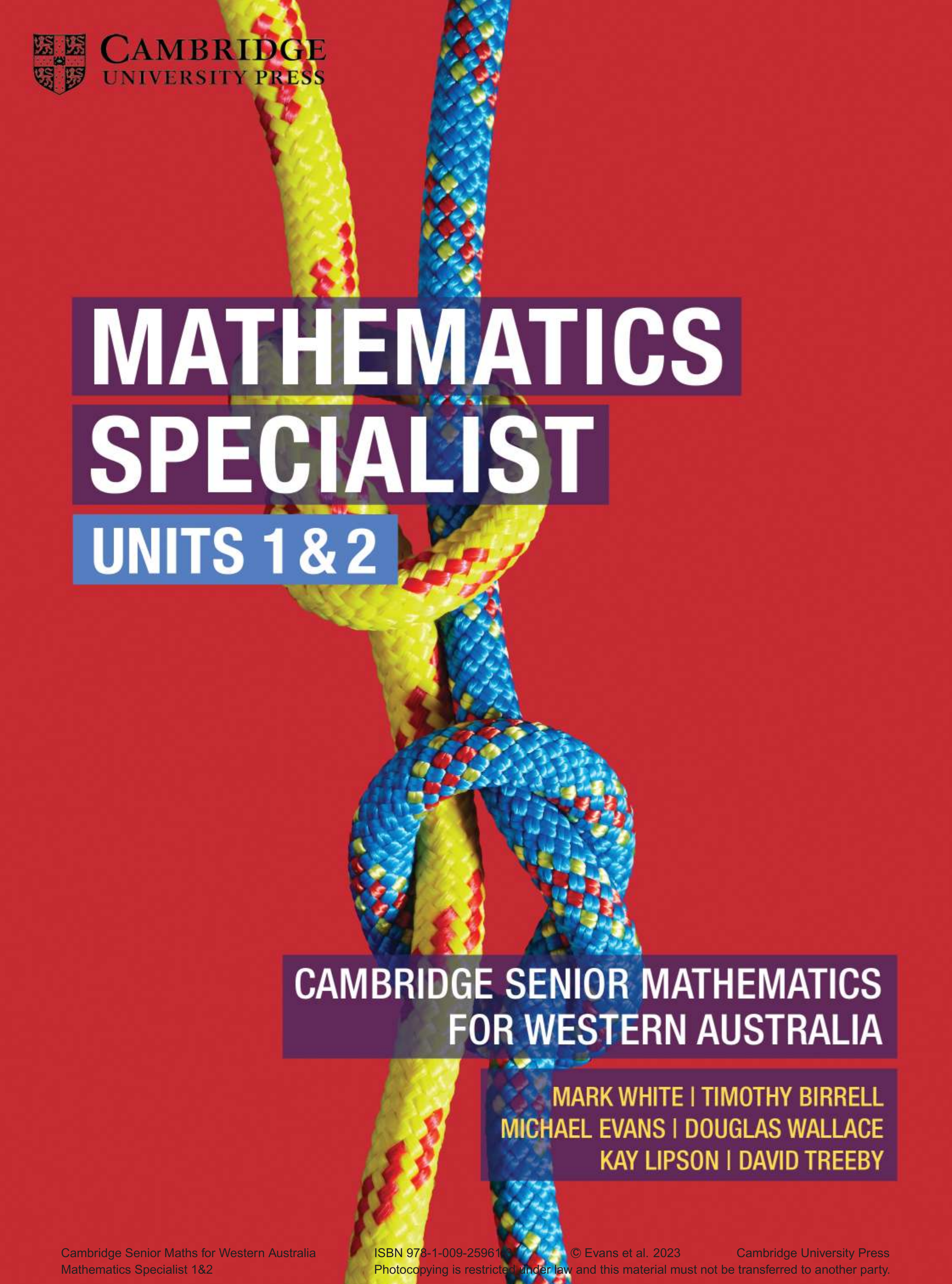

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# MATHEMATICS SPECIALIST

## UNITS 1 & 2

CAMBRIDGE SENIOR MATHEMATICS FOR WESTERN AUSTRALIA

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# Contents





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Online appendices accessed through the Interactive Textbook or PDF Textbook




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# Introduction

*Cambridge Mathematics Specialist for Western Australia Units 1 & 2* is a new edition aligned specifically to the Western Australian Mathematics Specialist Year 11 syllabus. Covering both Units 1 and 2 in one resource, it has been written with understanding as its chief aim and with ample practice offered through the worked examples and exercises.

Beginning with counting methods and an introduction to sets, the textbook then provides a study of vectors with application to statics of a particle, proof technique, circle geometry, trigonometric functions and identities, matrices, transformations of the plane and an introduction to complex numbers, all with a variety of practical and theoretical contexts. Worked examples utilising CAS calculators are provided throughout, with screenshots and detailed user instructions for both ClassPad and TI-Nspire included for each CAS example.

Compared to the previous Australian Curriculum edition, this WA edition has undergone a number of revisions. Careful adjustments to notation and language have been made throughout to match that used in the WA syllabus and the Year 12 exam and in WA classrooms more generally. Sections on algebra, geometry in the plane, kinematics, sampling and sampling distributions as well as logic have all been removed for this edition. All multiple-choice questions that were formerly located in the chapter reviews and revision chapters have also been removed.

The book contains three revision chapters. These chapters provide technology-free and extended-response questions and are intended to help prepare students for examinations and other assessments, and the grading of the questions and the inclusion of challenging problems ensure that WA students have the opportunity to achieve at the highest standards.

The TI-Nspire calculator examples and instructions have been completed by Russell Brown and those for the Casio ClassPad have been completed by Maria Schaffner.

The integration of the features of the textbook and the new digital components of the package, powered by Cambridge HOTmaths, are illustrated on pages vii to xi.

## About Cambridge HOTmaths

Cambridge HOTmaths is a comprehensive, award-winning mathematics learning system – an interactive online maths learning, teaching and assessment resource for students and teachers, for individuals or whole classes, for school and at home. Its digital engine or platform is used to host and power the Interactive Textbook and the Online Teaching Suite. All this is included in the price of the textbook.


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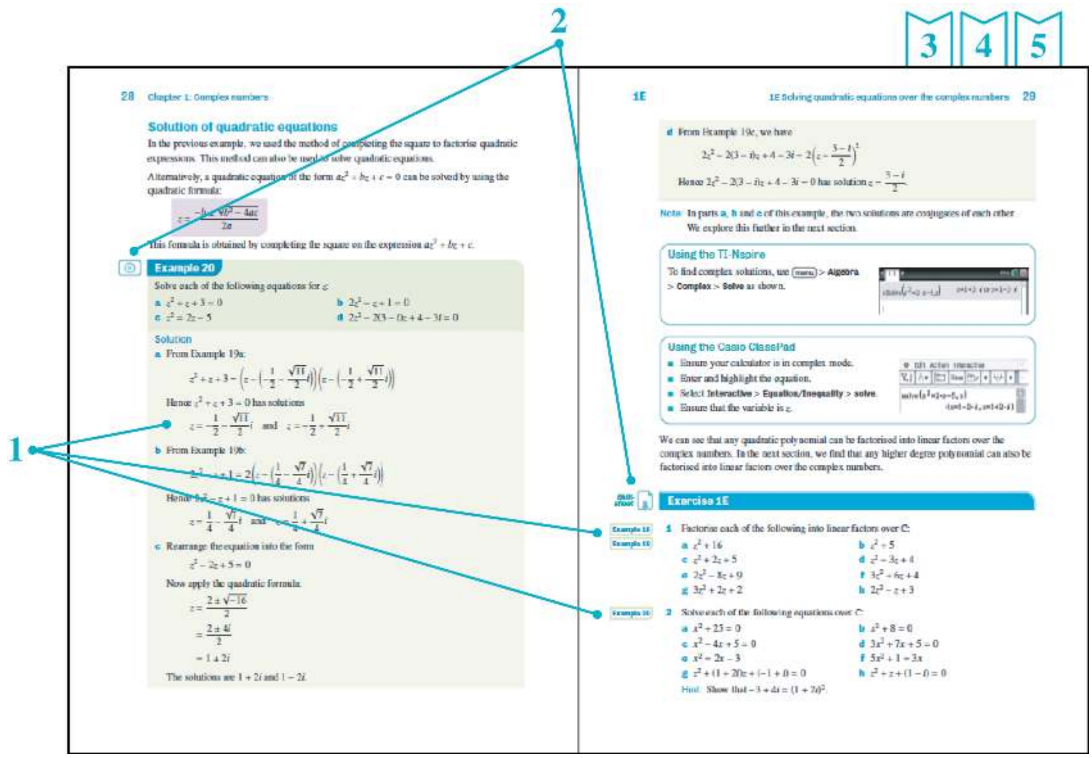
# Overview

## Overview of the print book

1. Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercise questions.
2. Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.
3. Chapter reviews contain a chapter summary and short-answer and extended-response questions.
4. Revision chapters provide comprehensive revision and preparation for assessment.
5. The glossary includes page numbers of the main explanation of each term.

*Numbers refer to descriptions above.*





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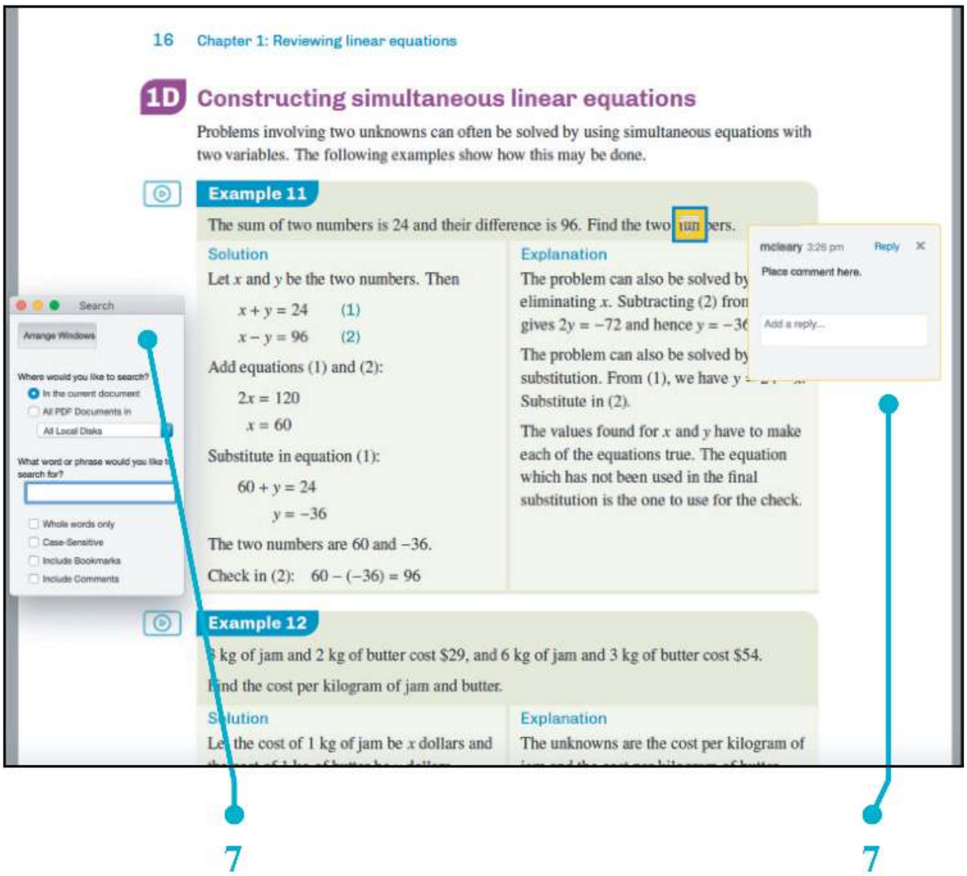
## Overview of the downloadable PDF textbook

**6** The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

**7** PDF annotation and search features are enabled.




## Overview of the Interactive Textbook

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

**8** The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

**9** **Workspaces** for all questions, which can be enabled or disabled by the teacher, allow students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing done on paper.

**10** **Self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.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**11** All worked examples have **video versions** to encourage independent learning.

**12** **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

**13** An expanded and revised set of **Desmos interactives** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

**14** The **Desmos graphics calculator, scientific calculator**, and **geometry tool** are also embedded for students to use for their own calculations and exploration.

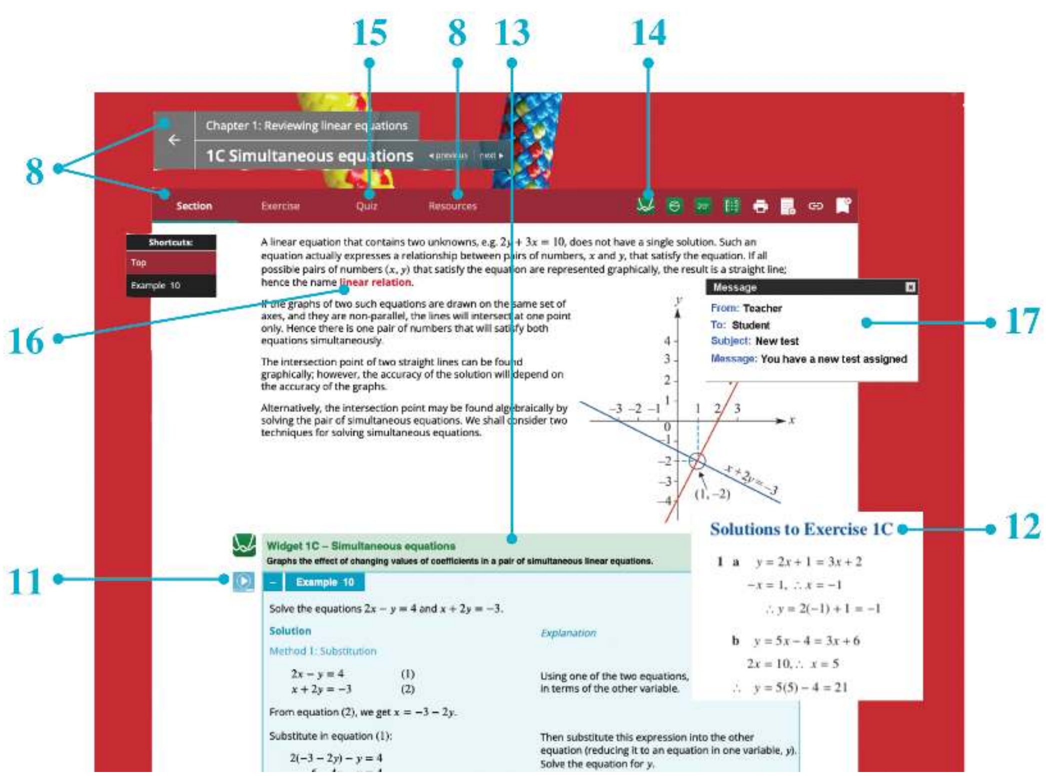
**15** **Quick quizzes** containing automarked multiple-choice questions have been thoroughly expanded and revised, enabling students to check their understanding.

**16** **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

**17** Messages from the teacher assign tasks and tests.

## INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

*A selection of features is shown. Numbers refer to the descriptions on pages ix–x. HOTmaths platform features are updated regularly*





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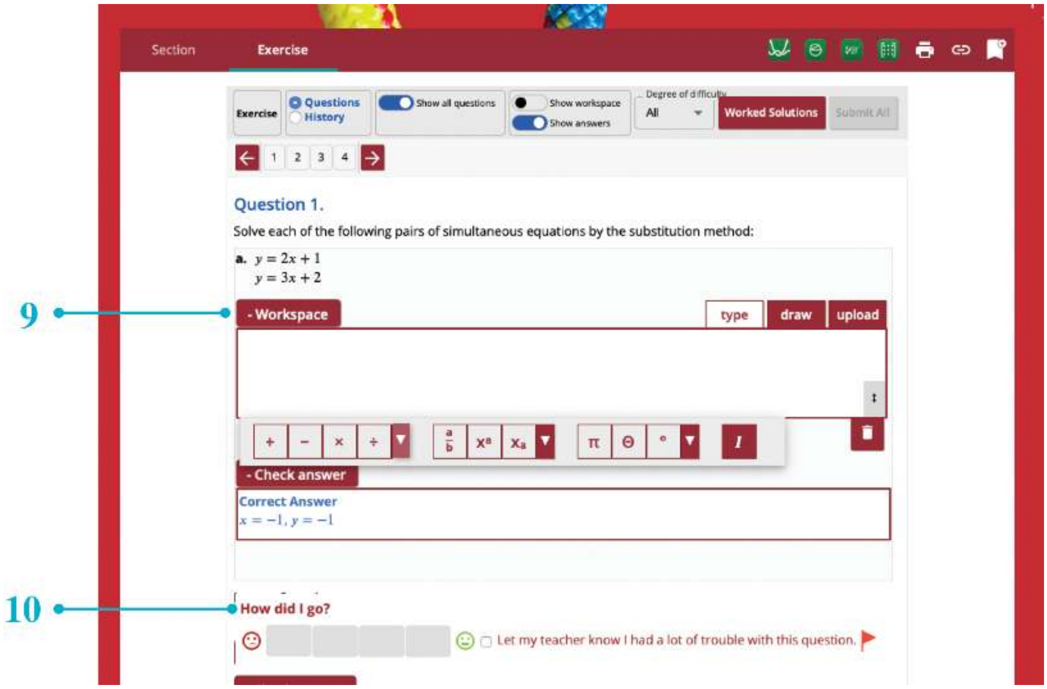
## WORKSPACES AND SELF-ASSESSMENT




## Overview of the Online Teaching Suite powered by the HOTmaths platform

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

**18** The HOTmaths learning management system with class and student analytics and reports, and communication tools.

**19** Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.

**20** A HOTmaths-style test generator.

**21** A suite of chapter tests and assignments.

**22** Editable curriculum grids and teaching programs.

**23** A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).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## More about the Exam Generator

The Online Teaching Suite includes a comprehensive bank of SCSA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- SCSA marking scheme
- All custom exams can be printed and completed under exam-like conditions or used as revision.


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# 1 Principles of counting

## In this chapter



## Syllabus references

**Topics:** Permutations (ordered arrangements); The pigeon-hole principle; Combinations (unordered selections)

**Subtopics:** 1.2.1 – 1.2.4, 1.2.6 – 1.2.9

Take a deck of 52 playing cards. This simple, familiar deck can be arranged in so many ways that if you and every other living human were to shuffle a deck once per second from the beginning of time, then by now only a tiny fraction of all possible arrangements would have been obtained. So, remarkably, every time you shuffle a deck you are likely to be the first person to have created that particular arrangement of cards!

To see this, note that we have 52 choices for the first card, and then 51 choices for the second card, and so on. This gives a total of

$$52 \times 51 \times \cdots \times 2 \times 1 \approx 8.1 \times 10^{67}$$

arrangements. This is quite an impressive number, especially in light of the fact that the universe is estimated to be merely $1.4 \times 10^{10}$ years old.

Combinatorics is concerned with counting the number of ways of doing something. Our goal is to find clever ways of doing this without explicitly listing all the possibilities. This is particularly important in the study of probability. For instance, we can use combinatorics to explain why certain poker hands are more likely to occur than others without considering all 2 598 960 possible hands.


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# 1A Basic counting methods

## Tree diagrams

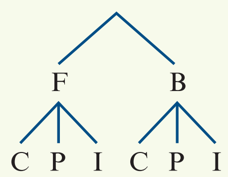
In most combinatorial problems, we are interested in the *number of solutions* to a given problem, rather than the solutions themselves. Nonetheless, for simple counting problems it is sometimes practical to list and then count all the solutions. Tree diagrams provide a systematic way of doing this, especially when the problem involves a small number of steps.

### Example 1

A restaurant has a fixed menu, offering a choice of fish or beef for the main meal, and cake, pudding or ice-cream for dessert. How many different meals can be chosen?

**Solution**

We illustrate the possibilities on a tree diagram:




This gives six different meals, which we can write as

FC, FP, FI, BC, BP, BI

## The multiplication principle

In the above example, for each of the two ways of selecting the main meal, there were three ways of selecting the dessert. This gives a total of $2 \times 3 = 6$ ways of choosing a meal. This is an example of the **multiplication principle**, which will be used extensively throughout this chapter.

**Multiplication principle**

If there are $m$ ways of performing one task and then there are $n$ ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

### Example 2

Sandra has three different skirts, four different tops and five different pairs of shoes. How many choices does she have for a complete outfit?

| Solution | Explanation |
|---|---|
| $3 \times 4 \times 5 = 60$ | Using the multiplication principle, we multiply the number of ways of making each choice. |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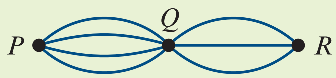
**Example 3**

How many paths are there from point $P$ to point $R$ travelling from left to right?




**Solution**

$4 \times 3 = 12$

**Explanation**

For each of the four paths from $P$ to $Q$, there are three paths from $Q$ to $R$.

## The addition principle

In some instances, we have to count the number of ways of choosing between two alternative tasks. In this case, we use the **addition principle**.

**Addition principle**

Suppose there are $m$ ways of performing one task and $n$ ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

**Example 4**

To travel from Canberra to Sydney tomorrow, Kara has a choice between three different flights and two different trains. How many choices does she have?

**Solution**

$3 + 2 = 5$

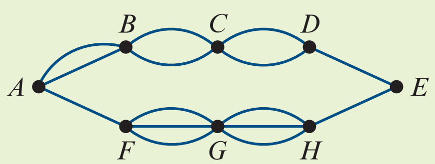
**Explanation**

The addition principle applies because Kara cannot travel by both plane and train. Therefore, we add the number of ways of making each choice.

Some problems will require use of both the multiplication and the addition principles.

**Example 5**

How many paths are there from point $A$ to point $E$ travelling from left to right?




**Solution**

We can take *either* an upper path *or* a lower path:

- Going from $A$ to $B$ to $C$ to $D$ to $E$ there are $2 \times 2 \times 2 \times 1 = 8$ paths.
- Going from $A$ to $F$ to $G$ to $H$ to $E$ there are $1 \times 3 \times 3 \times 1 = 9$ paths.

Using the addition principle, there is a total of $8 + 9 = 17$ paths from $A$ to $E$.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## Harder problems involving tree diagrams

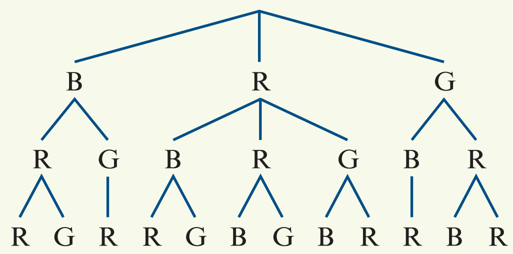
For some problems, a straightforward application of the multiplication and addition principles is not possible.

### Example 6

A bag contains one blue token, two red tokens and one green token. Three tokens are removed from the bag and placed in a row. How many arrangements are possible?

#### Solution

The three tokens are selected without replacement. So once a blue or green token is taken, these cannot appear again. We use a tree diagram to systematically find every arrangement.




The complete set of possible arrangements can be read by tracing out each path from top to bottom of the diagram. This gives 12 different arrangements:

BRR, BRG, BGR, RBR, RBG, RRB, RRG, RGB, RGR, GBR, GRB, GRR

### Summary 1A

Three useful approaches to solving simple counting problems:

- **Tree diagrams**
  These can be used to systematically list all solutions to a problem.
- **Multiplication principle**
  If there are $m$ ways of performing one task and then there are $n$ ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.
- **Addition principle**
  Suppose there are $m$ ways of performing one task and $n$ ways of performing another task. If we cannot perform both tasks, there are $m + n$ ways to perform one of the tasks.

Some problems require use of both the addition and the multiplication principles.

Skill-sheet

## Exercise 1A

Example 2

1 Sam has five T-shirts, three pairs of pants and three pairs of shoes. How many different outfits can he assemble using these clothes?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Example 4

**2** A restaurant offers five beef dishes and three chicken dishes. How many selections of one main meal does a customer have?

**3** Each of the 10 boys at a party shakes hands with each of the 12 girls. How many handshakes take place?

**4** Draw a tree diagram showing all the two-digit numbers that can be formed using the digits 7, 8 and 9 if each digit:

**a** cannot be repeated **b** can be repeated.

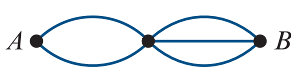
**5** How many different three-digit numbers can be formed using the digits 2, 4 and 6 if each digit can be used:

**a** as many times as you would like **b** at most once?

**6** Jack wants to travel from Sydney to Perth via Adelaide. There are four flights and two trains from Sydney to Adelaide. There are two flights and three trains from Adelaide to Perth. How many ways can Jack travel from Sydney to Perth?

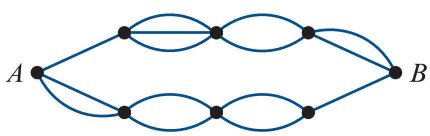
**7** Travelling from left to right, how many paths are there from point $A$ to point $B$ in each of the following diagrams?

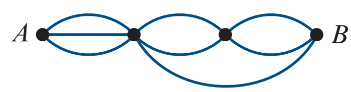
Example 3

**a** 


**b** 


Example 5

**c** 


**d** 


Example 6

**8** A bag contains two blue, one red and two green tokens. Two tokens are removed from the bag and placed in a row. With the help of a tree diagram, list all the different arrangements.

**9** How many ways can you make change for 50 cents using 5, 10 and 20 cent pieces?

**10** Four teachers decide to swap desks at work. How many ways can this be done if no teacher sits at their previous desk?

**11** Three runners compete in a race. In how many ways can the runners complete the race assuming:

**a** there are no tied places **b** the runners can tie places?

**12** A six-sided die has faces labelled with the numbers 0, 2, 3, 5, 7 and 11. If the die is rolled twice and the two results are multiplied, how many different answers can be obtained?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# 1B Factorial notation and permutations

## Factorial notation

**Factorial notation** provides a convenient way of expressing products of consecutive natural numbers. For each natural number $n$, we define

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$$

where the notation $n!$ is read as '$n$ factorial'.

We also define $0! = 1$. Although it might seem strange at first, this definition will turn out to be very convenient, as it is compatible with formulas that we will establish shortly.

Another very useful identity is

$$n! = n \cdot (n-1)!$$

### Example 7

Evaluate:

**a** $3!$

**b** $\dfrac{50!}{49!}$

**c** $\dfrac{10!}{2!\,8!}$

**Solution**

**a** $3! = 3 \cdot 2 \cdot 1$
$= 6$

**b** $\dfrac{50!}{49!} = \dfrac{50 \cdot 49!}{49!}$
$= 50$

**c** $\dfrac{10!}{2!\,8!} = \dfrac{10 \cdot 9 \cdot 8!}{2! \cdot 8!}$
$= \dfrac{10 \cdot 9}{2 \cdot 1}$
$= 45$

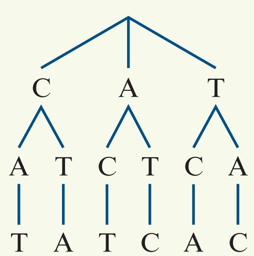
## Permutations of $n$ objects

A **permutation** is an ordered arrangement of a collection of objects.

### Example 8

Using a tree diagram, list all the permutations of the letters in the word CAT.

**Solution**




There are six permutations:

CAT, CTA, ACT, ATC, TCA, TAC

**Explanation**

There are three choices for the first letter. This leaves only two choices for the second letter, and then one for the third.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Another way to find the number of permutations for the previous example is to draw three boxes, corresponding to the three positions. In each box, we write the number of choices we have for that position.

- We have 3 choices for the first letter (C, A or T).
- We have 2 choices for the second letter (because we have already used one letter).
- We have 1 choice for the third letter (because we have already used two letters).

| | | |
|---|---|---|
| 3 | 2 | 1 |

By the multiplication principle, the total number of arrangements is

$$3 \times 2 \times 1 = 3!$$

So three objects can be arranged in 3! ways. More generally:

> The number of permutations of $n$ objects is $n!$.

**Proof** The reason for this is simple:

- The first item can be chosen in $n$ ways.
- The second item can be chosen in $n - 1$ ways, since only $n - 1$ objects remain.
- The third item can be chosen in $n - 2$ ways, since only $n - 2$ objects remain.

⋮

- The last item can be chosen in 1 way, since only 1 object remains.

Therefore, by the multiplication principle, there are

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$$

permutations of $n$ objects.

### Example 9

How many ways can six different books be arranged on a shelf?

**Solution**

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

**Explanation**

Six books can be arranged in 6! ways.

### Example 10

Using your calculator, find how many ways 12 students can be lined up in a row.

**Using the TI-Nspire**

Evaluate 12! as shown.


Note: The factorial symbol (!) can be accessed using ?!▸, the Symbols palette ( ctrl 📖 ) or menu > **Probability** > **Factorial**.


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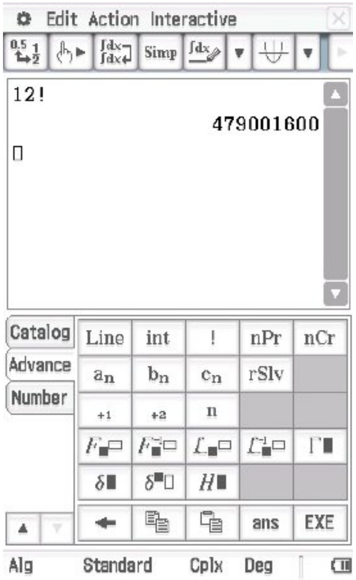
### Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, open the keyboard.
- Enter the number 12, followed by the factorial symbol. Tap EXE.

Note: The factorial symbol (!) is found in the Advance keyboard; you need to scroll down to see this.




**Example 11**

How many four-digit numbers can be formed using the digits 1, 2, 3 and 4 if:

**a** they cannot be repeated

**b** they can be repeated?

**Solution**

**a** $4! = 4 \times 3 \times 2 \times 1 = 24$

**b** $4^4 = 4 \times 4 \times 4 \times 4 = 256$

**Explanation**

Four numbers can be arranged in 4! ways.

Using the multiplication principle, there are 4 choices for each of the 4 digits.

## Permutations of $n$ objects taken $r$ at a time

Imagine a very small country with very few cars. Licence plates consist of a sequence of four digits, and repetitions of the digits are not allowed. How many such licence plates are there?

Here, we are asking for the number of permutations of 10 digits taken four at a time. We will denote this number by $^{10}P_4$.

To solve this problem, we draw four boxes. In each box, we write the number of choices we have for that position. For the first digit, we have a choice of 10 digits. Once chosen, we have only 9 choices for the second digit, then 8 choices for the third and 7 choices for the fourth.

| | | | |
|---|---|---|---|
| 10 | 9 | 8 | 7 |

By the multiplication principle, the total number of licence plates is

$$10 \times 9 \times 8 \times 7$$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There is a clever way of writing this product as a fraction involving factorials:

$$\begin{aligned} {}^{10}P_4 &= 10 \cdot 9 \cdot 8 \cdot 7 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{10!}{6!} \\ &= \frac{10!}{(10-4)!} \end{aligned}$$

More generally:

**Number of permutations**

The number of permutations of $n$ objects taken $r$ at a time is denoted by ${}^nP_r$ and is given by the formula

$${}^nP_r = \frac{n!}{(n-r)!}$$

**Proof** To establish this formula we note that:

- The 1st item can be chosen in $n$ ways.
- The 2nd item can be chosen in $n - 1$ ways.

⋮

- The $r$th item can be chosen in $n - r + 1$ ways.

Therefore, by the multiplication principle, the number of permutations of $n$ objects taken $r$ at a time is

$$\begin{aligned} {}^nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\ &= \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \cdot (n-r) \cdot \cdots \cdot 2 \cdot 1}{(n-r) \cdot \cdots \cdot 2 \cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

**Notes:**

- If $r = n$, then we have ${}^nP_n$, which is simply the number of permutations of $n$ objects and so must equal $n!$. The formula still works in this instance, since

$$\begin{aligned} {}^nP_n &= \frac{n!}{(n-n)!} \\ &= \frac{n!}{0!} \\ &= n! \end{aligned}$$

  Note that this calculation depends crucially on our decision to define $0! = 1$.
- If $r = 1$, then we obtain ${}^nP_1 = n$. Given $n$ objects, there are $n$ choices of one object, and each of these can be arranged in just one way.


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## Example 12

**a** Using the letters A, B, C, D and E without repetition, how many different two-letter arrangements are there?

**b** Six runners compete in a race. In how many ways can the gold, silver and bronze medals be awarded?

### Solution

**a** There are five letters to arrange in two positions:

$$\begin{aligned} {}^5P_2 &= \frac{5!}{(5-2)!} \\ &= \frac{5!}{3!} \\ &= \frac{5 \cdot 4 \cdot 3!}{3!} \\ &= 20 \end{aligned}$$

**b** There are six runners to arrange in three positions:

$$\begin{aligned} {}^6P_3 &= \frac{6!}{(6-3)!} \\ &= \frac{6!}{3!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 120 \end{aligned}$$

Although the formula developed for ${}^nP_r$ will have an important application later in this chapter, you do not actually have to use it when solving problems. It is often more convenient to simply draw boxes corresponding to the positions, and to write in each box the number of choices for that position.

## Example 13

How many ways can seven friends sit along a park bench with space for only four people?

### Solution

| | | | |
|---|---|---|---|
| 7 | 6 | 5 | 4 |

By the multiplication principle, the total number of arrangements is

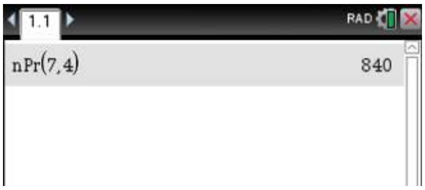
$$7 \times 6 \times 5 \times 4 = 840$$

### Explanation

We draw four boxes, representing the positions to be filled. In each box we write the number of ways we can fill that position.

### Using the TI-Nspire

- To evaluate ${}^7P_4$, use (menu) > **Probability** > **Permutations** as shown.




Note: Alternatively, you can simply type npr(7, 4). The command is not case sensitive.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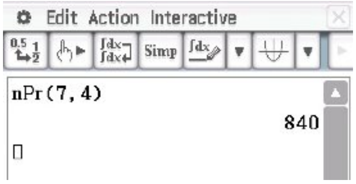
## Using the Casio ClassPad

To evaluate $^7P_4$:

- In Main $\sqrt{\alpha}$, select nPr from the Advance keyboard. (You need to scroll down to find this keyboard.)
- In the brackets, enter the numbers 7 and 4, separated by a comma. Then tap EXE.




## Summary 1B

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$
- $n! = n \cdot (n-1)!$
- A **permutation** is an ordered arrangement of objects.
- The number of permutations of $n$ objects is $n!$.
- The number of permutations of $n$ objects taken $r$ at a time is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

## Exercise 1B

**1** Evaluate $n!$ for $n = 0, 1, 2, \dots, 10$.

Example 7

**2** Evaluate each of the following:

**a** $\dfrac{5!}{4!}$ **b** $\dfrac{10!}{8!}$ **c** $\dfrac{12!}{10!\,2!}$ **d** $\dfrac{100!}{97!\,3!}$

**3** Simplify the following expressions:

**a** $\dfrac{(n+1)!}{n!}$ **b** $\dfrac{(n+2)!}{(n+1)!}$ **c** $\dfrac{n!}{(n-2)!}$ **d** $\dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

**4** Evaluate $^4P_r$ for $r = 0, 1, 2, 3, 4$.

Example 8

**5** Use a tree diagram to find all the permutations of the letters in the word DOG.

Example 9

**6** How many ways can five books on a bookshelf be arranged?

**7** How many ways can the letters in the word HYPERBOLA be arranged?

Example 12

**8** Write down all the two-letter permutations of the letters in the word FROG.

Example 13

**9** How many ways can six students be arranged along a park bench if the bench has:

**a** six seats **b** five seats **c** four seats?

**10** Using the digits 1, 2, 5, 7 and 9 without repetition, how many numbers can you form that have:

**a** five digits **b** four digits **c** three digits?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**11** How many ways can six students be allocated to eight vacant desks?

**12** How many ways can three letters be posted in five mailboxes if each mailbox can receive:

**a** more than one letter **b** at most one letter?

**13** Using six differently coloured flags without repetition, how many signals can you make using:

**a** three flags in a row **b** four flags in a row **c** five flags in a row?

**14** You are in possession of four flags, each coloured differently. How many signals can you make using at least two flags arranged in a row?

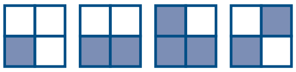
**15** Many Australian car licence plates consist of a sequence of three letters followed by a sequence of three digits.

**a** How many different car licence plates have letters and numbers arranged this way?

**b** How many of these have no repeated letters or numbers?

**16** **a** The three tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?

**b** The four tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



**17** Find all possible values of $m$ and $n$ if $m! \cdot n! = 720$ and $m > n$.

**18** Show that $n! = (n^2 - n) \cdot (n - 2)!$ for $n \geq 2$.

**19** Given six different colours, how many ways can you paint a cube so that all the faces have different colours? Two colourings are considered to be the same when one can be obtained from the other by rotating the cube.

## 1C Permutations with restrictions

Suppose we want to know how many three-digit numbers have no repeated digits. The answer is *not* simply ${}^{10}P_3$, the number of permutations of 10 digits taken three at a time. This is because the digit 0 cannot be used in the hundreds place.

- There are 9 choices for the first digit (1, 2, 3, . . . , 9).
- There are 9 choices for the second digit (0 and the eight remaining non-zero digits).
- This leaves 8 choices for the third digit.

| 100s | 10s | units |
|---|---|---|
| 9 | 9 | 8 |

By the multiplication principle, there are $9 \times 9 \times 8 = 648$ different three-digit numbers.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When considering permutations with restrictions, we deal with the restrictions first.

## Example 14

**a** How many arrangements of the word DARWIN begin and end with a vowel?

**b** Using the digits 0, 1, 2, 3, 4 and 5 without repetition, how many odd four-digit numbers can you form?

### Solution

**a** We draw six boxes. In each box, we write the number of choices we have for that position. We first consider restrictions. There are two choices of vowel (A or I) for the first letter, leaving only one choice for the last letter.

| | | | | | |
|---|---|---|---|---|---|
| 2 | | | | | 1 |

This leaves four choices for the second letter, three for the next, and so on.

| | | | | | |
|---|---|---|---|---|---|
| 2 | 4 | 3 | 2 | 1 | 1 |

By the multiplication principle, the number of arrangements is

$$2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$$

**b** We draw four boxes. Again, we first consider restrictions. The last digit must be odd (1, 3 or 5), giving three choices. We cannot use 0 in the first position, so this leaves four choices for that position.

| | | | |
|---|---|---|---|
| 4 | | | 3 |

Once these two digits have been chosen, this leaves four choices and then three choices for the remaining two positions.

| | | | |
|---|---|---|---|
| 4 | 4 | 3 | 3 |

Thus the number of arrangements is

$$4 \times 4 \times 3 \times 3 = 144$$

## Permutations with items grouped together

For some arrangements, we may want certain items to be grouped together. In this case, the trick is to initially treat each group of items as a single object. We then multiply by the numbers of arrangements within each group.

## Example 15

**a** How many arrangements of the word EQUALS are there if the vowels are kept together?

**b** How many ways can two chemistry, four physics and five biology books be arranged on a shelf if the books of each subject are kept together?


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| Solution | Explanation |
| --- | --- |
| **a** $4! \times 3! = 144$ | We group the three vowels together so that we have four items to arrange: (E, U, A), Q, L, S. They can be arranged in 4! ways. Then the three vowels can be arranged among themselves in 3! ways. We use the multiplication principle. |
| **b** $3! \times 2! \times 4! \times 5! = 34\,560$ | There are three groups and so they can be arranged in 3! ways. The two chemistry books can be arranged among themselves in 2! ways, the four physics books in 4! ways and the five biology books in 5! ways. We use the multiplication principle. |

## Summary 1C

- To count permutations that are subject to restrictions, we draw a series of boxes. In each box, we write the number of choices we have for that position. We always consider the restrictions first.
- When items are to be grouped together, we initially treat each group as a single object. We find the number of arrangements of the groups, and then multiply by the numbers of arrangements within each group.

Skill-sheet

## Exercise 1C

Example 14

**1** Using the digits 1, 2, 3, 4 and 5 without repetition, how many five-digit numbers can you form:

**a** without restriction
**b** that are odd
**c** that begin with 5
**d** that do not begin with 5?

Example 15

**2** In how many ways can three girls and two boys be arranged in a row:

**a** without restriction
**b** if the two boys sit together
**c** if the two boys do not sit together
**d** if girls and boys alternate?

**3** How many permutations of the word QUEASY:

**a** begin with a vowel
**b** begin and end with a vowel
**c** keep the vowels together
**d** keep the vowels and consonants together?

**4** How many ways can four boys and four girls be arranged in a row if:

**a** boys and girls sit in alternate positions
**b** boys sit together and girls sit together?

**5** The digits 0, 1, 2, 3, 4 and 5 can be combined without repetition to form new numbers. In how many ways can you form:

**a** a six-digit number
**b** a four-digit number divisible by 5
**c** a number less than 6000
**d** an even three-digit number?


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**6** Two parents and four children are seated in a cinema along six consecutive seats. How many ways can this be done:

- **a** without restriction
- **b** if the two parents sit at either end
- **c** if the children sit together
- **d** if the parents sit together and the children sit together
- **e** if the youngest child must sit between and next to both parents?

**7** 12321 is a **palindromic number** because it reads the same backwards as forwards. How many palindromic numbers have:

- **a** five digits
- **b** six digits?

**8** How many arrangements of the letters in VALUE do not begin and end with a vowel?

**9** Using each of the digits 1, 2, 3 and 4 at most once, how many even numbers can you form?

**10** How many ways can six girls be arranged in a row so that two of the girls, $A$ and $B$:

- **a** do not sit together
- **b** have one person between them?

**11** How many ways can three girls and three boys be arranged in a row if no two girls sit next to each other?

## 1D Permutations of like objects

The name for the Sydney suburb of WOOLLOOMOOLOO has the unusual distinction of having 13 letters in total, of which only four are different. Finding the number of permutations of the letters in this word is not as simple as evaluating 13!. This is because switching like letters does not result in a new permutation.

Our aim is to find an expression for $P$, where $P$ is the number of permutations of the letters in the word WOOLLOOMOOLOO. First notice that the word has

1 letter W, 1 letter M, 3 letter Ls, 8 letter Os

Replace the three identical Ls with $L_1$, $L_2$ and $L_3$. These three letters can be arranged in 3! different ways. Therefore, by the multiplication principle, there are now

$P \cdot 3!$ permutations.

Likewise, replace the eight identical Os with $O_1, O_2, \ldots, O_8$. These eight letters can be arranged in 8! different ways. Therefore there are now

$P \cdot 3! \cdot 8!$ permutations.


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On the other hand, notice that the 13 letters are now distinct, so there are 13! permutations of these letters. Therefore

$$P \cdot 3! \cdot 8! = 13! \quad \text{and so} \quad P = \frac{13!}{3!\,8!}$$

We can easily generalise this procedure to give the following result.

### Permutations of like objects

The number of permutations of $n$ objects of which $n_1$ are alike, $n_2$ are alike, ... and $n_r$ are alike is given by

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

## Example 16

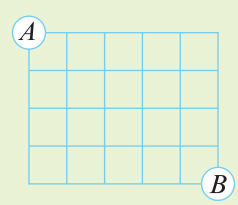
**a** Find the number of permutations of the letters in the word RIFFRAFF.

**b** There are four identical knives, three identical forks and two identical spoons in a drawer. They are taken out of the drawer and lined up in a row. How many ways can this be done?

| Solution | Explanation |
|---|---|
| **a** $\frac{8!}{4!\,2!} = 840$ | There are 8 letters of which 4 are alike and 2 are alike. |
| **b** $\frac{9!}{4!\,3!\,2!} = 1260$ | There are 9 items of which 4 are alike, 3 are alike and 2 are alike. |

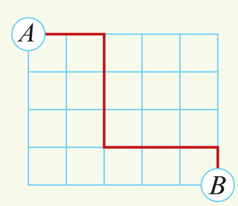
## Example 17

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point $A$ to point $B$?




### Solution

Each path from $A$ to $B$ can be described by a sequence of four Ds and five Rs in some order. For example, the path shown can be described by the sequence RRDDDRRRD.




There are

$$\frac{9!}{4!\,5!} = 126$$

permutations of these letters, since there are 9 letters of which 4 are alike and 5 are alike.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## Summary 1D

- Switching like objects does not give a new arrangement.
- The number of permutations of $n$ objects of which $n_1$ are alike, $n_2$ are alike, ... and $n_r$ are alike is given by

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

Skill-sheet

## Exercise 1D

Example 16

**1** Ying has four identical 20 cent pieces and three identical 10 cent pieces. How many ways can she arrange these coins in a row?

**2** How many ways can the letters in the word MISSISSIPPI be arranged?

**3** Find the number of permutations of the letters in the word WARRNAMBOOL.

**4** Using five 9s and three 7s, how many eight-digit numbers can be made?

**5** Using three As, four Bs and five Cs, how many sequences of 12 letters can be made?

**6** How many ways can two red, two black and four blue flags be arranged in a row:

**a** without restriction

**b** if the first flag is red

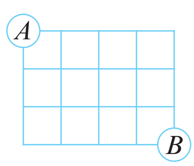
**c** if the first and last flags are blue

**d** if every alternate flag is blue

**e** if the two red flags are adjacent?

Example 17

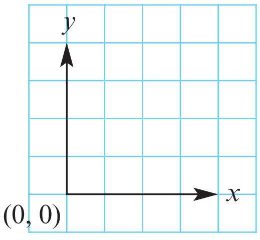
**7** The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point $A$ to point $B$?




**8** The grid shown consists of unit squares. By travelling only along the grid lines, how many paths are there:

**a** of length 6 from $(0, 0)$ to the point $(2, 4)$

**b** of length $m + n$ from $(0, 0)$ to the point $(m, n)$, where $m$ and $n$ are natural numbers?




**9** Consider a deck of 52 playing cards.

**a** How many ways can the deck be arranged? Express your answer in the form $a!$.

**b** If two identical decks are combined, how many ways can the cards be arranged? Express your answer in the form $\dfrac{a!}{(b!)^c}$.

**c** If $n$ identical decks are combined, find an expression for the number of ways that the cards can be arranged.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**10** An ant starts at position (0, 0) and walks north, east, south or west, one unit at a time. How many different paths of length 8 units finish at (0, 0)?

**11** Jessica is about to walk up a flight of 10 stairs. She can take either one or two stairs at a time. How many different ways can she walk up the flight of stairs?

# 1E Combinations

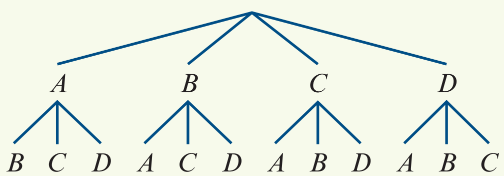
We have seen that a permutation is an ordered arrangement of objects. In contrast, a **combination** is a selection made regardless of order. We use the notation ${}^nP_r$ to denote the number of permutations of $n$ distinct objects taken $r$ at a time. Similarly, we use the notation ${}^nC_r$ to denote the number of combinations of $n$ distinct objects taken $r$ at a time.

**Example 18**

How many ways can two letters be chosen from the set $\{A, B, C, D\}$?

**Solution**

The tree diagram below shows the ways that the first and second choices can be made.




This gives 12 arrangements. But there are only six selections, since

$\{A, B\}$ is the same as $\{B, A\}$, $\{A, C\}$ is the same as $\{C, A\}$, $\{A, D\}$ is the same as $\{D, A\}$,
$\{B, C\}$ is the same as $\{C, B\}$, $\{B, D\}$ is the same as $\{D, B\}$ $\{C, D\}$ is the same as $\{D, C\}$

Suppose we want to count the number of ways that three students can be chosen from a group of seven. Let's label the students with the letters $\{A, B, C, D, E, F, G\}$. One such combination might be $BDE$. Note that this combination corresponds to 3! permutations:

$$BDE, \quad BED, \quad DBE, \quad DEB, \quad EBD, \quad EDB$$

In fact, each combination of three items corresponds to 3! permutations, and so there are 3! times as many permutations as combinations. Therefore

$${}^7P_3 = 3! \times {}^7C_3 \quad \text{and so} \quad {}^7C_3 = \frac{{}^7P_3}{3!}$$

Since we have already established that ${}^7P_3 = \dfrac{7!}{(7-3)!}$, we obtain

$${}^7C_3 = \frac{7!}{3!\,(7-3)!}$$

This argument generalises easily so that we can establish a formula for ${}^nC_r$.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### Number of combinations

The number of combinations of $n$ objects taken $r$ at a time is given by the formula

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

## Example 19

**a** A pizza can have three toppings chosen from nine options. How many different pizzas can be made?

**b** How many subsets of $\{1, 2, 3, \dots, 20\}$ have exactly two elements?

### Solution

**a** Three objects are to be chosen from nine options. This can be done in $^{9}C_3$ ways, and

$$\begin{aligned} ^{9}C_3 &= \frac{9!}{3!\,(9-3)!} \\ &= \frac{9!}{3!\,6!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! \cdot 6!} \\ &= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} \\ &= 84 \end{aligned}$$

**b** Two objects are to be chosen from 20 options. This can be done in $^{20}C_2$ ways, and

$$\begin{aligned} ^{20}C_2 &= \frac{20!}{2!\,(20-2)!} \\ &= \frac{20!}{2!\,18!} \\ &= \frac{20 \cdot 19 \cdot 18!}{2! \cdot 18!} \\ &= \frac{20 \cdot 19}{2 \cdot 1} \\ &= 190 \end{aligned}$$

## Example 20

Using your calculator, find how many ways 10 students can be selected from a class of 20 students.

### Using the TI-Nspire

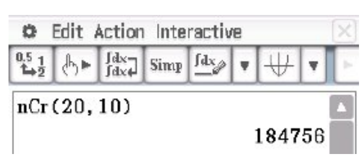
- To evaluate $^{20}C_{10}$, use menu > **Probability** > **Combinations** as shown.


Note: Alternatively, you can simply type ncr(20, 10). The command is not case sensitive.

### Using the Casio ClassPad

To evaluate $^{20}C_{10}$:

- In Main $\sqrt{\alpha}$, select nCr from the Advance keyboard.
- In the brackets, enter the numbers 20 and 10, separated by a comma. Then tap EXE.





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### Example 21

Consider a group of six students. In how many ways can a group of:

**a** two students be selected

**b** four students be selected?

**Solution**

**a** $^6C_2 = \dfrac{6!}{2!\,(6-2)!}$

$= \dfrac{6!}{2!\,4!}$

$= \dfrac{6 \cdot 5 \cdot 4!}{2! \cdot 4!}$

$= \dfrac{6 \cdot 5}{2 \cdot 1}$

$= 15$

**b** $^6C_4 = \dfrac{6!}{4!\,(6-4)!}$

$= \dfrac{6!}{4!\,2!}$

$= \dfrac{6 \cdot 5 \cdot 4!}{4! \cdot 2!}$

$= \dfrac{6 \cdot 5}{2 \cdot 1}$

$= 15$

The fact that parts **a** and **b** of the previous example have the same answer is not a coincidence. Choosing two students out of six is the same as *not choosing* the other four students out of six. Therefore $^6C_2 = {}^6C_4$.

More generally:

$$^nC_r = {}^nC_{n-r}$$

## Quick calculations

In some instances, you can avoid unnecessary calculations by noting that:

- $^nC_0 = 1$, since there is only one way to select no objects from $n$ objects
- $^nC_n = 1$, since there is only one way to select $n$ objects from $n$ objects
- $^nC_1 = n$, since there are $n$ ways to select one object from $n$ objects
- $^nC_{n-1} = n$, since this corresponds to the number of ways of not selecting one object from $n$ objects.

### Example 22

**a** Six points lie on a circle. How many triangles can you make using these points as the vertices?

**b** Each of the 20 people at a party shakes hands with every other person. How many handshakes take place?

| Solution | Explanation |
| --- | --- |
| **a** $^6C_3 = 20$ | This is the same as asking how many ways three vertices can be chosen out of six. |
| **b** $^{20}C_2 = 190$ | This is the same as asking how many ways two people can be chosen to shake hands out of 20 people. |


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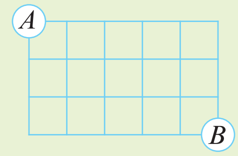
## Example 23

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point $A$ to point $B$?




### Solution

Each path from $A$ to $B$ can be described by a sequence of three Ds and five Rs in some order. Therefore, the number of paths is equal to the number of ways of selecting three of the eight boxes below to be filled with the three Ds. (The rest will be Rs.) This can be done in ${}^8C_3 = 56$ ways.

### Alternative notation

We will consistently use the notation ${}^nC_r$ to denote the number of ways of selecting $r$ objects from $n$ objects, regardless of order. However, it is also common to denote this number by $\binom{n}{r}$.

For example:

$$\binom{6}{4} = \frac{6!}{4!\,2!} = 15$$

### Summary 1E

- A **combination** is a selection made regardless of order.
- The number of combinations of $n$ objects taken $r$ at a time is given by

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

## Exercise 1E

**1** Evaluate ${}^5C_r$ for $r = 0, 1, 2, 3, 4, 5$.

**2** Evaluate each of the following without the use of your calculator:

**a** ${}^7C_1$ **b** ${}^6C_5$ **c** ${}^{12}C_{10}$

**d** ${}^8C_5$ **e** ${}^{100}C_{99}$ **f** ${}^{1000}C_{998}$

**3** Simplify each of the following:

**a** ${}^nC_1$ **b** ${}^nC_2$ **c** ${}^nC_{n-1}$

**d** ${}^{n+1}C_1$ **e** ${}^{n+2}C_n$ **f** ${}^{n+1}C_{n-1}$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Example 19

**4** A playlist contains ten of Nandi's favourite songs. How many ways can he:

**a** arrange three songs in a list

**b** select three songs for a list?

**5** How many ways can five cards be selected from a deck of 52 playing cards?

**6** How many subsets of $\{1, 2, 3, \dots, 10\}$ contain exactly:

**a** 1 element

**b** 2 elements

**c** 8 elements

**d** 9 elements?

**7** A lottery consists of drawing seven balls out of a barrel of balls numbered from 1 to 45. How many ways can this be done if their order does not matter?

Example 22

**8** Eight points lie on a circle. How many triangles can you make using these points as the vertices?

**9** **a** In a hockey tournament, each of the 10 teams plays every other team once. How many games take place?

**b** In another tournament, each team plays every other team once and 120 games take place. How many teams competed?

**10** At a party, every person shakes hands with every other person. Altogether there are 105 handshakes. How many people are at the party?

**11** Prove that ${}^nC_r = {}^nC_{n-r}$.

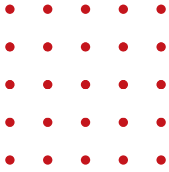
**12** Explain why the number of diagonals in a regular polygon with $n$ sides is ${}^nC_2 - n$.

**13** Ten students are divided into two teams of five. Explain why the number of ways of doing this is $\dfrac{{}^{10}C_5}{2}$.

**14** Twelve students are to be divided into two teams of six. In how many ways can this be done? (Hint: First complete the previous question.)

**15** Using the formula for ${}^nC_r$, prove that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 \le r < n$.

**16** Consider the $5 \times 5$ grid shown.



**a** How many ways can three dots be chosen?

**b** How many ways can three dots be chosen so that they lie on a straight line?

**c** How many ways can three dots be chosen so that they are the vertices of a triangle? (Hint: Use parts **a** and **b**.)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# 1F Combinations with restrictions

## Combinations including specific items

In some problems, we want to find the number of combinations that include specific items. This reduces both the number of items we have to select and the number of items from which we are selecting.

### Example 24

**a** Grace belongs to a group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team?

**b** A hand of cards consists of five cards drawn from a deck of 52 playing cards. How many hands contain both the queen and the king of hearts?

| Solution | Explanation |
|---|---|
| **a** ${}^{7}C_3 = 35$ | Grace must be in the selection. Therefore three more workers are to be selected from the remaining seven workers. |
| **b** ${}^{50}C_3 = 19\,600$ | The queen and king of hearts must be in the selection. So three more cards are to be selected from the remaining 50 cards. |

In some other problems, it can be more efficient to count the selections that we don't want.

### Example 25

Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group, Sam and Tess, do not get along and cannot both be on the team. How many ways can the team be selected?

**Solution**

There are ${}^{8}C_4$ ways of selecting four students from eight. We then subtract the number of combinations that include both Sam and Tess. If Sam and Tess are on the team, then we can select two more students from the six that remain in ${}^{6}C_2$ ways. This gives

$$ {}^{8}C_4 - {}^{6}C_2 = 55 $$

## Combinations from multiple groups

Sometimes we are required to make multiple selections from separate groups. In this case, the multiplication principle dictates that we simply multiply the number of ways of performing each task.

### Example 26

From seven women and four men in a workplace, how many groups of five can be chosen:

**a** without restriction

**b** containing three women and two men

**c** containing at least one man

**d** containing at most one man?


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**Solution**

**a** There are 11 people in total, from which we must select five. This gives

$$^{11}C_5 = 462$$

**b** There are $^7C_3$ ways of selecting three women from seven. There are $^4C_2$ ways of selecting two men from four. We then use the multiplication principle to give

$$^7C_3 \cdot {}^4C_2 = 210$$

**c** **Method 1**

If you select at least one man, then you select 1, 2, 3 or 4 men and fill the remaining positions with women. We use the multiplication and addition principles to give

$$^4C_1 \cdot {}^7C_4 + {}^4C_2 \cdot {}^7C_3 + {}^4C_3 \cdot {}^7C_2 + {}^4C_4 \cdot {}^7C_1 = 441$$

**Method 2**

It is more efficient to consider all selections of 5 people from 11 and then subtract the number of combinations containing all women. This gives

$$^{11}C_5 - {}^7C_5 = 441$$

**d** If there is at most one man, then either there are no men or there is one man. If there are no men, then there are $^7C_5$ ways of selecting all women. If there is one man, then there are $^4C_1$ ways of selecting one man and $^7C_4$ ways of selecting four women. This gives

$$^7C_5 + {}^4C_1 \cdot {}^7C_4 = 161$$

## Permutations and combinations combined

In the following example, we first select the items and then arrange them.

### Example 27

**a** How many arrangements of the letters in the word DUPLICATE can be made that have two vowels and three consonants?

**b** A president, vice-president, secretary and treasurer are to be chosen from a group containing seven women and six men. How many ways can this be done if exactly two women are chosen?

| Solution | Explanation |
|---|---|
| **a** $^4C_2 \cdot {}^5C_3 \cdot 5! = 7200$ | There are $^4C_2$ ways of selecting 2 of 4 vowels and $^5C_3$ ways of selecting 3 of 5 consonants. Once chosen, the 5 letters can be arranged in 5! ways. |
| **b** $^7C_2 \cdot {}^6C_2 \cdot 4! = 7560$ | There are $^7C_2$ ways of selecting 2 of 7 women and $^6C_2$ ways of selecting 2 of 6 men. Once chosen, the 4 people can be arranged into the positions in 4! ways. |


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## Summary 1F

- If a selection must include specific items, then this reduces both the number of items that we have to select and the number of items that we select from.
- If we are required to make multiple selections from separate groups, then we multiply the number of ways of performing each task.
- Some problems will require us to select and then arrange objects.

Skill-sheet

## Exercise 1F

Example 24

**1** Jane and Jenny belong to a class of 20 students. How many ways can you select a group of four students from the class if both Jane and Jenny are to be included?

**2** How many subsets of $\{1, 2, 3, \ldots, 10\}$ have exactly five elements and contain the number 5?

**3** Five cards are dealt from a deck of 52 playing cards. How many hands contain the jack, queen and king of hearts?

Example 25

**4** Six students are to be chosen from a group of 10 students for the school basketball team. Two members of the group, Rachel and Nethra, do not get along and cannot both be on the team. How many ways can the team be selected?

Example 26

**5** From eight girls and five boys, a team of seven is selected for a mixed netball team. How many ways can this be done if:

- **a** there are no restrictions
- **b** there are four girls and three boys on the team
- **c** there must be at least three boys and three girls on the team
- **d** there are at least two boys on the team?

**6** There are 10 student leaders at a secondary school. Four are needed for a fundraising committee and three are needed for a social committee. How many ways can the students be selected if they can serve on:

- **a** both committees
- **b** at most one committee?

**7** There are 18 students in a class. Seven are required for a basketball team and eight are required for a netball team. How many ways can the teams be selected if students can play in:

- **a** both teams
- **b** at most one team?


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**8** From 10 Labor senators and 10 Liberal senators, a committee of five is formed. How many ways can this be done if:

**a** there are no restrictions

**b** there are at least two senators from each political party

**c** there is at least one Labor senator?

**9** Consider the set of numbers $\{1, 2, 3, 4, 5, 6, 7\}$.

**a** How many subsets have exactly five elements?

**b** How many five-element subsets contain the numbers 2 and 3?

**c** How many five-element subsets do not contain both 2 and 3?

**10** Four letters are selected from the English alphabet. How many of these selections will contain exactly two vowels?

**11** A seven-card hand is dealt from a deck of 52 playing cards. How many distinct hands contain:

**a** four hearts and three spades

**b** exactly two hearts and three spades?

**12** A committee of five people is chosen from four doctors, four dentists and three physiotherapists. How many ways can this be done if the committee contains:

**a** exactly three doctors and one dentist

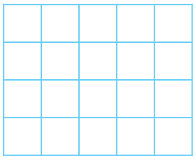
**b** exactly two doctors?

Example 27 **13** There are four girls and five boys. Two of each are chosen and then arranged on a bench. How many ways can this be done?

**14** A president, vice-president, secretary and treasurer are to be chosen from a group containing six women and five men. How many ways can this be done if exactly two women must be chosen?

**15** Using five letters from the word TRAMPOLINE, how many arrangements contain two vowels and three consonants?

**16** How many rectangles are there in the grid shown on the right?
Hint: Every rectangle is determined by a choice of two vertical and two horizontal lines.



**17** Five cards are dealt from a deck of 52 playing cards. A full house is a hand that contains 3 cards of one rank and 2 cards of another rank (for example, 3 kings and 2 sevens). How many ways can a full house be dealt?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# 1G Pascal's triangle

The diagram below consists of the binomial coefficients ${}^nC_r$ for $0 \le n \le 5$. They form the first 6 rows of **Pascal's triangle**, named after the seventeenth century French mathematician Blaise Pascal, one of the founders of probability theory.

Interestingly, the triangle was well known to Chinese and Indian mathematicians many centuries earlier.

| | | |
|---|---|---|
| $n = 0$: | ${}^0C_0$ | 1 |
| $n = 1$: | ${}^1C_0 \quad {}^1C_1$ | 1 1 |
| $n = 2$: | ${}^2C_0 \quad {}^2C_1 \quad {}^2C_2$ | 1 2 1 |
| $n = 3$: | ${}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$ | 1 3 3 1 |
| $n = 4$: | ${}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4$ | 1 4 6 4 1 |
| $n = 5$: | ${}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$ | 1 5 10 10 5 1 |

## Pascal's rule

Pascal's triangle has many remarkable properties. Most importantly:

Each entry in Pascal's triangle is the sum of the two entries immediately above.

Pascal's triangle has this property because of the following identity.

### Pascal's rule

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \qquad \text{where } 1 \le r < n$$

**Proof** In Question 15 of Exercise 1E, you are asked to prove Pascal's rule using the formula for ${}^nC_r$. However, there is a much nicer argument.

The number of subsets of $\{1, 2, \dots, n\}$ containing exactly $r$ elements is ${}^nC_r$. Each of these subsets can be put into one of two groups:

1. those that contain $n$
2. those that do not contain $n$.

If the subset contains $n$, then each of the remaining $r - 1$ elements must be chosen from $\{1, 2, \dots, n-1\}$. Therefore the first group contains ${}^{n-1}C_{r-1}$ subsets.

If the subset does not contain $n$, then we still have to choose $r$ elements from $\{1, 2, \dots, n-1\}$. Therefore the second group contains ${}^{n-1}C_r$ subsets.

The two groups together contain all ${}^nC_r$ subsets and so

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

which establishes Pascal's rule.


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## Example 28

Given that ${}^{17}C_2 = 136$ and ${}^{17}C_3 = 680$, evaluate ${}^{18}C_3$.

**Solution**

$$\begin{aligned}{}^{18}C_3 &= {}^{17}C_2 + {}^{17}C_3\\ &= 136 + 680\\ &= 816\end{aligned}$$

**Explanation**

We let $n = 18$ and $r = 3$ in Pascal's rule:

$${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

## Example 29

Write down the $n = 6$ row of Pascal's triangle and then write down the value of ${}^{6}C_3$.

**Solution**

$n = 6$: 1 6 15 **20** 15 6 1

${}^{6}C_3 = 20$

**Explanation**

Each entry in the $n = 6$ row is the sum of the two entries immediately above.

Note that ${}^{6}C_3$ is the fourth entry in the row, since the first entry corresponds to ${}^{6}C_0$.

## Subsets of a set

Suppose your friend says to you: 'I have five books that I no longer need, take any that you want.' How many different selections are possible?

We will look at two solutions to this problem.

### Solution 1

You could select none of the books (${}^{5}C_0$ ways), or one out of five (${}^{5}C_1$ ways), or two out of five (${}^{5}C_2$ ways), and so on. This gives the answer

$${}^{5}C_0 + {}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 + {}^{5}C_4 + {}^{5}C_5 = 32$$

Note that this is simply the sum of the entries in the $n = 5$ row of Pascal's triangle.

### Solution 2

For each of the five books we have two options: either accept or reject the book. Using the multiplication principle, we obtain the answer

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

There are two important conclusions that we can draw from this example.

1. The sum of the entries in row $n$ of Pascal's triangle is $2^n$. That is,

$${}^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$$

2. A set of size $n$ has $2^n$ subsets, including the empty set and the set itself.


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### Example 30

**a** Your friend offers you any of six books that she no longer wants. How many selections are possible assuming that you take at least one book?

**b** How many subsets of $\{1, 2, 3, \ldots, 10\}$ have at least two elements?

| Solution | Explanation |
|---|---|
| **a** $2^6 - 1 = 63$ | There are $2^6$ subsets of a set of size 6. We subtract 1 because we discard the empty set of no books. |
| **b** $2^{10} - {}^{10}C_1 - {}^{10}C_0$<br>$= 2^{10} - 10 - 1$<br>$= 1013$ | There are $2^{10}$ subsets of a set of size 10. There are ${}^{10}C_1$ subsets containing 1 element and ${}^{10}C_0$ subsets containing 0 elements. |

### Summary 1G

- The values of ${}^nC_r$ can be arranged to give Pascal's triangle.
- Each entry in Pascal's triangle is the sum of the two entries immediately above.
- Pascal's rule: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$
- The sum of the entries in row $n$ of Pascal's triangle is $2^n$. That is,

$${}^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

- A set of size $n$ has $2^n$ subsets, including the empty set and the set itself.

## Exercise 1G

Example 28

**1** Evaluate ${}^7C_2$, ${}^6C_2$ and ${}^6C_1$, and verify that the first is the sum of the other two.

Example 29

**2** Write down the $n = 7$ row of Pascal's triangle. Use your answer to write down the values of ${}^7C_2$ and ${}^7C_4$.

**3** Write down the $n = 8$ row of Pascal's triangle. Use your answer to write down the values of ${}^8C_4$ and ${}^8C_6$.

Example 30

**4** Your friend offers you any of six different DVDs that he no longer wants. How many different selections are possible?

**5** How many subsets does the set $\{A, B, C, D, E\}$ have?

**6** How many subsets does the set $\{1, 2, 3, \ldots, 10\}$ have?

**7** How many subsets of $\{1, 2, 3, 4, 5, 6\}$ have at least one element?

**8** How many subsets of $\{1, 2, 3, \ldots, 8\}$ have at least two elements?

**9** How many subsets of $\{1, 2, 3, \ldots, 10\}$ contain the numbers 9 and 10?


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**10** You have one 5 cent, one 10 cent, one 20 cent and one 50 cent piece. How many different sums of money can you make assuming that at least one coin is used?

**11** Let's call a set **selfish** if it contains its size as an element. For example, the set $\{1, 2, 3\}$ is selfish because the set has size 3 and the number 3 belongs to the set.

**a** How many subsets of $\{1, 2, 3, \ldots, 8\}$ are selfish?

**b** How many subsets of $\{1, 2, 3, \ldots, 8\}$ have the property that both the subset and its complement are selfish?

# 1H The pigeonhole principle

The **pigeonhole principle** is an intuitively obvious counting technique which can be used to prove some remarkably counterintuitive results. It gets its name from the following simple observation: If $n + 1$ pigeons are placed into $n$ holes, then some hole contains at least two pigeons. Obviously, in most instances we will not be working with pigeons, so we will recast the principle as follows.

**Pigeonhole principle**

If $n + 1$ or more objects are placed into $n$ holes, then some hole contains at least two objects.

**Proof** Suppose that each of the $n$ holes contains at most one object. Then the total number of objects is at most $n$, which is a contradiction.

We are now in a position to prove a remarkable fact: There are at least two people in Australia with the same number of hairs on their head. The explanation is simple. No one has more than 1 million hairs on their head, so let's make 1 million holes labelled with the numbers from 1 to 1 million. We now put each of the 24 million Australians into the hole corresponding to the number of hairs on their head. Clearly, some hole contains at least two people, and all the people in that hole will have the same number of hairs on their head.

## Example 31

You have thirteen red, ten blue and eight green socks. How many socks need to be selected at random to ensure that you have a matching pair?

**Solution**

Label three holes with the colours red, blue and green.

| R | B | G |
|---|---|---|

Selecting just three socks is clearly not sufficient, as you might pick one sock of each colour. Select four socks and place each sock into the hole corresponding to the colour of the sock. As there are four socks and three holes, the pigeonhole principle guarantees that some hole contains at least two socks. This is the required pair.

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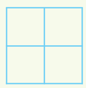
**Example 32**

**a** Show that for any five points chosen inside a $2 \times 2$ square, at least two of them will be no more than $\sqrt{2}$ units apart.

**b** Seven football teams play 22 games of football. Show that some pair of teams play each other at least twice.

**Solution**

**a** Split the $2 \times 2$ square into four unit squares.



Now we have four squares and five points. By the pigeonhole principle, some square contains at least two points. The distance between any two of these points cannot exceed the length of the square's diagonal, $\sqrt{1^2 + 1^2} = \sqrt{2}$.

**b** There are ${}^7C_2 = 21$ ways that two teams can be chosen to compete from seven. There are 22 games of football, and so some pair of teams play each other at least twice.

## The generalised pigeonhole principle

Suppose that 13 pigeons are placed into four holes. By the pigeonhole principle, there is some hole with at least two pigeons. In fact, some hole must contain at least four pigeons. The reason is simple: If each of the four holes contained no more than three pigeons, then there would be no more than 12 pigeons.

This observation generalises as follows.

**Generalised pigeonhole principle**

If at least $mn + 1$ objects are placed into $n$ holes, then some hole contains at least $m + 1$ objects.

**Proof** Again, let's suppose that the statement is false. Then each of the $n$ holes contains no more than $m$ objects. However, this means that there are no more than $mn$ objects, which is a contradiction.

**Example 33**

Sixteen natural numbers are written on a whiteboard. Prove that at least four numbers will leave the same remainder when divided by 5.

**Solution**

We label five holes with each of the possible remainders on division by 5.

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|

There are 16 numbers to be placed into five holes. Since $16 = 3 \times 5 + 1$, there is some hole with at least four numbers, each of which leaves the same remainder when divided by 5.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## Pigeons in multiple holes

In some instances, objects can be placed into more than one hole.

### Example 34

Seven people sit at a round table with 10 chairs. Show that there are three consecutive chairs that are occupied.

**Solution**

Number the chairs from 1 to 10. There are 10 groups of three consecutive chairs:

$$\begin{array}{lllll} \{1,2,3\}, & \{2,3,4\}, & \{3,4,5\}, & \{4,5,6\}, & \{5,6,7\}, \\ \{6,7,8\}, & \{7,8,9\}, & \{8,9,10\}, & \{9,10,1\}, & \{10,1,2\} \end{array}$$

Each of the seven people will belong to three of these groups, and so 21 people have to be allocated to 10 groups. Since $21 = 2 \times 10 + 1$, the generalised pigeonhole principle guarantees that some group must contain three people.

### Summary 1H

- **Pigeonhole principle**
  If $n + 1$ or more objects are placed into $n$ holes, then some hole contains at least two objects.
- **Generalised pigeonhole principle**
  If at least $mn + 1$ objects are placed into $n$ holes, then some hole contains at least $m + 1$ objects.

## Exercise 1H

Example 31

**1** You have twelve red, eight blue and seven green socks. How many socks need to be selected at random to ensure that you have a matching pair?

**2** A sentence contains 27 English words. Show that there are at least two words that begin with the same letter.

**3** Show that in any collection of five natural numbers, at least two will leave the same remainder when divided by 4.

**4** How many cards need to be dealt from a deck of 52 playing cards to be certain that you will obtain at least two cards of the same:

**a** colour **b** suit **c** rank?

**5** Eleven points on the number line are located somewhere between 0 and 1. Show that there are at least two points no more than 0.1 apart.


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Example 32

**6** An equilateral triangle has side length 2 units. Choose any five points inside the triangle. Prove that there are at least two points that are no more than 1 unit apart.

**7** Thirteen points are located inside a rectangle of length 6 and width 8. Show that there are at least two points that are no more than $2\sqrt{2}$ units apart.

**8** The **digital sum** of a natural number is defined to be the sum of its digits. For example, the digital sum of 123 is $1 + 2 + 3 = 6$.

**a** Nineteen two-digit numbers are selected. Prove that at least two of them have the same digital sum.

**b** Suppose that 82 three-digit numbers are selected. Prove that at least four of them have the same digital sum.

Example 33

**9** Whenever Eva writes down 13 integers, she notices that at least four of them leave the same remainder when divided by 4. Explain why this is always the case.

**10** Twenty-nine games of football are played among eight teams. Prove that there is some pair of teams who play each other more than once.

**11** A teacher instructs each member of her class to write down a different whole number between 1 and 49. She says that there will be at least one pair of students such that the sum of their two numbers is 50. How many students must be in her class?

Example 34

**12** There are 10 students seated at a round table with 14 chairs. Show that there are three consecutive chairs that are occupied.

**13** There are four points on a circle. Show that three of these points lie on a half-circle.
Hint: Pick any one of the four points and draw a diameter through that point.

**14** There are 35 players on a football team and each player has a different number chosen from 1 to 99. Prove that there are at least four pairs of players whose numbers have the same sum.

**15** Seven boys and five girls sit evenly spaced at a round table. Prove that some pair of boys are sitting opposite each other.

**16** There are $n$ guests at a party and some of these guests shake hands when they meet. Use the pigeonhole principle to show that there is a pair of guests who shake hands with the same number of people.
Hint: Place the $n$ guests into holes labelled from 0 to $n - 1$, corresponding to the number of hands that they shake. Why must either the first or the last hole be empty?


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Review

Assign-ment

Nrich

## Chapter summary

- The addition and multiplication principles provide efficient methods for counting the number of ways of performing multiple tasks.
- The number of **permutations** (or arrangements) of $n$ objects taken $r$ at a time is given by
$$^nP_r = \frac{n!}{(n-r)!}$$
- The number of **combinations** (or selections) of $n$ objects taken $r$ at a time is given by
$$^nC_r = \frac{n!}{r!\,(n-r)!}$$
- When permutations or combinations involve restrictions, we deal with them first.
- The values of $^nC_r$ can be arranged to give **Pascal's triangle**, where each entry is the sum of the two entries immediately above.
- The sum of the entries in row $n$ of Pascal's triangle is $2^n$. That is,
$$^nC_0 + {^nC_1} + \cdots + {^nC_{n-1}} + {^nC_n} = 2^n$$
- A set of size $n$ has $2^n$ subsets.
- The **pigeonhole principle** is used to show that some pair or group of objects have the same property.

## Short-answer questions

1 Evaluate:
  a $^6C_3$ b $^{20}C_2$ c $^{300}C_1$ d $^{100}C_{98}$

2 Find the value of $n$ if $^nC_2 = 55$.

3 How many three-digit numbers can be formed using the digits 1, 2 and 3 if the digits:
  a can be repeated b cannot be repeated?

4 How many ways can six students be arranged on a bench seat with space for three?

5 How many ways can three students be allocated to five vacant desks?

6 There are four Year 11 and three Year 12 students in a school debating club. How many ways can a team of four be selected if two are chosen from each year level?

7 There are three boys and four girls in a group. How many ways can three children be selected if at least one of them is a boy?

8 On a ship's mast are two identical red and three identical black flags that can be arranged to send messages to nearby ships. How many different arrangements using all five flags are possible?

9 There are 53 English words written on a page. How many are guaranteed to share the same first letter?


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## Extended-response questions

**1** A six-digit number is formed using the digits 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done if:

**a** the first digit is 5

**b** the first digit is even

**c** even and odd digits alternate

**d** the even digits are kept together?

**2** Three letters from the word AUNTIE are arranged in a row. How many ways can this be done if:

**a** the first letter is E

**b** the first letter is a vowel

**c** the letter E is used?

**3** A student leadership team consists of four boys and six girls. A group of four students is required to organise a social function. How many ways can the group be selected:

**a** without restriction

**b** if the school captain is included

**c** if there are two boys

**d** if there is at least one boy?

**4** Consider the eight letters N, N, J, J, T, T, T, T. How many ways can all eight letters be arranged if:

**a** there is no restriction

**b** the first and last letters are both N

**c** the two Js are adjacent

**d** no two Ts are adjacent?

**5** A pizza restaurant offers the following toppings: onion, capsicum, mushroom, olives, ham and pineapple.

**a** How many different kinds of pizza can be ordered with:

**i** three different toppings

**ii** three different toppings including ham

**iii** any number of toppings (between none and all six)?

**b** Another pizza restaurant boasts that they can make more than 200 varieties of pizza. What is the smallest number of toppings that they could use?


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**6** In how many ways can a group of four people be chosen from five married couples if:

**a** there is no restriction

**b** any two women and two men are chosen

**c** any two married couples are chosen

**d** a husband and wife cannot both be selected?

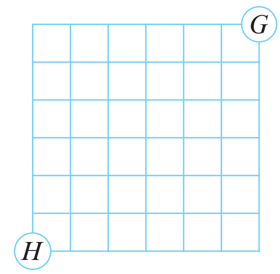
**7** The name David Smith has initials DS.

**a** How many different two-letter initials are possible?

**b** How many different two-letter initials contain at least one vowel?

**c** Given 50 000 people, how many of them can be guaranteed to share the same two-letter initials?

**8** Every morning, Milly walks from her home $H(0, 0)$ to the gym $G(6, 6)$ along city streets that are laid out in a square grid as shown. She always takes a path of shortest distance.




**a** How many paths are there from $H$ to $G$?

**b** Show that there is some path that she takes at least twice in the course of three years.

**c** On her way to the gym, she often purchases a coffee at a cafe located at point $C(2, 2)$. How many paths are there from:

**i** $H$ to $C$

**ii** $C$ to $G$

**iii** $H$ to $C$ to $G$?

**9** A box contains 400 balls, each of which is blue, red, green, yellow or orange. The ratio of blue to red to green balls is 1 : 4 : 2. The ratio of green to yellow to orange balls is 1 : 3 : 6. What is the smallest number of balls that must be drawn to ensure that at least 50 balls of one colour are selected?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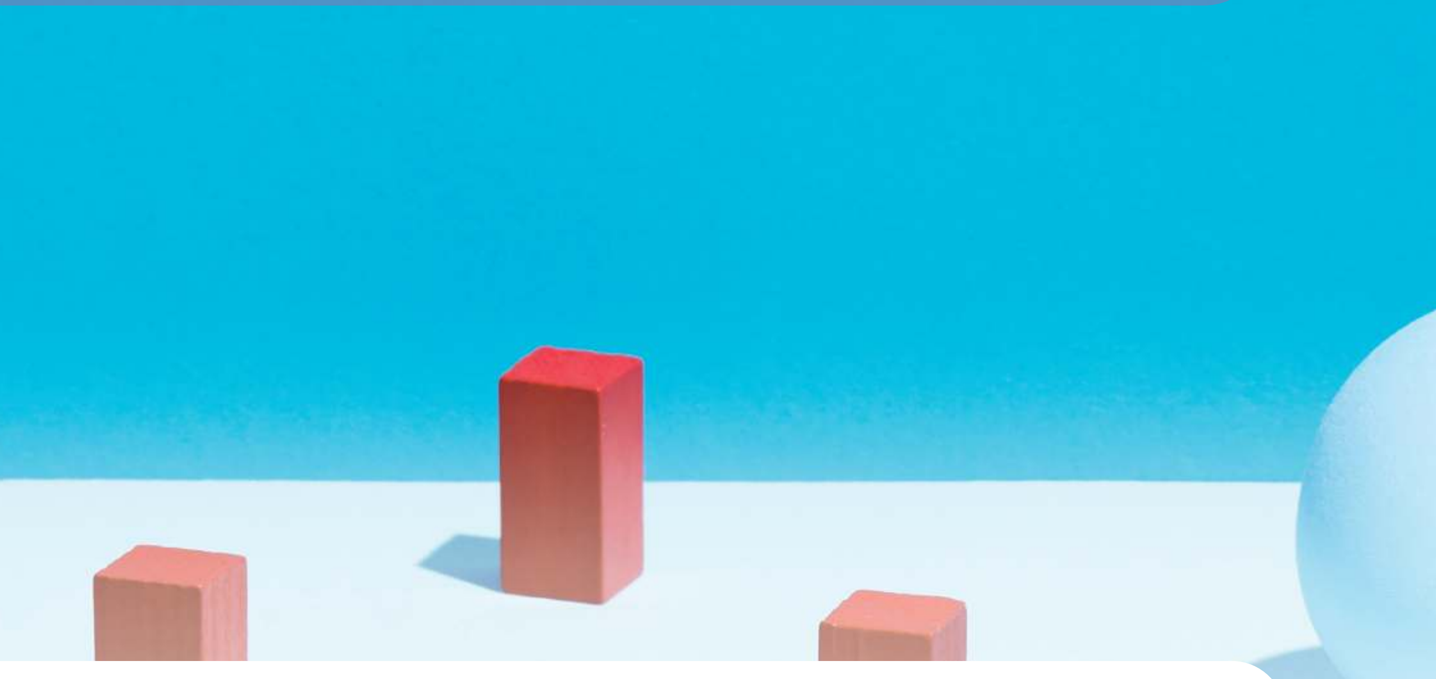
# 2 Number systems and sets



## In this chapter



## Syllabus references

**Topics:** The inclusion-exclusion principle for the union of two sets and three sets; Rational and irrational numbers

**Subtopics:** 1.2.5, 2.3.2

This chapter introduces set notation and discusses sets of numbers and their properties. Set notation is used widely in mathematics and in this book it is employed where appropriate. In this chapter we discuss natural numbers, integers and rational numbers, and then continue on to consider irrational numbers.

Irrational numbers such as $\sqrt{2}$ naturally arise when applying Pythagoras' theorem. When solving a quadratic equation, using the method of completing the square or the quadratic formula, we obtain answers such as $x = \frac{1}{2}(1 \pm \sqrt{5})$. These numbers involve surds.

Since these numbers are irrational, we cannot express them in exact form using decimals or fractions. Sometimes we may wish to approximate them using decimals, but mostly we prefer to leave them in exact form. Thus we need to be able to manipulate these types of numbers and to simplify combinations of them which arise when solving a problem.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# 2A Set notation

A **set** is a general name for any collection of things or numbers. There must be a way of deciding whether any particular object is a member of the set or not. This may be done by referring to a list of the members of the set or a statement describing them.

For example: $A = \{-3, 3\} = \{ x : x^2 = 9 \}$

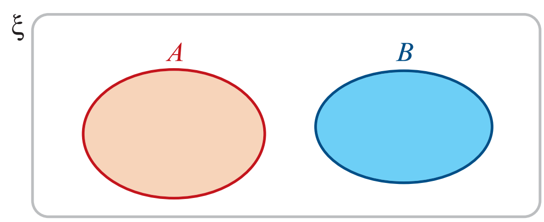
Note: $\{ x : \dots \}$ is read as 'the set of all $x$ such that ...'.

- The symbol $\in$ means 'is a member of' or 'is an element of'.

  For example: $3 \in$ {prime numbers} is read '3 is a member of the set of prime numbers'.
- The symbol $\notin$ means 'is not a member of' or 'is not an element of'.

  For example: $4 \notin$ {prime numbers} is read '4 is not a member of the set of prime numbers'.
- Two sets are **equal** if they contain exactly the same elements, not necessarily in the same order. For example: if $A$ = {prime numbers less than 10} and $B = \{2, 3, 5, 7\}$, then $A = B$.
- The set with no elements is called the **empty set** and is denoted by $\varnothing$.
- The **universal set** will be denoted by $\xi$. The universal set is the set of all elements which are being considered.
- If all the elements of a set $B$ are also elements of a set $A$, then the set $B$ is called a **subset** of $A$. This is written $B \subseteq A$. For example: $\{a, b, c\} \subseteq \{a, b, c, d, e, f, g\}$ and $\{3, 9, 27\} \subseteq$ {multiples of 3}. We note also that $A \subseteq A$ and $\varnothing \subseteq A$.

**Venn diagrams** are used to illustrate sets. For example, the diagram on the right shows two subsets $A$ and $B$ of a universal set $\xi$ such that $A$ and $B$ have no elements in common. Two such sets are said to be **disjoint**.




## The union of two sets

The set of all the elements that are members of set $A$ or set $B$ (or both) is called the **union** of $A$ and $B$. The union of $A$ and $B$ is written as $A \cup B$.

**Example 1**

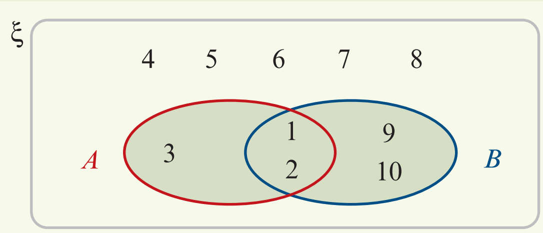
Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$ and $B = \{1, 2, 9, 10\}$.

Find $A \cup B$ and illustrate on a Venn diagram.

**Solution**

$A \cup B = \{1, 2, 3, 9, 10\}$

The shaded area illustrates $A \cup B$.





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## The intersection of two sets

The set of all the elements that are members of both set $A$ and set $B$ is called the **intersection** of $A$ and $B$. The intersection of $A$ and $B$ is written as $A \cap B$.

### Example 2

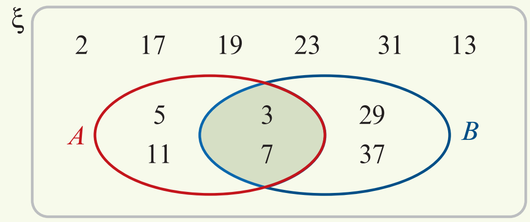
Let $\xi$ = {prime numbers less than 40}, $A = \{3, 5, 7, 11\}$ and $B = \{3, 7, 29, 37\}$.

Find $A \cap B$ and illustrate on a Venn diagram.

**Solution**

$A \cap B = \{3, 7\}$

The shaded area illustrates $A \cap B$.




## The complement of a set

The **complement** of a set $A$ is the set of all elements of $\xi$ that are not members of $A$. The complement of $A$ is denoted by $A'$ (or $\bar{A}$).

If $\xi$ = {students at Highland Secondary College} and $A$ = {students with blue eyes}, then $A'$ is the set of all students at Highland Secondary College who do not have blue eyes.

Similarly, if $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8, 10\}$.

### Example 3

Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A$ = {odd numbers} = $\{1, 3, 5, 7, 9\}$

$B$ = {multiples of 3} = $\{3, 6, 9\}$

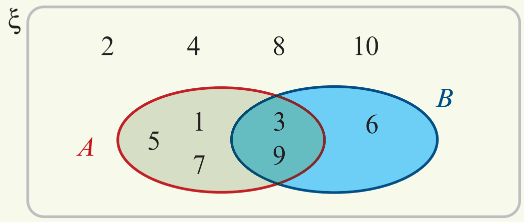
**a** Show these sets on a Venn diagram.

**b** Use the diagram to list the following sets:

**i** $A'$ **ii** $B'$ **iii** $A \cup B$ **iv** the complement of $A \cup B$, i.e. $(A \cup B)'$ **v** $A' \cap B'$

**Solution**

**a**




**b** From the diagram:

**i** $A' = \{2, 4, 6, 8, 10\}$

**ii** $B' = \{1, 2, 4, 5, 7, 8, 10\}$

**iii** $A \cup B = \{1, 3, 5, 6, 7, 9\}$

**iv** $(A \cup B)' = \{2, 4, 8, 10\}$

**v** $A' \cap B' = \{2, 4, 8, 10\}$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## Finite and infinite sets

When all the elements of a set may be counted, the set is called a **finite** set. For example, the set $A$ = {months of the year} is finite. The number of elements of a set $A$ will be denoted by $|A|$. In this example, $|A| = 12$. If $C$ = {letters of the alphabet}, then $|C| = 26$.

Sets which are not finite are called **infinite** sets. For example, the set of real numbers, $\mathbb{R}$, and the set of integers, $\mathbb{Z}$, are infinite sets.

### Summary 2A

- If $x$ is an element of a set $A$, we write $x \in A$.
- If $x$ is not an element of a set $A$, we write $x \notin A$.
- The **empty set** is denoted by $\varnothing$ and the **universal set** by $\xi$.
- If every element of $B$ is an element of $A$, we say $B$ is a **subset** of $A$ and write $B \subseteq A$.
- The set $A \cup B$ is the **union** of $A$ and $B$, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- The set $A \cap B$ is the **intersection** of $A$ and $B$, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The **complement** of $A$, denoted by $A'$, is the set of all elements of $\xi$ that are not in $A$.
- If two sets $A$ and $B$ have no elements in common, we say that they are **disjoint** and write $A \cap B = \varnothing$.

## Exercise 2A

Example 1
Example 2
Example 3

**1** Let $\xi = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 5\}$ and $B = \{2, 4\}$.
Show these sets on a Venn diagram and use the diagram to find:

**a** $A'$ **b** $B'$ **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

**2** Let $\xi$ = {natural numbers less than 17}, $P$ = {multiples of 3} and $Q$ = {even numbers}.
Show these sets on a Venn diagram and use it to find:

**a** $P'$ **b** $Q'$ **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

**3** Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A$ = {multiples of 4} and $B$ = {even numbers}.
Show these sets on a Venn diagram and use this diagram to list the sets:

**a** $A'$ **b** $B'$ **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

**4** Let $\xi$ = {natural numbers from 10 to 25}, $P$ = {multiples of 4} and $Q$ = {multiples of 5}.
Show these sets on a Venn diagram and use this diagram to list the sets:

**a** $P'$ **b** $Q'$ **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

**5** Let $\xi = \{p, q, r, s, t, u, v, w\}$, $X = \{r, s, t, w\}$ and $Y = \{q, s, t, u, v\}$.
Show $\xi$, $X$ and $Y$ on a Venn diagram, entering all members. Hence list the sets:

**a** $X'$ **b** $Y'$ **c** $X' \cap Y'$ **d** $X' \cup Y'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?


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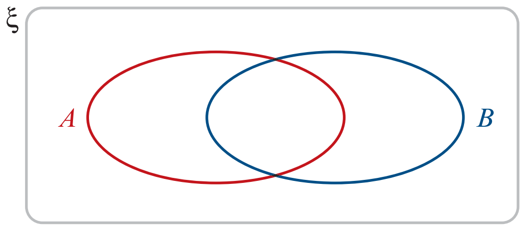
**6** Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $X = \{\text{factors of } 12\}$ and $Y = \{\text{even numbers}\}$. Show $\xi$, $X$ and $Y$ on a Venn diagram, entering all members. Hence list the sets:

**a** $X'$ **b** $Y'$ **c** $X' \cup Y'$ **d** $(X \cap Y)'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

**7** Draw this diagram six times.
Use shading to illustrate each of the following sets:

**a** $A'$ **b** $B'$ **c** $A' \cap B'$
**d** $A' \cup B'$ **e** $A \cup B$ **f** $(A \cup B)'$




**8** Let $\xi = \{\text{different letters in the word } GENERAL\}$,
$A = \{\text{different letters in the word } ANGEL\}$,
$B = \{\text{different letters in the word } LEAN\}$

Show these sets on a Venn diagram and use this diagram to list the sets:

**a** $A'$ **b** $B'$ **c** $A \cap B$ **d** $A \cup B$ **e** $(A \cup B)'$ **f** $A' \cup B'$

**9** Let $\xi = \{\text{different letters in the word } MATHEMATICS\}$
$A = \{\text{different letters in the word } ATTIC\}$
$B = \{\text{different letters in the word } TASTE\}$

Show $\xi$, $A$ and $B$ on a Venn diagram, entering all the elements. Hence list the sets:

**a** $A'$ **b** $B'$ **c** $A \cap B$ **d** $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

## 2B Sets of numbers

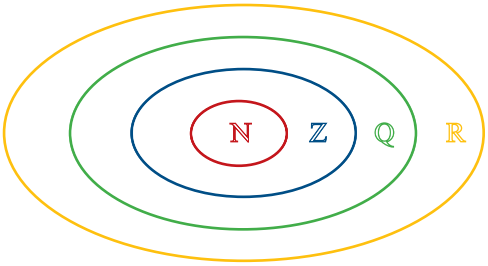
Recall that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

The numbers of the form $\frac{p}{q}$, with $p$ and $q$ integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational**. Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, $\pi$, $\pi + 2$ and $\sqrt{6} + \sqrt{7}$. These numbers cannot be written in the form $\frac{p}{q}$, for integers $p, q$; the decimal representations of these numbers do not terminate or repeat.

- The set of real numbers is denoted by $\mathbb{R}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of integers is denoted by $\mathbb{Z}$.
- The set of natural numbers is denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.





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We can use set notation to describe subsets of the real numbers.

For example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x > 0,\ x \in \mathbb{Q} \}$ is the set of all positive rational numbers
- $\{ 2n : n = 0, 1, 2, \dots \}$ is the set of all non-negative even numbers.

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. That is,

$$\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$$

This set is known as the **Cartesian product** of $\mathbb{R}$ with itself.

## Rational numbers

Every rational number can be expressed as a terminating or recurring decimal.

To find the decimal representation of a rational number $\frac{m}{n}$, perform the division $m \div n$.

For example, to find the decimal representation of $\frac{3}{7}$, divide $3.0000000\dots$ by 7.

$$\begin{array}{r|l} & 0.\ \ 4\ \ 2\ \ 8\ \ 5\ \ 7\ \ 1\ \ 4\ \dots \\ \hline 7 & 3.\ {}^{3}0\ {}^{2}0\ {}^{6}0\ {}^{4}0\ {}^{5}0\ {}^{1}0\ {}^{3}0\ \dots \end{array}$$

Therefore $\frac{3}{7} = 0.\dot{4}2857\dot{1}$.

**Theorem**

Every rational number can be written as a terminating or recurring decimal.

**Proof** Consider any two natural numbers $m$ and $n$. At each step in the division of $m$ by $n$, there is a remainder. If the remainder is 0, then the division algorithm stops and the decimal is a terminating decimal.

If the remainder is never 0, then it must be one of the numbers $1, 2, 3, \dots, n - 1$. (In the above example, $n = 7$ and the remainders can only be 1, 2, 3, 4, 5 and 6.) Hence the remainder must repeat after at most $n - 1$ steps.

Further examples:

$$\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\dot{3}, \quad \frac{1}{7} = 0.\dot{1}4285\dot{7}$$

**Theorem**

A real number has a terminating decimal representation if and only if it can be written as

$$\frac{m}{2^{\alpha} \times 5^{\beta}}$$

for some $m \in \mathbb{Z}$ and some $\alpha, \beta \in \mathbb{N} \cup \{0\}$.


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**Proof** Assume that $x = \dfrac{m}{2^\alpha \times 5^\beta}$ with $\alpha \ge \beta$. Multiply the numerator and denominator by $5^{\alpha-\beta}$. Then

$$x = \frac{m \times 5^{\alpha-\beta}}{2^\alpha \times 5^\alpha} = \frac{m \times 5^{\alpha-\beta}}{10^\alpha}$$

and so $x$ can be written as a terminating decimal. The case $\alpha < \beta$ is similar.

Conversely, if $x$ can be written as a terminating decimal, then there is $m \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{0\}$ such that $x = \dfrac{m}{10^\alpha} = \dfrac{m}{2^\alpha \times 5^\alpha}$.

The method for finding a rational number $\dfrac{m}{n}$ from its decimal representation is demonstrated in the following example.

**Example 4**

Write each of the following in the form $\dfrac{m}{n}$, where $m$ and $n$ are integers:

**a** 0.05 **b** $0.\dot{4}2857\dot{1}$

**Solution**

**a** $0.05 = \dfrac{5}{100} = \dfrac{1}{20}$

**b** We can write

$$0.\dot{4}2857\dot{1} = 0.428571428571\ldots \qquad (1)$$

Multiply both sides by $10^6$:

$$0.\dot{4}2857\dot{1} \times 10^6 = 428571.428571428571\ldots \qquad (2)$$

Subtract (1) from (2):

$$0.\dot{4}2857\dot{1} \times (10^6 - 1) = 428571$$

$$\therefore \quad 0.\dot{4}2857\dot{1} = \frac{428571}{10^6 - 1} = \frac{3}{7}$$

## Real numbers

The set of real numbers is made up of two important subsets: the **algebraic numbers** and the **transcendental numbers**.

An algebraic number is a solution to a polynomial equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad \text{where } a_0, a_1, \ldots, a_n \text{ are integers}$$

Every rational number is algebraic. The irrational number $\sqrt{2}$ is algebraic, as it is a solution of the equation

$$x^2 - 2 = 0$$


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It can be shown that $\pi$ is not an algebraic number; it is a transcendental number. The proof is too difficult to be given here.

The proof that $\sqrt{2}$ is irrational is presented in Chapter 13.

## Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that $a$ and $b$ are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 5.

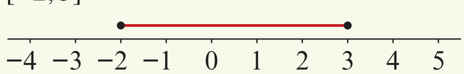
### Example 5

Illustrate each of the following intervals of real numbers:

**a** $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 2]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

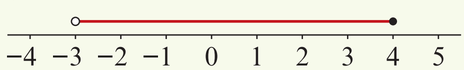
#### Solution

**a** $[-2, 3]$



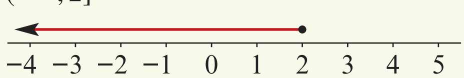

**Explanation:** The square brackets indicate that the endpoints are included; this is shown with closed circles.

**b** $(-3, 4]$



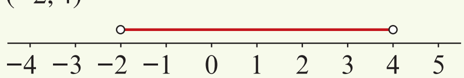

**Explanation:** The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

**c** $(-\infty, 2]$



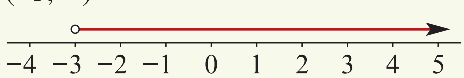

**Explanation:** The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as 'negative infinity'. The right endpoint is included.

**d** $(-2, 4)$




**Explanation:** Both brackets are round; the endpoints are not included.

**e** $(-3, \infty)$




**Explanation:** The symbol $\infty$ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as 'infinity'. The left endpoint is not included.

Notes:

- The 'closed' circle (●) indicates that the number is included.
- The 'open' circle (○) indicates that the number is not included.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## Summary 2B

- Sets of numbers
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers $a$ and $b$ with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

## Exercise 2B

**1** **a** Is the sum of two rational numbers also rational?
**b** Is the product of two rational numbers also rational?
**c** Is the quotient of two rational numbers also rational (if defined)?

**2** **a** Is the sum of two irrational numbers always irrational?
**b** Is the product of two irrational numbers always irrational?
**c** Is the quotient of two irrational numbers always irrational?

Example 4

**3** Write each of the following in the form $\frac{m}{n}$, where $m$ and $n$ are integers:

**a** $0.45$ **b** $0.\dot{2}\dot{7}$ **c** $0.12$
**d** $0.\dot{2}8571\dot{4}$ **e** $0.\dot{3}\dot{6}$ **f** $0.\dot{2}$

**4** Give the decimal representation of each of the following rational numbers:

**a** $\frac{2}{7}$ **b** $\frac{5}{11}$ **c** $\frac{7}{20}$ **d** $\frac{4}{13}$ **e** $\frac{1}{17}$

Example 5

**5** Illustrate each of the following intervals of real numbers:

**a** $[-1, 4]$ **b** $(-2, 2]$ **c** $(-\infty, 3]$ **d** $(-1, 5)$ **e** $(-2, \infty)$

**6** Write each of the following sets using interval notation:

**a** $\{ x : x < 3 \}$ **b** $\{ x : x \ge -3 \}$
**c** $\{ x : x \le -3 \}$ **d** $\{ x : x > 5 \}$
**e** $\{ x : -2 \le x < 3 \}$ **f** $\{ x : -2 \le x \le 3 \}$
**g** $\{ x : -2 < x \le 3 \}$ **h** $\{ x : -5 < x < 3 \}$


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# 2C Problems involving sets

Sets can be used to help sort information, as each of the following examples demonstrates. Recall that, if $A$ is a finite set, then the number of elements in $A$ is denoted by $|A|$.

## Example 6

Two hundred and eighty students each travel to school by either train or bus or both. Of these students, 150 travel by train, and 60 travel by both train and bus.

**a** Show this information on a Venn diagram.

**b** Hence find the number of students who travel by:

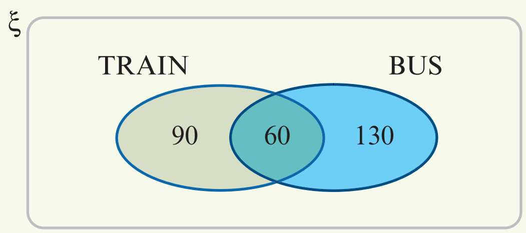
**i** bus

**ii** train but not bus

**iii** just one of these modes of transport.

### Solution

**a**




$|\xi| = 280$

**b** **i** $|\text{BUS}| = 130 + 60 = 190$

**ii** $|\text{TRAIN} \cap (\text{BUS})'| = 90$

**iii** $|\text{TRAIN} \cap (\text{BUS})'| + |(\text{TRAIN})' \cap \text{BUS}| = 90 + 130 = 220$

## Example 7

An athletics team has 18 members. Each member competes in at least one of three events: sprints ($S$), jumps ($J$) and hurdles ($H$). Every hurdler also jumps or sprints. The following additional information is available:

$$|S| = 11, \quad |J| = 10, \quad |J \cap H' \cap S'| = 5, \quad |J' \cap H' \cap S| = 5 \quad \text{and} \quad |J \cap H'| = 7$$

**a** Draw a Venn diagram.

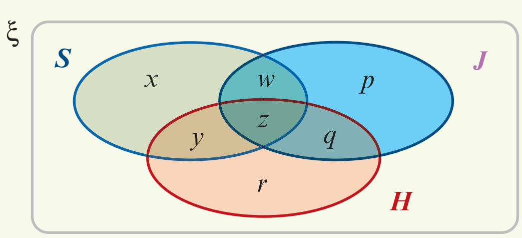
**b** Find:

**i** $|H|$ **ii** $|S \cap H \cap J|$ **iii** $|S \cup J|$ **iv** $|S \cap J \cap H'|$

### Solution

**a** Assign a variable to the number of members in each region of the Venn diagram.

The information in the question can be summarised in terms of these variables:





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$$x + y + z + w = 11 \quad \text{as } |S| = 11 \qquad (1)$$
$$p + q + z + w = 10 \quad \text{as } |J| = 10 \qquad (2)$$
$$x + y + z + w + p + q + r = 18 \quad \text{as all members compete} \qquad (3)$$
$$p = 5 \quad \text{as } |J \cap H' \cap S'| = 5 \qquad (4)$$
$$x = 5 \quad \text{as } |J' \cap H' \cap S| = 5 \qquad (5)$$
$$r = 0 \quad \text{as every hurdler also jumps or sprints} \qquad (6)$$
$$w + p = 7 \quad \text{as } |J \cap H'| = 7 \qquad (7)$$

From (4) and (7): $w = 2$.

Equation (3) now becomes

$$5 + y + z + 2 + 5 + q = 18$$
$$\therefore \quad y + z + q = 6 \qquad (8)$$

Equation (1) becomes

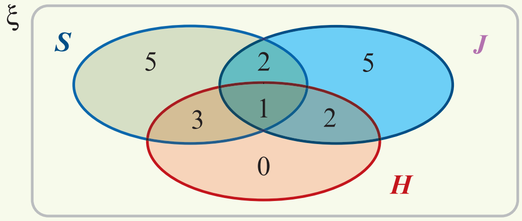
$$y + z = 4$$

Therefore from (8): $q = 2$.

Equation (2) becomes

$$5 + 2 + z + 2 = 10$$
$$\therefore \quad z = 1$$
$$\therefore \quad y = 3$$

The Venn diagram can now be completed as shown.




**b** **i** $|H| = 6$ **ii** $|S \cap H \cap J| = 1$ **iii** $|S \cup J| = 18$ **iv** $|S \cap J \cap H'| = 2$

Skill-sheet

## Exercise 2C

Example 6

**1** There are 28 students in a class, all of whom take either History or Economics or both. Of the 14 students who take History, five also take Economics.

**a** Show this information on a Venn diagram.

**b** Hence find the number of students who take:

**i** Economics **ii** History but not Economics **iii** just one of these subjects.

**2** **a** Draw a Venn diagram to show three sets $A$, $B$ and $C$ in a universal set $\xi$. Enter numbers in the correct parts of the diagram using the following information:

$$|A \cap B \cap C| = 2, \quad |A \cap B| = 7, \quad |B \cap C| = 6,$$
$$|A \cap C| = 8, \quad |A| = 16, \quad |B| = 20, \quad |C| = 19, \quad |\xi| = 50$$

**b** Use the diagram to find:

**i** $|A' \cap C'|$ **ii** $|A \cup B'|$ **iii** $|A' \cap B \cap C'|$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**3** In a border town in the Balkans, 60% of people speak Bulgarian, 40% speak Greek and 20% speak neither. What percentage of the town speak both Bulgarian and Greek?

**4** A survey of a class of 40 students showed that 16 own at least one dog and 25 at least one cat. Six students have neither. How many students own both?

**5** At an international conference there were 105 delegates. Seventy spoke English, 50 spoke French and 50 spoke Japanese. Twenty-five spoke English and French, 15 spoke French and Japanese and 30 spoke Japanese and English.

**a** How many delegates spoke all three languages?

**b** How many spoke Japanese only?

**6** A restaurant serves lunch to 350 people. It offers three desserts: profiteroles, gelati and fruit. Forty people have all three desserts, 70 have gelati only, 50 have profiteroles only and 60 have fruit only. Forty-five people have fruit and gelati only, 30 people have gelati and profiteroles only and 10 people have fruit and profiteroles only. How many people do not have a dessert?

Example 7

**7** Forty travellers were questioned about the various methods of transport they had used the previous day. Every traveller used at least one of the following methods: car ($C$), bus ($B$), train ($T$). Of these travellers:

- eight had used all three methods of transport
- four had travelled by bus and car only
- two had travelled by car and train only
- the number ($x$) who had travelled by train only was equal to the number who had travelled by bus and train only.

If 20 travellers had used a train and 33 had used a bus, find:

**a** the value of $x$

**b** the number who travelled by bus only

**c** the number who travelled by car only.

**8** Let $\xi$ be the set of all integers and let

$$X = \{ x : 21 < x < 37 \}, \quad Y = \{ 3y : 0 < y \leq 13 \}, \quad Z = \{ z^2 : 0 < z < 8 \}$$

**a** Draw a Venn diagram representing these sets.

**b** **i** Find $X \cap Y \cap Z$. **ii** Find $|X \cap Y|$.

**9** A number of students bought red, green and black pens. Three bought one of each colour. Of the students who bought two colours, three did not buy red, five not green and two not black. The same number of students bought red only as bought red with other colours. The same number bought black only as bought green only. More students bought red and black but not green than bought black only. More bought only green than bought green and black but not red. How many students were there and how many pens of each colour were sold?


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**10** For three subsets $B$, $M$ and $F$ of a universal set $\xi$,

$$|B \cap M| = 12, \quad |M \cap F \cap B| = |F'|, \quad |F \cap B| > |M \cap F|,$$
$$|B \cap F' \cap M'| = 5, \quad |M \cap B' \cap F'| = 5, \quad |F \cap M' \cap B'| = 5, \quad |\xi| = 28$$

Find $|M \cap F|$.

**11** A group of 80 students were interviewed about which sports they play. It was found that 23 do athletics, 22 swim and 18 play football. If 10 students do athletics and swim only, 11 students do athletics and play football only, six students swim and play football only and 46 students do none of these activities on a regular basis, how many students do all three?

**12** At a certain secondary college, students have to be proficient in at least one of the languages Italian, French and German. In a particular group of 33 students, two are proficient in all three languages, three in Italian and French only, four in French and German only and five in German and Italian only. The number of students proficient in Italian only is $x$, in French only is $x$ and in German only is $x + 1$. Find $x$ and then find the total number of students proficient in Italian.

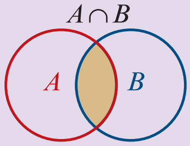
**13** At a certain school, 201 students study one or more of Mathematics, Physics and Chemistry. Of these students: 35 take Chemistry only, 50% more students study Mathematics only than study Physics only, four study all three subjects, 25 study both Mathematics and Physics but not Chemistry, seven study both Mathematics and Chemistry but not Physics, and 20 study both Physics and Chemistry but not Mathematics. Find the number of students studying Mathematics.

## 2D The inclusion–exclusion principle

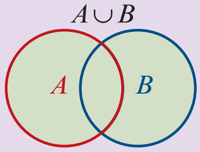
Recall, that the set with no elements is called the **empty set** and is denoted by $\varnothing$. We say that set $B$ is a **subset** of set $A$ if each element of $B$ is also in $A$. In this case, we can write $B \subseteq A$. Note that $\varnothing \subseteq A$ and $A \subseteq A$.

Earlier in the chapter we saw that given any two sets $A$ and $B$ we can define two important sets:

**1** The **intersection** of sets $A$ and $B$ is denoted by $A \cap B$ and consists of elements belonging to $A$ *and* $B$.




**2** The **union** of sets $A$ and $B$ is denoted by $A \cup B$ and consists of elements belonging to $A$ *or* $B$.




Note: It is important to realise that $A \cup B$ includes elements belonging to $A$ and $B$.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**Example 8**

Consider the three sets of numbers $A = \{2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{3, 4, 5\}$.

**a** Find $B \cap C$.
**b** Find $A \cup C$.
**c** Find $A \cap B \cap C$.
**d** Find $A \cup B \cup C$.
**e** Find $|A|$.
**f** List all the subsets of $C$.

**Solution**

**a** $B \cap C = \{3, 4\}$
**b** $A \cup C = \{2, 3, 4, 5\}$
**c** $A \cap B \cap C = \{3\}$
**d** $A \cup B \cup C = \{1, 2, 3, 4, 5\}$
**e** $|A| = 2$
**f** $\emptyset$, $\{3\}$, $\{4\}$, $\{5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$, $\{3, 4, 5\}$

Earlier in Chapter 1 we encountered the addition principle. This principle can be concisely expressed using set notation.

**Addition principle**

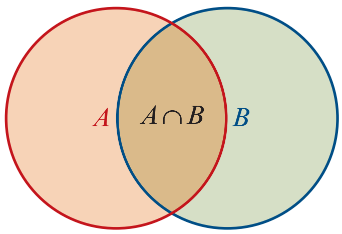
If $A$ and $B$ are two finite sets of objects such that $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|$$

Our aim is to extend this rule for instances where $A \cap B \neq \emptyset$.

## Two sets

To count the number of elements in the set $A \cup B$, we first add (include) $|A|$ and $|B|$. However, this counts the elements in $A \cap B$ twice, and so we subtract (exclude) $|A \cap B|$.




**Inclusion–exclusion principle for two sets**

If $A$ and $B$ are two finite sets of objects, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

**Example 9**

Each of the 25 students in a Year 11 class studies Physics or Chemistry. Of these students, 15 study Physics and 18 study Chemistry. How many students study both subjects?

**Solution**

$$|P \cup C| = |P| + |C| - |P \cap C|$$
$$25 = 15 + 18 - |P \cap C|$$
$$25 = 33 - |P \cap C|$$
$$\therefore \quad |P \cap C| = 8$$

**Explanation**

Let $P$ and $C$ be the sets of students who study Physics and Chemistry respectively.

Since each student studies Physics or Chemistry, we know that $|P \cup C| = 25$.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## Example 10

A bag contains 100 balls labelled with the numbers from 1 to 100. How many ways can a ball be chosen that is a multiple of 2 or 5?

**Solution**

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= 50 + 20 - 10 \\ &= 60\end{aligned}$$

**Explanation**

Within the set of numbers $\{1, 2, 3, \ldots, 100\}$, let $A$ be the set of multiples of 2 and let $B$ be the set of multiples of 5.

Then $A \cap B$ consists of numbers that are multiples of both 2 and 5, that is, multiples of 10.

Therefore $|A| = 50$, $|B| = 20$ and $|A \cap B| = 10$. We then use the inclusion–exclusion principle.

## Example 11

A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly:

**a** two clubs

**b** three spades

**c** two clubs and three spades

**d** two clubs or three spades?

**Solution**

**a** ${}^{13}C_2 \cdot {}^{39}C_3 = 712\,842$

There are ${}^{13}C_2$ ways of choosing 2 clubs from 13 and ${}^{39}C_3$ ways of choosing 3 more cards from the 39 non-clubs.

**b** ${}^{13}C_3 \cdot {}^{39}C_2 = 211\,926$

There are ${}^{13}C_3$ ways of choosing 3 spades from 13 and ${}^{39}C_2$ ways of choosing 2 more cards from the 39 non-spades.

**c** ${}^{13}C_2 \cdot {}^{13}C_3 = 22\,308$

There are ${}^{13}C_2$ ways of choosing 2 clubs from 13 and ${}^{13}C_3$ ways of choosing 3 spades from 13.

**d** $|A \cup B|$

$$\begin{aligned}&= |A| + |B| - |A \cap B| \\ &= 712\,842 + 211\,926 - 22\,308 \\ &= 902\,460\end{aligned}$$

We let $A$ be the set of all hands with 2 clubs and let $B$ be the set of all hands with 3 spades. Then $A \cap B$ is the set of all hands with 2 clubs and 3 spades. We use the inclusion–exclusion principle to find $|A \cup B|$.


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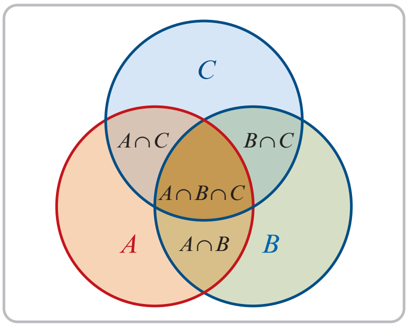
## Three sets

For three sets $A$, $B$ and $C$, the formula for $|A \cup B \cup C|$ is slightly harder to establish.

We first add $|A|$, $|B|$ and $|C|$. However, we have counted the elements in $A \cap B$, $A \cap C$ and $B \cap C$ twice, and the elements in $A \cap B \cap C$ three times.

Therefore we subtract $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ to compensate. But then the elements in $A \cap B \cap C$ will have been excluded once too often, and so we add $|A \cap B \cap C|$.




### Inclusion–exclusion principle for three sets

If $A$, $B$ and $C$ are three finite sets of objects, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

### Example 12

How many integers from 1 to 140 inclusive are not divisible by 2, 5 or 7?

**Solution**

Let $A$, $B$ and $C$ be the sets of all integers from 1 to 140 that are divisible by 2, 5 and 7 respectively. We then have

| | | |
|---|---|---|
| $A$ | multiples of 2 | $\|A\| = 140 \div 2 = 70$ |
| $B$ | multiples of 5 | $\|B\| = 140 \div 5 = 28$ |
| $C$ | multiples of 7 | $\|C\| = 140 \div 7 = 20$ |
| $A \cap B$ | multiples of 10 | $\|A \cap B\| = 140 \div 10 = 14$ |
| $A \cap C$ | multiples of 14 | $\|A \cap C\| = 140 \div 14 = 10$ |
| $B \cap C$ | multiples of 35 | $\|B \cap C\| = 140 \div 35 = 4$ |
| $A \cap B \cap C$ | multiples of 70 | $\|A \cap B \cap C\| = 140 \div 70 = 2$ |

We use the inclusion–exclusion principle to give

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 70 + 28 + 20 - 14 - 10 - 4 + 2 \\ &= 92\end{aligned}$$

Therefore the number of integers not divisible by 2, 5 or 7 is $140 - 92 = 48$.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## Summary 2D

- The **inclusion–exclusion principle** extends the addition principle to instances where the two sets have objects in common.
- The principle works by ensuring that objects belonging to multiple sets are not counted more than once.
- The inclusion–exclusion principles for two sets and three sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

## Exercise 2D

Example 8

**1** Consider the three sets of numbers $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 4, 6\}$.

**a** Find $B \cap C$.

**b** Find $A \cup C$.

**c** Find $A \cap B \cap C$.

**d** Find $A \cup B \cup C$.

**e** Find $|A|$.

**f** List all the subsets of $A$.

Example 9

**2** In an athletics team, each athlete competes in track or field events. There are 25 athletes who compete in track events, 23 who compete in field events and 12 who compete in both track and field events. How many athletes are in the team?

**3** Fifty patients at a medical clinic are being treated for a disease using two types of medication, $A$ and $B$. There are 25 patients using medication $A$ and 29 patients using medication $B$. How many patients are using both types of medication?

Example 10

**4** How many integers from 1 to 630 inclusive are multiples of 7 or 9?

**5** Consider the integers from 1 to 96 inclusive. How many of these are:

**a** divisible by 2 or 3

**b** not divisible by 2 or 3?

Example 11

**6** How many five-letter arrangements of the word ALIEN:

**a** begin with a vowel

**b** end with a vowel

**c** begin and end with a vowel

**d** begin or end with a vowel?


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**7** **a** How many of the integers from 1 to 100 inclusive are perfect squares or perfect cubes?

**b** How many of the integers from 1 to 1000 inclusive are perfect squares or perfect cubes?

Example 12

**8** How many of the integers from 1 to 120 inclusive are multiples of 2, 3 or 5?

**9** How many of the integers from 1 to 220 inclusive are not divisible by 2, 5 or 11?

**10** There are 90 Year 11 students at a secondary school and each of them must study at least one of Biology, Physics or Chemistry. There are 36 students who study Biology, 42 who study Physics and 40 who study Chemistry. Moreover, 9 study Biology and Physics, 8 study Biology and Chemistry and 7 study Physics and Chemistry. How many students study all three subjects?

**11** A group of six students is selected from four students in Year 10, five in Year 11 and four in Year 12. How many selections have exactly:

**a** two Year 10 students

**b** two Year 11 students

**c** two Year 10 and two Year 11 students

**d** two Year 10 or two Year 11 students?

**12** A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly one heart or exactly two diamonds?

**13** Find the sum of all the integers from 1 to 100 inclusive that are divisible by 2 or 3.

Hint: Use the formula $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

**14** There are seventy Year 11 students at a school and each of them must study at least one of three languages. Thirty are studying French, forty-five are studying Chinese, thirty are studying German and fifteen are studying all three languages. How many students are studying exactly two languages?


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Review

## Chapter summary

Assign-ment

Nrich

### Sets

- Set notation

| | |
|---|---|
| $x \in A$ | $x$ is an element of $A$ |
| $x \notin A$ | $x$ is not an element of $A$ |
| $\xi$ | the universal set |
| $\varnothing$ | the empty set |
| $A \subseteq B$ | $A$ is a subset of $B$ |
| $A \cup B$ | the union of $A$ and $B$ consists of all elements that are in either $A$ or $B$ or both |
| $A \cap B$ | the intersection of $A$ and $B$ consists of all elements that are in both $A$ and $B$ |
| $A'$ | the complement of $A$ consists of all elements of $\xi$ that are not in $A$ |

- Sets of numbers

| | | | |
|---|---|---|---|
| $\mathbb{N}$ | Natural numbers | $\mathbb{Z}$ | Integers |
| $\mathbb{Q}$ | Rational numbers | $\mathbb{R}$ | Real numbers |

- The **inclusion–exclusion principle** allows us to count the number of elements in a union of sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

## Short-answer questions

**1** Express the following as fractions in their simplest form:

**a** $0.0\dot{7}$ **b** $0.\dot{4}\dot{5}$ **c** $0.005$ **d** $0.405$ **e** $0.2\dot{6}$ **f** $0.1\dot{7}1428\dot{5}$

**2** In a class of 100 students, 55 are girls, 45 have blue eyes, 40 are blond, 25 are blond girls, 15 are blue-eyed blonds, 20 are blue-eyed girls, and five are blue-eyed blond girls. Find:

**a** the number of blond boys

**b** the number of boys who are neither blond nor blue-eyed.

**3** A group of 30 students received prizes in at least one of the subjects of English, Mathematics and French. Two students received prizes in all three subjects. Fourteen received prizes in English and Mathematics but not French. Two received prizes in English alone, two in French alone and five in Mathematics alone. Four received prizes in English and French but not Mathematics.

**a** How many received prizes in Mathematics and French but not English?

**b** How many received prizes in Mathematics?

**c** How many received prizes in English?


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Review

**4** Fifty people are interviewed. Twenty-three people like Brand $X$, 25 like Brand $Y$ and 19 like Brand $Z$. Eleven like $X$ and $Z$. Eight like $Y$ and $Z$. Five like $X$ and $Y$. Two like all three. How many like none of them?

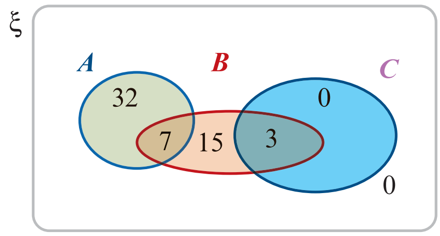
**5** Three rectangles $A$, $B$ and $C$ overlap (intersect). Their areas are $20\text{ cm}^2$, $10\text{ cm}^2$ and $16\text{ cm}^2$ respectively. The area common to $A$ and $B$ is $3\text{ cm}^2$, that common to $A$ and $C$ is $6\text{ cm}^2$ and that common to $B$ and $C$ is $4\text{ cm}^2$. How much of the area is common to all three if the total area covered is $35\text{ cm}^2$?

**6** $A$, $B$ and $C$ are three sets and $\xi = A \cup B \cup C$. The number of elements in the regions of the Venn diagram are as shown. Find:

**a** the number of elements in $A \cup B$

**b** the number of elements in $C$

**c** the number of elements in $B' \cap A$.




**7** Each of the twenty students in a class plays netball or basketball. Twelve play basketball and four play both sports. How many students play netball?

**8** Six people are to be seated in a row. Calculate the number of ways this can be done so that two particular people, $A$ and $B$, always have exactly one person between them.

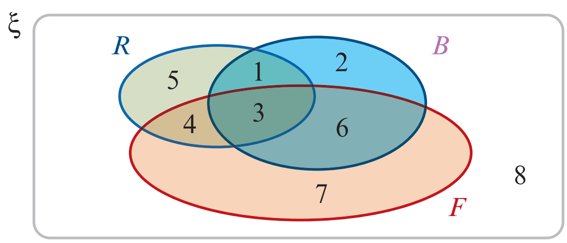
## Extended-response questions

**1** Use Venn diagrams to illustrate:

**a** $|A \cup B| = |A| + |B| - |A \cap B|$

**b** $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

**2** **a** The Venn diagram shows the set $\xi$ of all students enrolled at Argos Secondary College. Set $R$ is the set of all students with red hair. Set $B$ is the set of all students with blue eyes. Set $F$ is the set of all female students.




The numbers on the diagram are to label the eight different regions.

**i** Identify the region in the Venn diagram which represents male students who have neither red hair nor blue eyes.

**ii** Describe the gender, hair colour and eye colour of students represented in region 1 of the diagram.

**iii** Describe the gender, hair colour and eye colour of students represented in region 2 of the diagram.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**b** It is known that, at Argos Secondary College, 250 students study French ($F$), Greek ($G$) or Japanese ($J$). Forty-one students do not study French. Twelve students study French and Japanese but not Greek. Thirteen students study Japanese and Greek but not French. Thirteen students study only Greek. Twice as many students study French and Greek but not Japanese as study all three. The number studying only Japanese is the same as the number studying both French and Greek.

**i** How many students study all three languages?

**ii** How many students study only French?

**3** Consider the universal set $\xi$ as the set of all students enrolled at Sounion Secondary College. Let $B$ denote the set of students taller than 180 cm and let $A$ denote the set of female students.

**a** Give a brief description of each of the following sets:

**i** $B'$

**ii** $A \cup B$

**iii** $A' \cap B'$

**b** Use a Venn diagram to show $(A \cup B)' = A' \cap B'$.

**c** Hence show that $A \cup B \cup C = (A' \cap B' \cap C')'$, where $C$ is the set of students who play sport.

**4** In a certain city, three Sunday newspapers ($A$, $B$ and $C$) are available. In a sample of 500 people from this city, it was found that:

- nobody regularly reads both $A$ and $C$
- a total of 100 people regularly read $A$
- 205 people regularly read only $B$
- of those who regularly read $C$, exactly half of them also regularly read $B$
- 35 people regularly read $A$ and $B$ but not $C$
- 35 people don't read any of the papers at all.

**a** Draw a Venn diagram showing the number of regular readers for each possible combination of $A$, $B$ and $C$.

**b** How many people in the sample were regular readers of $C$?

**c** How many people in the sample regularly read $A$ only?

**d** How many people are regular readers of $A$, $B$ and $C$?

**5** Consider the integers from 1 to 96 inclusive. Let sets $A$ and $B$ consist of those integers that are multiples of 6 and 8 respectively.

**a** What is the lowest common multiple of 6 and 8?

**b** How many integers belong to $A \cap B$?

**c** How many integers from 1 to 96 are divisible by 6 or 8?

**d** An integer from 1 to 96 is chosen at random. What is the probability that it is not divisible by 6 or 8?


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# 3 Vectors

## In this chapter



## Syllabus references

**Topics:** Representing vectors in the plane by directed line segments; Algebra of vectors in the plane; Geometric vectors in the plane, including proof and use

**Subtopics:** 1.3.1 – 1.3.13, 1.1.16 – 1.1.18

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

| | |
|---|---|
| length | 30 cm is the length of the page of a particular book |
| time | 10 s is the time for one athlete to run 100 m |

More is required to describe velocity, displacement or force. The direction must be recorded as well as the magnitude.

| | |
|---|---|
| displacement | 30 km in a direction north |
| velocity | 60 km/h in a direction south-east |

A quantity that has both a magnitude and a direction is called a **vector**. Our study of vectors will tie together different ideas from geometry and trigonometry.



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# 3A Introduction to vectors

Suppose that you are asked: ‘Where is your school in relation to your house?’

It is not enough to give an answer such as ‘four kilometres’. You need to specify a direction as well as a distance. You could give the answer ‘four kilometres north-east’.

Position is an example of a vector quantity.

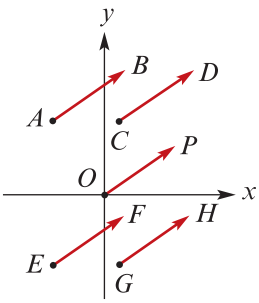
## Directed line segments

A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

The five directed line segments shown all have the same length and direction, and so they are equivalent.



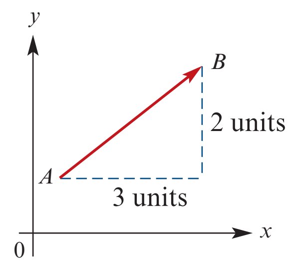

A directed line segment from a point $A$ to a point $B$ is denoted by $\overrightarrow{AB}$.

For simplicity of language, this is also called vector $\overrightarrow{AB}$. That is, the set of equivalent segments can be named through one member of the set.

Note: The five directed line segments in the diagram all name the same vector: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$.

## Column vectors

We can represent each vector by a column of numbers. In this form, the top number of the column represents the horizontal ($x$-axis) component of the vector, and we take a positive number to mean across to the right and a negative number to mean across to the left. Similarly, the bottom number of the column represents the vertical ($y$-axis) component of the vector, and we take a positive number to mean up and a negative number to mean down.




In this approach, the numbers in the column correspond to a set of equivalent directed line segments that all have the same $x$ and $y$-axis components.

For example, the column vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the set of directed line segments which go 3 across and 2 up.

## Vector notation

A vector is often denoted by a single bold lowercase letter. The vector from $A$ to $B$ can be denoted by $\overrightarrow{AB}$ or by a single letter $\boldsymbol{v}$. That is, $\boldsymbol{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underset{\sim}{v}$.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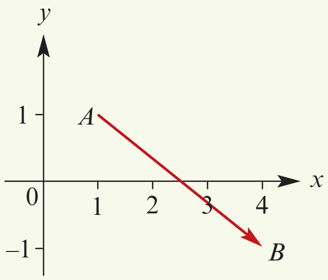
## Example 1

Draw a directed line segment corresponding to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

### Solution




### Explanation

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 across to the right and 2 down'.

Note: Here the segment starts at (1, 1) and goes to (4, −1). It can start at any point.

## Example 2

The vector $\boldsymbol{u}$ is defined by the directed line segment from (2, 6) to (3, 1).

If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find $a$ and $b$.

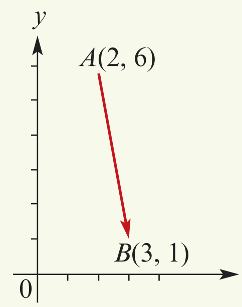
### Solution

The vector is

$$\boldsymbol{u} = \begin{bmatrix} 3-2 \\ 1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Hence $a = 1$ and $b = -5$.

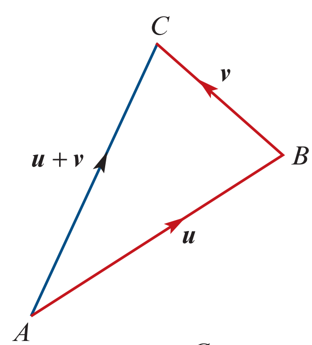
### Explanation




# Addition of vectors

## Adding vectors geometrically

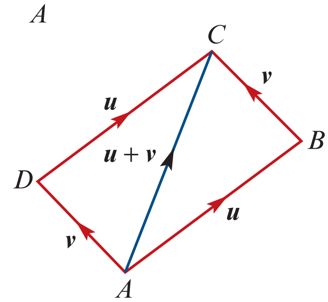
Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from $A$ to $B$ and then a line segment representing $\boldsymbol{v}$ from $B$ to $C$.




The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from $A$ to $C$. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$$

The same result is achieved if the order is reversed. This is represented in the diagram on the right:




$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{AC} \\ &= \boldsymbol{v} + \boldsymbol{u} \end{aligned}$$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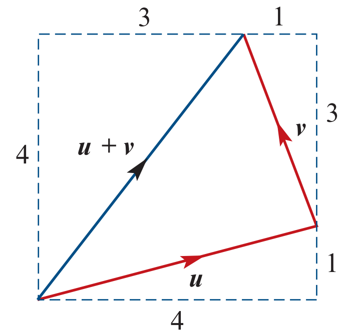
### Adding column vectors

Two vectors can be added using column-vector notation.

For example, if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

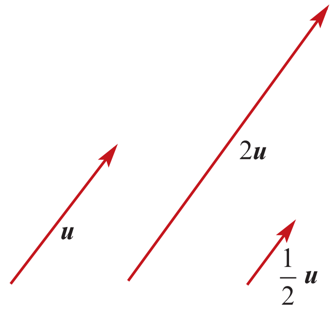
$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$




## Scalar multiplication

Multiplication by a real number (**scalar**) changes the length of the vector. For example:

- $2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$
- $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$

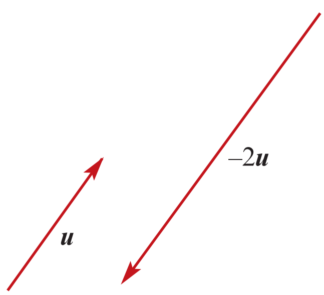



We have $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$.

In general, for $k \in (0, \infty)$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of $k$.

When a vector is multiplied by $-2$, the vector's direction is reversed and the length is doubled.

When a vector is multiplied by $-1$, the vector's direction is reversed and the length remains the same.




If $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

If $\boldsymbol{u} = \overrightarrow{AB}$, then

$$-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$$

The directed line segment $-\overrightarrow{AB}$ goes from $B$ to $A$.

## Zero vector

The **zero vector** is denoted by $\mathbf{0}$ and represents a line segment of zero length. The zero vector has no direction.

## Subtraction of vectors

To find $\boldsymbol{u} - \boldsymbol{v}$, we add $-\boldsymbol{v}$ to $\boldsymbol{u}$.



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## Example 3

For the vectors $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, find $2\boldsymbol{u} + 3\boldsymbol{v}$.

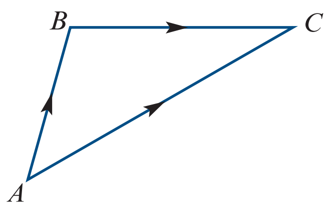
### Solution

$$\begin{aligned} 2\boldsymbol{u} + 3\boldsymbol{v} &= 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{aligned}$$

## Polygons of vectors

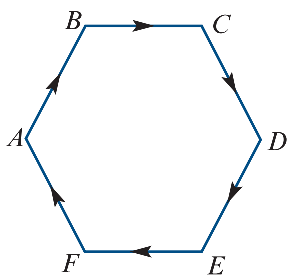
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$




- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$

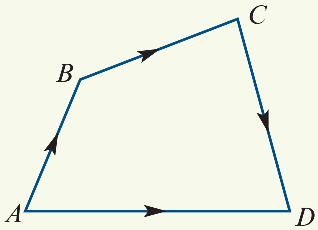



## Example 4

Illustrate the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, where $A$, $B$, $C$ and $D$ are points in the plane.

### Solution

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$




## Parallel vectors

Two **parallel vectors** have the same direction or opposite directions.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \neq 0$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

For example, if $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel as $\boldsymbol{v} = 3\boldsymbol{u}$.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## Position vectors

We can use a point $O$, the origin, as a starting point for a vector to indicate the position of a point $A$ in space relative to $O$.

For most of this chapter, we study vectors in two dimensions and the point $O$ is the origin of the Cartesian plane.

For a point $A$, the **position vector** is $\overrightarrow{OA}$.

## Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

**Proof** Assume that $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$. Then

$$m\boldsymbol{a} - p\boldsymbol{a} = q\boldsymbol{b} - n\boldsymbol{b}$$

$$\therefore \quad (m - p)\boldsymbol{a} = (q - n)\boldsymbol{b}$$

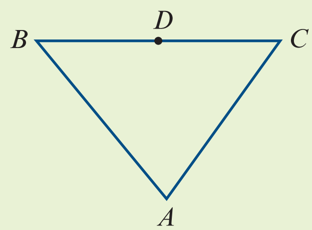
If $m \neq p$ or $n \neq q$, we could therefore write

$$\boldsymbol{a} = \frac{q - n}{m - p}\boldsymbol{b} \quad \text{or} \quad \boldsymbol{b} = \frac{m - p}{q - n}\boldsymbol{a}$$

But this is not possible, as $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel. Therefore $m = p$ and $n = q$.

### Example 5

Let $A$, $B$ and $C$ be the vertices of a triangle, and let $D$ be the midpoint of $BC$.




Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$.

Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

**a** $\overrightarrow{BD}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{AC}$

**d** $\overrightarrow{AD}$ **e** $\overrightarrow{CA}$

| Solution | Explanation |
|---|---|
| **a** $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\boldsymbol{b}$ | same direction and half the length |
| **b** $\overrightarrow{DC} = \overrightarrow{BD} = \frac{1}{2}\boldsymbol{b}$ | equivalent vectors |
| **c** $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$ | |
| **d** $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$ | |
| **e** $\overrightarrow{CA} = -\overrightarrow{AC} = -(\boldsymbol{a} + \boldsymbol{b})$ | since $\overrightarrow{CA} + \overrightarrow{AC} = \boldsymbol{0}$ |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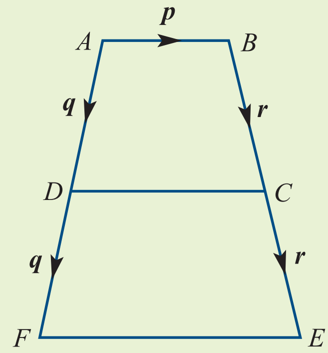
## Example 6

In the figure, $\overrightarrow{DC} = k\boldsymbol{p}$ where $k \neq 0$.

**a** Express $\boldsymbol{p}$ in terms of $k$, $\boldsymbol{q}$ and $\boldsymbol{r}$.

**b** Express $\overrightarrow{FE}$ in terms of $k$ and $\boldsymbol{p}$ to show that $FE$ is parallel to $DC$.

**c** If $\overrightarrow{FE} = 4\overrightarrow{AB}$, find the value of $k$.




### Solution

**a** $\boldsymbol{p} = \overrightarrow{AB}$

$= \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$

$= \boldsymbol{q} + k\boldsymbol{p} - \boldsymbol{r}$

Therefore

$(1 - k)\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{r}$

**b** $\overrightarrow{FE} = -2\boldsymbol{q} + \boldsymbol{p} + 2\boldsymbol{r}$

$= 2(\boldsymbol{r} - \boldsymbol{q}) + \boldsymbol{p}$

From part **a**, we have

$\boldsymbol{r} - \boldsymbol{q} = k\boldsymbol{p} - \boldsymbol{p}$

$= (k - 1)\boldsymbol{p}$

Therefore

$\overrightarrow{FE} = 2(k - 1)\boldsymbol{p} + \boldsymbol{p}$

$= 2k\boldsymbol{p} - 2\boldsymbol{p} + \boldsymbol{p}$

$= (2k - 1)\boldsymbol{p}$

**c** $\overrightarrow{FE} = 4\overrightarrow{AB}$

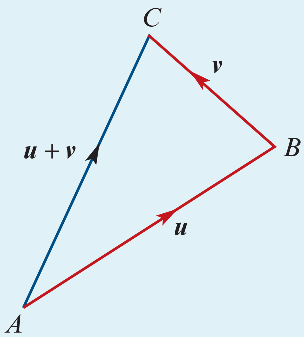
$(2k - 1)\boldsymbol{p} = 4\boldsymbol{p}$

$2k - 1 = 4$

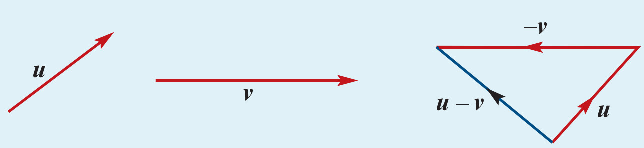
$\therefore \quad k = \dfrac{5}{2}$

## Summary 3A

- A **vector** is a set of equivalent **directed line segments**.
- Addition of vectors
  If $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{BC}$, then $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.




- Scalar multiplication
  - For $k \in (0, \infty)$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of $k$.
  - If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.
- Zero vector
  The **zero vector**, denoted by **0**, has zero length and has no direction.
- Subtraction of vectors
  $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$




- Parallel vectors
  Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \neq 0$ such that $\boldsymbol{u} = k\boldsymbol{v}$.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Skillsheet

## Exercise 3A

Example 1

**1** On the same graph, draw arrows which represent the following vectors:

**a** $\begin{bmatrix}1\\5\end{bmatrix}$ **b** $\begin{bmatrix}0\\-2\end{bmatrix}$ **c** $\begin{bmatrix}-1\\-2\end{bmatrix}$ **d** $\begin{bmatrix}-4\\3\end{bmatrix}$

Example 2

**2** The vector $\boldsymbol{u}$ is defined by the directed line segment from $(1, 5)$ to $(6, 6)$.
If $\boldsymbol{u} = \begin{bmatrix}a\\b\end{bmatrix}$, find $a$ and $b$.

**3** The vector $\boldsymbol{v}$ is defined by the directed line segment from $(-1, 5)$ to $(2, -10)$.
If $\boldsymbol{v} = \begin{bmatrix}a\\b\end{bmatrix}$, find $a$ and $b$.

**4** Let $A = (1, -2)$, $B = (3, 0)$ and $C = (2, -3)$ and let $O$ be the origin.
Express each of the following vectors in the form $\begin{bmatrix}a\\b\end{bmatrix}$:

**a** $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

Example 3

**5** Let $\boldsymbol{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}1\\-3\end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix}-2\\1\end{bmatrix}$.

**a** Find:

**i** $\boldsymbol{a} + \boldsymbol{b}$ **ii** $2\boldsymbol{c} - \boldsymbol{a}$ **iii** $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

**b** Show that $\boldsymbol{a} + \boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

Example 4

**6** If $A = (2, -3)$, $B = (4, 0)$, $C = (1, -4)$ and $O$ is the origin, sketch the following vectors:

**a** $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

**7** On graph paper, sketch the vectors joining the following pairs of points in the direction indicated:

**a** $(0, 0) \to (2, 1)$ **b** $(3, 4) \to (0, 0)$ **c** $(1, 3) \to (3, 4)$
**d** $(2, 4) \to (4, 3)$ **e** $(-2, 2) \to (5, -1)$ **f** $(-1, -3) \to (3, 0)$

**8** Identify vectors from Question **7** which are parallel to each other.

**9** **a** Plot the points $A(-1, 0)$, $B(1, 4)$, $C(4, 3)$ and $D(2, -1)$ on a set of coordinate axes.

**b** Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$.

**c** Show that:

**i** $\overrightarrow{AB} = \overrightarrow{DC}$ **ii** $\overrightarrow{BC} = \overrightarrow{AD}$

**d** Describe the shape of the quadrilateral $ABCD$.

**10** Find the values of $m$ and $n$ such that $m\begin{bmatrix}3\\-3\end{bmatrix} + n\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}-19\\61\end{bmatrix}$


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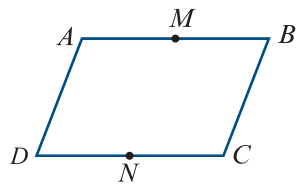
**11** Points $A$, $B$, $C$, $D$ are the vertices of a parallelogram, and $M$ and $N$ are the midpoints of $AB$ and $DC$ respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AD}$.

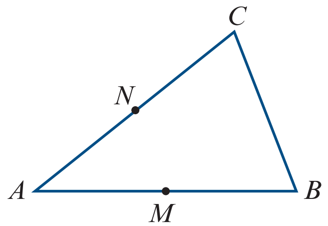



**a** Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

**i** $\overrightarrow{MD}$ **ii** $\overrightarrow{MN}$

**b** Find the relationship between $\overrightarrow{MN}$ and $\overrightarrow{AD}$.

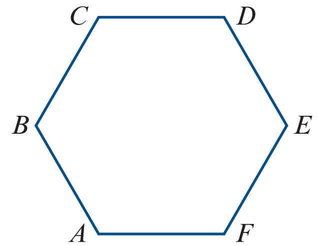
Example 5 **12** The figure represents the triangle $ABC$, where $M$ and $N$ are the midpoints of $AB$ and $AC$ respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AC}$.




**a** Express $\overrightarrow{CB}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

**b** Hence describe the relation between the two vectors (or directed line segments).

Example 6 **13** The figure shows a regular hexagon $ABCDEF$. Let $\boldsymbol{a} = \overrightarrow{AF}$ and $\boldsymbol{b} = \overrightarrow{AB}$. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:




**a** $\overrightarrow{CD}$ **b** $\overrightarrow{ED}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{FC}$

**e** $\overrightarrow{FA}$ **f** $\overrightarrow{FB}$ **g** $\overrightarrow{FE}$

**14** In parallelogram $ABCD$, let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

**a** $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{CA}$ **e** $\overrightarrow{BD}$

**15** In triangle $OAB$, let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. The point $P$ on $AB$ is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$ and the point $Q$ is such that $\overrightarrow{OP} = 3\overrightarrow{PQ}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

**a** $\overrightarrow{BA}$ **b** $\overrightarrow{PB}$ **c** $\overrightarrow{OP}$ **d** $\overrightarrow{PQ}$ **e** $\overrightarrow{BQ}$

**16** $PQRS$ is a quadrilateral in which $\overrightarrow{PQ} = \boldsymbol{u}$, $\overrightarrow{QR} = \boldsymbol{v}$ and $\overrightarrow{RS} = \boldsymbol{w}$. Express each of the following vectors in terms of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$:

**a** $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$

**17** $OABC$ is a parallelogram. Let $\boldsymbol{u} = \overrightarrow{OA}$ and $\boldsymbol{v} = \overrightarrow{OC}$. Let $M$ be the midpoint of $AB$.

**a** Express $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

**b** Express $\overrightarrow{CM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

**c** If $P$ is a point on $CM$ and $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CP}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

**d** Find $\overrightarrow{OP}$ and hence show that $P$ lies on the line segment $OB$.

**e** Find the ratio $OP : PB$.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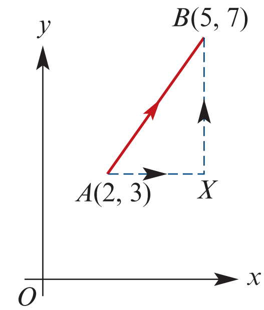
# 3B Components of vectors

The vector $\overrightarrow{AB}$ in the diagram is described by the column vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.




From the diagram, we see that the vector $\overrightarrow{AB}$ can also be expressed as the sum

$$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$$

Using column-vector notation:

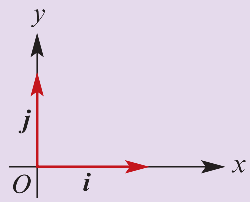
$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

This suggests the introduction of two important vectors.

### Standard unit vectors in two dimensions

- Let $\boldsymbol{i}$ be the vector of unit length in the positive direction of the $x$-axis.
- Let $\boldsymbol{j}$ be the vector of unit length in the positive direction of the $y$-axis.

Using column-vector notation, we have $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.




Note: These two vectors also play an important role in our study of linear transformations using matrices in Chapter 12.

For the example above, we have $\overrightarrow{AX} = 3\boldsymbol{i}$ and $\overrightarrow{XB} = 4\boldsymbol{j}$. Therefore

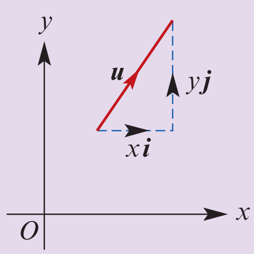
$$\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

It is possible to describe any two-dimensional vector in this way.

### Component form

- We can write the vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.

  We say that $\boldsymbol{u}$ is the sum of the two **components** $x\boldsymbol{i}$ and $y\boldsymbol{j}$.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|\boldsymbol{u}|$ and is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.




Operations with vectors now look more like basic algebra:

- $(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
- $k(x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$

Two vectors are **equal** if and only if their components are equal:

$$x\boldsymbol{i} + y\boldsymbol{j} = m\boldsymbol{i} + n\boldsymbol{j} \quad \text{if and only if} \quad x = m \text{ and } y = n$$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**Example 7**

**a** Find $\overrightarrow{AB}$ if $\overrightarrow{OA} = 3\boldsymbol{i}$ and $\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j}$.

**b** Find $|2\boldsymbol{i} - 3\boldsymbol{j}|$.

**Solution**

**a**
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= -3\boldsymbol{i} + (2\boldsymbol{i} - \boldsymbol{j})\\ &= -\boldsymbol{i} - \boldsymbol{j}\end{aligned}$$

**b**
$$\begin{aligned}|2\boldsymbol{i} - 3\boldsymbol{j}| &= \sqrt{2^2 + (-3)^2}\\ &= \sqrt{4+9}\\ &= \sqrt{13}\end{aligned}$$

**Example 8**

Let $A$ and $B$ be points on the Cartesian plane such that $\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 3\boldsymbol{j}$.

Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

**Solution**

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ \therefore\quad \overrightarrow{AB} &= -(2\boldsymbol{i} + \boldsymbol{j}) + \boldsymbol{i} - 3\boldsymbol{j}\\ &= -\boldsymbol{i} - 4\boldsymbol{j}\end{aligned}$$

$$\therefore\quad |\overrightarrow{AB}| = \sqrt{1+16} = \sqrt{17}$$

## Unit vectors

A **unit vector** is a vector of length one unit. For example, both $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors.

The unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$. (We say 'a hat'.)

Since $|\hat{\boldsymbol{a}}| = 1$, we have

$$|\boldsymbol{a}|\,\hat{\boldsymbol{a}} = \boldsymbol{a}$$

$$\therefore\quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

**Example 9**

Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$.

Find $|\boldsymbol{a}|$, the magnitude of $\boldsymbol{a}$, and hence find the unit vector in the direction of $\boldsymbol{a}$.

**Solution**

$$|\boldsymbol{a}| = \sqrt{9+16} = 5$$

$$\therefore\quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$$


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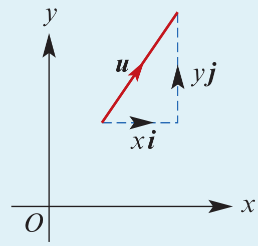
## Summary 3B

- A **unit vector** is a vector of length one unit.
- Each vector $\boldsymbol{u}$ in the plane can be written in **component form** as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the $x$-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the $y$-axis.




- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Skill-sheet

## Exercise 3B

Example 7a

**1** If $A$ and $B$ are points in the plane such that $\overrightarrow{OA} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\overrightarrow{OB} = 3\boldsymbol{i} - 5\boldsymbol{j}$, find $\overrightarrow{AB}$.

**2** $OAPB$ is a rectangle with $\overrightarrow{OA} = 5\boldsymbol{i}$ and $\overrightarrow{OB} = 6\boldsymbol{j}$. Express each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

**a** $\overrightarrow{OP}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

Example 7b

**3** Determine the magnitude of each of the following vectors:

**a** $5\boldsymbol{i}$ **b** $-2\boldsymbol{j}$ **c** $3\boldsymbol{i} + 4\boldsymbol{j}$ **d** $-5\boldsymbol{i} + 12\boldsymbol{j}$

**4** The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are given by $\boldsymbol{u} = 7\boldsymbol{i} + 8\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j}$.

**a** Find $|\boldsymbol{u} - \boldsymbol{v}|$. **b** Find constants $x$ and $y$ such that $x\boldsymbol{u} + y\boldsymbol{v} = 44\boldsymbol{j}$.

**5** Points $A$ and $B$ have position vectors $\overrightarrow{OA} = 10\boldsymbol{i}$ and $\overrightarrow{OB} = 4\boldsymbol{i} + 5\boldsymbol{j}$. If $M$ is the midpoint of $AB$, find $\overrightarrow{OM}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

**6** $OPAQ$ is a rectangle with $\overrightarrow{OP} = 2\boldsymbol{i}$ and $\overrightarrow{OQ} = \boldsymbol{j}$. Let $M$ be the point on $OP$ such that $OM = \frac{1}{5}OP$ and let $N$ be the point on $MQ$ such that $MN = \frac{1}{6}MQ$.

**a** Find each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

**i** $\overrightarrow{OM}$ **ii** $\overrightarrow{MQ}$ **iii** $\overrightarrow{MN}$ **iv** $\overrightarrow{ON}$ **v** $\overrightarrow{OA}$

**b** **i** Hence show that $N$ is on the diagonal $OA$.

**ii** State the ratio of the lengths $ON : NA$.

**7** The position vectors of $A$ and $B$ are given by $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Find the distance between $A$ and $B$.

**8** Find the pronumerals in the following equations:

**a** $2\boldsymbol{i} + 3\boldsymbol{j} = 2(\ell\boldsymbol{i} + k\boldsymbol{j})$ **b** $(x - 1)\boldsymbol{i} + y\boldsymbol{j} = 5\boldsymbol{i} + (x - 4)\boldsymbol{j}$

**c** $(x + y)\boldsymbol{i} + (x - y)\boldsymbol{j} = 6\boldsymbol{i}$ **d** $k(\boldsymbol{i} + \boldsymbol{j}) = 3\boldsymbol{i} - 2\boldsymbol{j} + \ell(2\boldsymbol{i} - \boldsymbol{j})$

Example 8

**9** Let $A = (2, 3)$ and $B = (5, 1)$. Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

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**10** Let $\overrightarrow{OA} = 3\boldsymbol{i}$, $\overrightarrow{OB} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{OC} = -3\boldsymbol{i} + \boldsymbol{j}$. Find:

**a** $\overrightarrow{AB}$ **b** $\overrightarrow{AC}$ **c** $|\overrightarrow{BC}|$

**11** Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Find:

**a** $D$ such that $\overrightarrow{AB} = \overrightarrow{CD}$ **b** $F$ such that $\overrightarrow{AF} = \overrightarrow{BC}$ **c** $G$ such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

**12** Let $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 2\boldsymbol{j}$. Points $A$, $B$ and $C$ are such that $\overrightarrow{AO} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{BC} = 2\boldsymbol{a}$, where $O$ is the origin. Find the coordinates of $A$, $B$ and $C$.

**13** $A$, $B$, $C$ and $D$ are the vertices of a parallelogram and $O$ is the origin. $A = (2, -1)$, $B = (-5, 4)$ and $C = (1, 7)$.

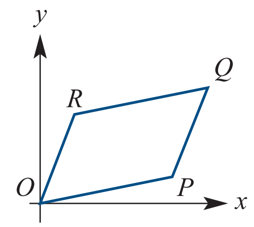
**a** Find:

**i** $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{BC}$ **v** $\overrightarrow{AD}$

**b** Hence find the coordinates of $D$.

**14** The diagram shows a parallelogram $OPQR$. The points $P$ and $Q$ have coordinates $(12, 5)$ and $(18, 13)$ respectively. Find:

**a** $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ **b** $|\overrightarrow{RQ}|$ and $|\overrightarrow{OR}|$




**15** $A(1, 6)$, $B(3, 1)$ and $C(13, 5)$ are the vertices of a triangle $ABC$.

**a** Find:

**i** $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

**b** Hence show that $ABC$ is a right-angled triangle.

**16** $A(4, 4)$, $B(3, 1)$ and $C(7, 3)$ are the vertices of a triangle $ABC$.

**a** Find the vectors:

**i** $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

**b** Find:

**i** $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

**c** Hence show that triangle $ABC$ is a right-angled isosceles triangle.

**17** $A(-3, 2)$ and $B(0, 7)$ are points on the Cartesian plane, $O$ is the origin and $M$ is the midpoint of $AB$.

**a** Find:

**i** $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{BA}$ **iv** $\overrightarrow{BM}$

**b** Hence find the coordinates of $M$. (Hint: $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$.)

Example 9

**18** Find the unit vector in the direction of each of the following vectors:

**a** $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ **b** $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$ **c** $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j}$

**d** $\boldsymbol{d} = \boldsymbol{i} - \boldsymbol{j}$ **e** $\boldsymbol{e} = \frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}$ **f** $\boldsymbol{f} = 6\boldsymbol{i} - 4\boldsymbol{j}$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# 3C Scalar product of vectors

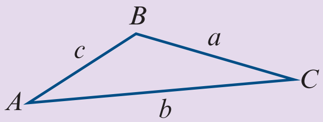
## The sine and cosine rules

In our study of vectors and their applications, we will make use of the sine and cosine rules, which are introduced in Mathematics Methods Units 1 & 2.

### Sine rule

For triangle $ABC$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$




The sine rule is used to find unknown quantities in a triangle in the following cases:

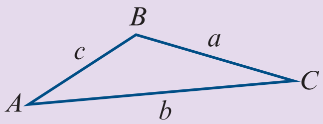
- one side and two angles are given
- two sides and a non-included angle are given.

In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

### Cosine rule

For triangle $ABC$:

$$a^2 = b^2 + c^2 - 2bc\cos A$$




The cosine rule is used to find unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

## Defining the scalar product

The scalar product is an operation that takes two vectors and gives a real number.

### Definition of the scalar product

We define the **scalar product** of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

For example:

$$\begin{aligned}(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} - 4\boldsymbol{j}) &= 2 \times 1 + 3 \times (-4) \\ &= -10\end{aligned}$$

The scalar product is often called the **dot product**.

Note: If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.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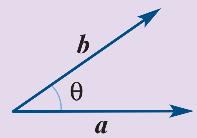
### Geometric description of the scalar product

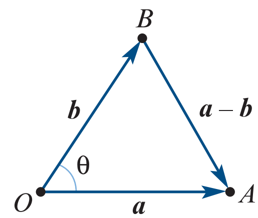
For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.




**Proof** Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$. Then using the cosine rule in $\triangle OAB$ gives




$$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = |\boldsymbol{a} - \boldsymbol{b}|^2$$
$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2$$
$$2(a_1b_1 + a_2b_2) = 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$
$$a_1b_1 + a_2b_2 = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$
$$\therefore\quad \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

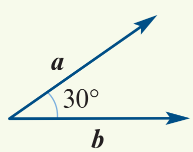
### Example 10

**a** If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 30°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

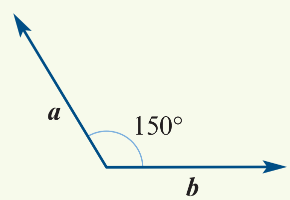
**b** If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 150°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

**Solution**

**a**
$$\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 30^\circ = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$$




**b**
$$\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 150^\circ = 20 \times \frac{-\sqrt{3}}{2} = -10\sqrt{3}$$




## Properties of the scalar product

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular**, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in the same direction} \\ -|\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in opposite directions} \end{cases}$$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## Finding the magnitude of the angle between two vectors

To find the angle θ between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we can use the two different forms of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta \quad \text{and} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

Therefore

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

**Example 11**

$A$, $B$ and $C$ are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \quad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$$

Find the magnitude of $\angle ABC$.

**Solution**

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \cos(\angle ABC)$$

We have

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\boldsymbol{i} + 2\boldsymbol{j}) \cdot (-\boldsymbol{i} - 3\boldsymbol{j}) = 1 - 6 = -5$$

$$|\overrightarrow{BA}| = \sqrt{1 + 4} = \sqrt{5}$$

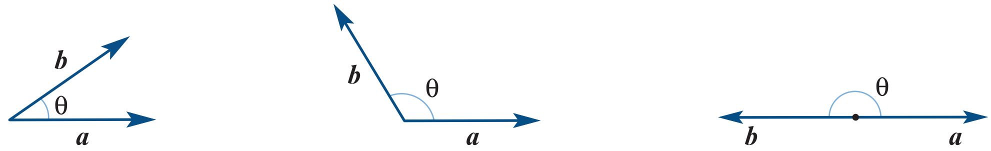
$$|\overrightarrow{BC}| = \sqrt{1 + 9} = \sqrt{10}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{-5}{\sqrt{5}\sqrt{10}} = \frac{-1}{\sqrt{2}}$$

Hence $\angle ABC = \dfrac{3\pi}{4}$.

Note: When two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are placed so that their initial points coincide, the angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is chosen as shown in the diagrams. Note that $0 \le \theta \le \pi$.





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## Summary 3C

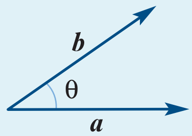
- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

- The scalar product can be described geometrically by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

  where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.




- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Skill-sheet

## Exercise 3C

**1** Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

**a** $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

**e** $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

**2** Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j}$. Find:

**a** $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{a} \cdot \boldsymbol{b}$

**d** $\boldsymbol{a} \cdot \boldsymbol{c}$ **e** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b})$

Example 10

**3** **a** If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $45^\circ$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

**b** If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $135^\circ$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

**4** Expand and simplify:

**a** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 2\boldsymbol{b})$ **b** $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$

**c** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} + \boldsymbol{b})$ **d** $\dfrac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

**5** If $A$ and $B$ are points defined by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ respectively, find:

**a** $\overrightarrow{AB}$

**b** $|\overrightarrow{AB}|$

**c** the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$.

**6** Let $C$ and $D$ be points with position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c} \cdot \boldsymbol{d} = 4$, find $|\overrightarrow{CD}|$.

**7** Solve each of the following equations:

**a** $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (5\boldsymbol{i} + x\boldsymbol{j}) = -6$ **b** $(x\boldsymbol{i} + 7\boldsymbol{j}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j}) = 10$

**c** $(x\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j}) = x$ **d** $x(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} + x\boldsymbol{j}) = 6$

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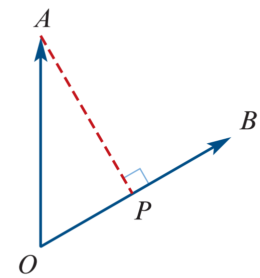
**8** Points $A$ and $B$ are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j}$. Let $P$ be the point on $OB$ such that $AP$ is perpendicular to $OB$. Then $\overrightarrow{OP} = q\boldsymbol{b}$, for a constant $q$.

- **a** Express $\overrightarrow{AP}$ in terms of $q$, $\boldsymbol{a}$ and $\boldsymbol{b}$.
- **b** Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to find the value of $q$.
- **c** Find the coordinates of the point $P$.




Example 11

**9** Find the angle, in degrees, between each of the following pairs of vectors, correct to two decimal places:

- **a** $\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{i} - 4\boldsymbol{j}$
- **b** $-2\boldsymbol{i} + \boldsymbol{j}$ and $-2\boldsymbol{i} - 2\boldsymbol{j}$
- **c** $2\boldsymbol{i} - \boldsymbol{j}$ and $4\boldsymbol{i}$
- **d** $7\boldsymbol{i} + \boldsymbol{j}$ and $-\boldsymbol{i} + \boldsymbol{j}$

**10** Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors such that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Use the geometric description of the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

*For Questions 11–12, find the angles in degrees correct to two decimal places.*

**11** Let $A$ and $B$ be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$ respectively. Let $M$ be the midpoint of $AB$. Find:

- **a** $\overrightarrow{OM}$
- **b** $\angle AOM$
- **c** $\angle BMO$

**12** Let $A$, $B$ and $C$ be the points defined by the position vectors $3\boldsymbol{i}$, $4\boldsymbol{j}$ and $-2\boldsymbol{i} + 6\boldsymbol{j}$ respectively. Let $M$ and $N$ be the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively. Find:

- **a** **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$
- **b** $\angle MON$
- **c** $\angle MOC$

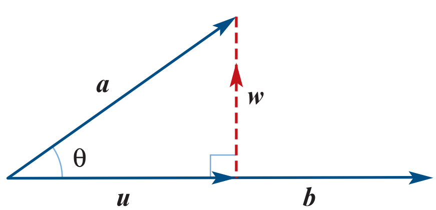
## 3D Vector projections

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram, it can be seen that

$$\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$$

where $\boldsymbol{u} = k\boldsymbol{b}$ and so $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$.




For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, we must have

$$\begin{aligned} \boldsymbol{w} \cdot \boldsymbol{b} &= 0 \\ (\boldsymbol{a} - k\boldsymbol{b}) \cdot \boldsymbol{b} &= 0 \\ \boldsymbol{a} \cdot \boldsymbol{b} - k(\boldsymbol{b} \cdot \boldsymbol{b}) &= 0 \end{aligned}$$

Hence $k = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}$ and therefore $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

This vector $\boldsymbol{u}$ is called the **vector projection** (or **vector resolute**) of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.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**Vector resolute**

The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be expressed in any one of the following equivalent forms:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Note: The quantity $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that, from our previous calculation, we have $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

Expressing $\boldsymbol{a}$ as the sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, gives

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**, one parallel to $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

## Example 12

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$. Find the vector resolute of:

**a** $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

**b** $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

### Solution

**a** $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 3 = -2$

$\boldsymbol{b} \cdot \boldsymbol{b} = 1 + 1 = 2$

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\begin{aligned}\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} &= \frac{-2}{2}(\boldsymbol{i} - \boldsymbol{j}) \\ &= -1(\boldsymbol{i} - \boldsymbol{j}) \\ &= -\boldsymbol{i} + \boldsymbol{j}\end{aligned}$$

**b** $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = -2$

$\boldsymbol{a} \cdot \boldsymbol{a} = 1 + 9 = 10$

The vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$$\begin{aligned}\frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} &= \frac{-2}{10}(\boldsymbol{i} + 3\boldsymbol{j}) \\ &= -\frac{1}{5}(\boldsymbol{i} + 3\boldsymbol{j})\end{aligned}$$

## Example 13

Find the scalar resolute of $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ in the direction of $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$.

### Solution

$\boldsymbol{a} \cdot \boldsymbol{b} = -2 + 6 = 4$

$|\boldsymbol{b}| = \sqrt{1 + 9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{10}}{5}$$


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## Example 14

Resolve $\boldsymbol{i} + 3\boldsymbol{j}$ into rectangular components, one of which is parallel to $2\boldsymbol{i} - 2\boldsymbol{j}$.

### Solution

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - 6 = -4$$
$$\boldsymbol{b} \cdot \boldsymbol{b} = 4 + 4 = 8$$

Therefore the vector resolute is

$$\frac{-4}{8}(2\boldsymbol{i} - 2\boldsymbol{j}) = -\frac{1}{2}(2\boldsymbol{i} - 2\boldsymbol{j})$$
$$= -\boldsymbol{i} + \boldsymbol{j}$$

The perpendicular component is

$$\boldsymbol{a} - (-\boldsymbol{i} + \boldsymbol{j}) = (\boldsymbol{i} + 3\boldsymbol{j}) - (-\boldsymbol{i} + \boldsymbol{j})$$
$$= 2\boldsymbol{i} + 2\boldsymbol{j}$$

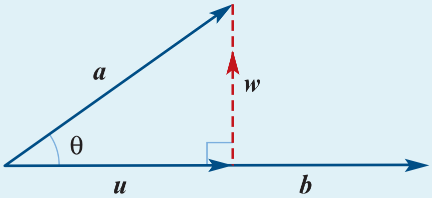
Hence we can write

$$\boldsymbol{i} + 3\boldsymbol{j} = (-\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} + 2\boldsymbol{j})$$

Check: We can check our calculation by verifying that the second component is indeed perpendicular to $\boldsymbol{b}$. We have $(2\boldsymbol{i} + 2\boldsymbol{j}) \cdot (2\boldsymbol{i} - 2\boldsymbol{j}) = 4 - 4 = 0$, as expected.

## Summary 3D

- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.




## Exercise 3D

**1** Points $A$ and $B$ are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$.

**a** Find $\hat{\boldsymbol{a}}$. **b** Find $\hat{\boldsymbol{b}}$. **c** Find $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = \overrightarrow{AB}$.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**2** Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$.

**a** Find:

**i** $\hat{\boldsymbol{a}}$ **ii** $|\boldsymbol{b}|$

**b** Find the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

**3** Points $A$ and $B$ are defined by the position vectors $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

**a** Find:

**i** $\hat{\boldsymbol{a}}$ **ii** $\hat{\boldsymbol{b}}$

**b** Find the unit vector which bisects $\angle AOB$.

Example 12

**4** For each pair of vectors, find the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:

**a** $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

**b** $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

**c** $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{i}$

Example 13

**5** For each of the following pairs of vectors, find the scalar resolute of the first vector in the direction of the second vector:

**a** $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i}$

**b** $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$

**c** $\boldsymbol{b} = 2\boldsymbol{j}$ and $\boldsymbol{a} = 2\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$

**d** $\boldsymbol{b} = \boldsymbol{i} - \sqrt{5}\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$

Example 14

**6** For each of the following pairs of vectors, find the resolution of the vector $\boldsymbol{a}$ into rectangular components, one of which is parallel to $\boldsymbol{b}$:

**a** $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 5\boldsymbol{i}$

**b** $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$

**c** $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$

**7** Let $A$ and $B$ be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$ respectively. Find:

**a** the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

**b** a unit vector through $A$ perpendicular to $OB$

**8** Let $A$ and $B$ be the points defined by the position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$ respectively. Find:

**a** the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

**b** the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$

**c** the shortest distance from $A$ to line $OB$

**9** Points $A$, $B$ and $C$ have position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

**a** **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

**b** the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

**c** the shortest distance from $B$ to line $AC$

**d** the area of triangle $ABC$


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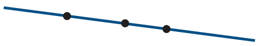
# 3E Geometric proofs

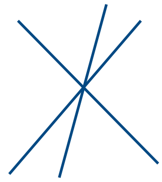
In this section we see how vectors can be used as a tool for proving geometric results.

We require the following two definitions.

**Collinear points** Three or more points are collinear if they all lie on a single line.



**Concurrent lines** Three or more lines are concurrent if they all pass through a single point.



Here are some properties of vectors that will be useful:

- For $k \in (0, \infty)$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \neq 0$. Conversely, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors such that $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \neq 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \neq 0$, then $A$, $B$ and $C$ are collinear.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

## Example 15

Three points $P$, $Q$ and $R$ have position vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $k(2\boldsymbol{p} + \boldsymbol{q})$ respectively, relative to a fixed origin $O$. The points $O$, $P$ and $Q$ are not collinear.

Find the value of $k$ if:

**a** $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$ **b** $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$ **c** $P$, $Q$ and $R$ are collinear.

### Solution

**a** 
$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR}\\ &= -\boldsymbol{q} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= 2k\boldsymbol{p} + (k-1)\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$, then there is some $\lambda \neq 0$ such that

$$2k\boldsymbol{p} + (k-1)\boldsymbol{q} = \lambda\boldsymbol{p}$$

This implies that

$$2k = \lambda \quad\text{and}\quad k - 1 = 0$$

Hence $k = 1$.

**b** 
$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PO} + \overrightarrow{OR}\\ &= -\boldsymbol{p} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= (2k-1)\boldsymbol{p} + k\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$, then there is some $m \neq 0$ such that

$$(2k-1)\boldsymbol{p} + k\boldsymbol{q} = m\boldsymbol{q}$$

This implies that

$$2k - 1 = 0 \quad\text{and}\quad k = m$$

Hence $k = \dfrac{1}{2}$.

**Note:** Since points $O$, $P$ and $Q$ are not collinear, the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are not parallel.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**c** If points $P$, $Q$ and $R$ are collinear, then there exists $n \neq 0$ such that

$$n\overrightarrow{PQ} = \overrightarrow{QR}$$

$$\therefore \quad n(-\boldsymbol{p} + \boldsymbol{q}) = 2k\boldsymbol{p} + (k-1)\boldsymbol{q}$$

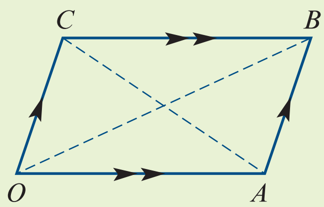
This implies that

$$-n = 2k \quad \text{and} \quad n = k - 1$$

Therefore $3k - 1 = 0$ and so $k = \frac{1}{3}$.

## Example 16

Suppose that $OABC$ is a parallelogram. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.




**a** Express each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

**i** $\overrightarrow{OB}$ **ii** $\overrightarrow{CA}$

**b** Find in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

**i** $|\overrightarrow{OB}|^2$ **ii** $|\overrightarrow{CA}|^2$

**c** Hence, prove that if the diagonals of a parallelogram are of equal length, then the parallelogram is a rectangle.

### Solution

**a** **i**
$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= \overrightarrow{OA} + \overrightarrow{OC}\\ &= \boldsymbol{a} + \boldsymbol{c}\end{aligned}$$

**ii**
$$\begin{aligned}\overrightarrow{CA} &= \overrightarrow{CB} + \overrightarrow{BA}\\ &= \overrightarrow{OA} - \overrightarrow{OC}\\ &= \boldsymbol{a} - \boldsymbol{c}\end{aligned}$$

**b** **i**
$$\begin{aligned}|\overrightarrow{OB}|^2 &= \overrightarrow{OB} \cdot \overrightarrow{OB}\\ &= (\boldsymbol{a}+\boldsymbol{c}) \cdot (\boldsymbol{a}+\boldsymbol{c})\\ &= \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{a} + \boldsymbol{c}\cdot\boldsymbol{c}\\ &= |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{c} + |\boldsymbol{c}|^2\end{aligned}$$

**ii**
$$\begin{aligned}|\overrightarrow{CA}|^2 &= \overrightarrow{CA} \cdot \overrightarrow{CA}\\ &= (\boldsymbol{a}-\boldsymbol{c}) \cdot (\boldsymbol{a}-\boldsymbol{c})\\ &= \boldsymbol{a}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{c}\cdot\boldsymbol{a} + \boldsymbol{c}\cdot\boldsymbol{c}\\ &= |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{c} + |\boldsymbol{c}|^2\end{aligned}$$

**c** Assume that the diagonals of the parallelogram $OABC$ are of equal length. Then $|\overrightarrow{OB}| = |\overrightarrow{CA}|$. This implies that

$$\begin{aligned}|\overrightarrow{OB}|^2 &= |\overrightarrow{CA}|^2\\ |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{c} + |\boldsymbol{c}|^2 &= |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{c} + |\boldsymbol{c}|^2\\ 4\boldsymbol{a}\cdot\boldsymbol{c} &= 0\\ \therefore \quad \boldsymbol{a}\cdot\boldsymbol{c} &= 0\end{aligned}$$

We have shown that $\overrightarrow{OA} \cdot \overrightarrow{OC} = 0$. So $\angle COA = 90°$. Hence the parallelogram $OABC$ is a rectangle.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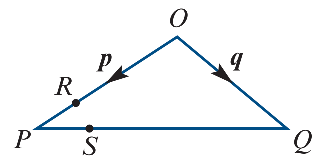
## Exercise 3E

**1** In the diagram, $OR = \frac{4}{5}OP$, $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and $PS : SQ = 1 : 4$.




**a** Express each of the following in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$:

**i** $\overrightarrow{OR}$ **ii** $\overrightarrow{RP}$ **iii** $\overrightarrow{PO}$ **iv** $\overrightarrow{PS}$ **v** $\overrightarrow{RS}$

**b** What can be said about line segments $RS$ and $OQ$?

**c** What type of quadrilateral is $ORSQ$?

**d** The area of triangle $PRS$ is $5\,\text{cm}^2$. What is the area of $ORSQ$?

**2** The position vectors of three points $A$, $B$ and $C$ relative to an origin $O$ are $\boldsymbol{a}$, $\boldsymbol{b}$ and $k\boldsymbol{a}$ respectively. The point $P$ lies on $AB$ and is such that $AP = 2PB$. The point $Q$ lies on $BC$ and is such that $CQ = 6QB$.

**a** Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

**i** the position vector of $P$ **ii** the position vector of $Q$

**b** Given that $OPQ$ is a straight line, find:

**i** the value of $k$ **ii** the ratio $\dfrac{OP}{PQ}$

**c** The position vector of a point $R$ is $\frac{7}{3}\boldsymbol{a}$. Show that $PR$ is parallel to $BC$.

Example 15

**3** The position vectors of two points $A$ and $B$ relative to an origin $O$ are $3\boldsymbol{i} + 3.5\boldsymbol{j}$ and $6\boldsymbol{i} - 1.5\boldsymbol{j}$ respectively.

**a** **i** Given that $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ and $\overrightarrow{AE} = \frac{1}{4}\overrightarrow{AB}$, write down the position vectors of $D$ and $E$.

**ii** Hence find $|\overrightarrow{ED}|$.

**b** Given that $OE$ and $AD$ intersect at $X$ and that $\overrightarrow{OX} = p\overrightarrow{OE}$ and $\overrightarrow{XD} = q\overrightarrow{AD}$, find the position vector of $X$ in terms of:

**i** $p$ **ii** $q$

**c** Hence determine the values of $p$ and $q$.

**4** Points $P$ and $Q$ have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, with reference to an origin $O$, and $M$ is the point on $PQ$ such that

$$\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$$

**a** Prove that the position vector of $M$ is given by $\boldsymbol{m} = \dfrac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha + \beta}$.

**b** Write the position vectors of $P$ and $Q$ as $\boldsymbol{p} = k\boldsymbol{a}$ and $\boldsymbol{q} = \ell\boldsymbol{b}$, where $k$ and $\ell$ are positive real numbers and $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors.

**i** Prove that the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\boldsymbol{a} + \boldsymbol{b})$.

**ii** If $M$ is the point where the internal bisector of $\angle POQ$ meets $PQ$, show that

$$\frac{\alpha}{\beta} = \frac{k}{\ell}$$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Example 16

**5** Suppose that $OABC$ is a parallelogram. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

**a** Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

**i** $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$

**b** Find $\overrightarrow{OB} \cdot \overrightarrow{AC}$.

**c** Hence, prove that the diagonals of a parallelogram intersect at right angles if and only if it is a rhombus.

**6** Suppose that $ORST$ is a parallelogram, where $O$ is the origin. Let $U$ be the midpoint of $RS$ and let $V$ be the midpoint of $ST$. Denote the position vectors of $R$, $S$, $T$, $U$ and $V$ by $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.

**a** Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$.

**b** Express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.

**c** Hence, or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

**7** Prove that the midpoints of any quadrilateral are the vertices of a parallelogram.

**8** Prove that the diagonals of a square are of equal length and bisect each other.

**9** Prove that the diagonals of a parallelogram bisect each other.

**10** Prove that the altitudes of a triangle are concurrent. That is, they meet at a point.

**11** Apollonius' theorem

For $\triangle OAB$, the point $C$ is the midpoint of side $AB$. Prove that:

**a** $4\overrightarrow{OC} \cdot \overrightarrow{OC} = OA^2 + OB^2 + 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

**b** $4\overrightarrow{AC} \cdot \overrightarrow{AC} = OA^2 + OB^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

**c** $2OC^2 + 2AC^2 = OA^2 + OB^2$

**12** If $P$ is any point in the plane of rectangle $ABCD$, prove that

$$PA^2 + PC^2 = PB^2 + PD^2$$

**13** Prove that the medians bisecting the equal sides of an isosceles triangle are equal.

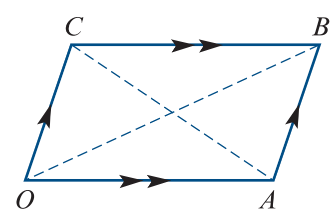
**14** **a** Prove that if $(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$ and $(\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} = 0$, then $(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{c} = 0$.

**b** Use part **a** to prove that the altitudes of a triangle meet at a point.

**15** For a parallelogram $OABC$, prove that

$$OB^2 + AC^2 = 2OA^2 + 2OC^2$$

That is, prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.





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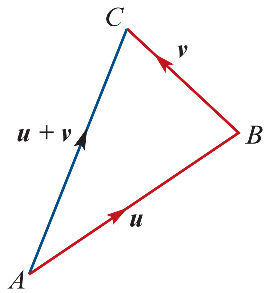
## Chapter summary

Assign-ment

Nrich

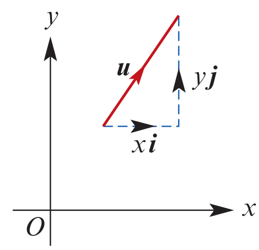
- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point $A$ to a point $B$ is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point $A$ is the vector $\overrightarrow{OA}$, where $O$ is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

### Basic operations on vectors




- Addition
  - If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.
  - The sum $\boldsymbol{u} + \boldsymbol{v}$ can also be obtained geometrically as shown.
- Scalar multiplication
  - For $k \in (0, \infty)$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of $k$.
  - The vector $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$, but the opposite direction.
  - Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there exists $k \neq 0$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- Subtraction $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

### Component form

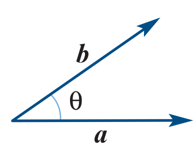



- In two dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the $x$-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the $y$-axis.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

### Scalar product and vector projections

- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$




- The scalar product is described geometrically by $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$, where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

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Review

## Short-answer questions

**1** Given that $\boldsymbol{a} = 7\boldsymbol{i} + 6\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + x\boldsymbol{j}$, find the values of $x$ for which:

**a** $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$

**b** $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same magnitude.

**2** $ABCD$ is a parallelogram where $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AD} = -2\boldsymbol{i} + 5\boldsymbol{j}$. Find the coordinates of the four vertices of the parallelogram.

**3** Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - 6\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$. Find the value of $p$ such that $\boldsymbol{a} + p\boldsymbol{b} + \boldsymbol{c}$ is parallel to the $x$-axis.

**4** The position vectors of $P$ and $Q$ are $2\boldsymbol{i} - 2\boldsymbol{j}$ and $3\boldsymbol{i} - 7\boldsymbol{j}$ respectively.

**a** Find $|\overrightarrow{PQ}|$.

**b** Find the unit vector in the direction of $\overrightarrow{PQ}$.

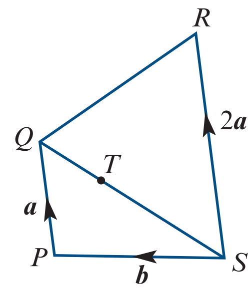
**5** The position vectors of $A$, $B$ and $C$ are $2\boldsymbol{j}$, $4\boldsymbol{i} + 10\boldsymbol{j}$ and $x\boldsymbol{i} + 14\boldsymbol{j}$ respectively. Find $x$ if $A$, $B$ and $C$ are collinear.

**6** $\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j}$ and $C$ is a point on $OA$ such that $|\overrightarrow{OC}| = \dfrac{16}{5}$.

**a** Find the unit vector in the direction of $\overrightarrow{OA}$.

**b** Hence find $\overrightarrow{OC}$.

**7** In the diagram, $ST = 2TQ$, $\overrightarrow{PQ} = \boldsymbol{a}$, $\overrightarrow{SR} = 2\boldsymbol{a}$ and $\overrightarrow{SP} = \boldsymbol{b}$.




**a** Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

**i** $\overrightarrow{SQ}$

**ii** $\overrightarrow{TQ}$

**iii** $\overrightarrow{RQ}$

**iv** $\overrightarrow{PT}$

**v** $\overrightarrow{TR}$

**b** Show that $P$, $T$ and $R$ are collinear.

**8** If $\boldsymbol{a} = 5\boldsymbol{i} - s\boldsymbol{j}$ and $\boldsymbol{b} = t\boldsymbol{i} + 2\boldsymbol{j}$ are equal vectors.

**a** Find $s$ and $t$.

**b** Find $|\boldsymbol{a}|$.

**9** The vector $\boldsymbol{p}$ has magnitude 7 units and bearing 050° and the vector $\boldsymbol{q}$ has magnitude 12 units and bearing 170°. (These are compass bearings on the horizontal plane.) Draw a diagram (not to scale) showing $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{p} + \boldsymbol{q}$. Calculate the magnitude of $\boldsymbol{p} + \boldsymbol{q}$.

**10** Let $O$, $A$ and $B$ be the points $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

**a** If $C$ is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$, find the coordinates of $C$.

**b** If $D$ is the point $(1, 24)$ and $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, find the values of $h$ and $k$.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**11** Let $\boldsymbol{p} = 3\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} - 5\boldsymbol{j}$. Find the values of $m$ and $n$ such that $m\boldsymbol{p} + n\boldsymbol{q} = 8\boldsymbol{i} + 9\boldsymbol{j}$.

**12** The points $A$, $B$ and $C$ have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ relative to an origin $O$. Write down an equation connecting $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for each of the following cases:

**a** $OABC$ is a parallelogram

**b** $B$ divides $AC$ in the ratio $3 : 2$. That is, $AB : BC = 3 : 2$.

**13** Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

**a** $\boldsymbol{a} \cdot \boldsymbol{a}$

**b** $\boldsymbol{b} \cdot \boldsymbol{b}$

**c** $\boldsymbol{c} \cdot \boldsymbol{c}$

**d** $\boldsymbol{a} \cdot \boldsymbol{b}$

**e** $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$

**f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

**g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

**14** Points $A$, $B$ and $C$ have position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{c} = -5\boldsymbol{i} + 3\boldsymbol{j}$ respectively. Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and hence show that $\triangle ABC$ is right-angled at $B$.

**15** Given the vectors $\boldsymbol{p} = 5\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} + t\boldsymbol{j}$, find the values of $t$ for which:

**a** $\boldsymbol{p} + \boldsymbol{q}$ is parallel to $\boldsymbol{p} - \boldsymbol{q}$

**b** $\boldsymbol{p} - 2\boldsymbol{q}$ is perpendicular to $\boldsymbol{p} + 2\boldsymbol{q}$

**c** $|\boldsymbol{p} - \boldsymbol{q}| = |\boldsymbol{q}|$

**16** Points $A$, $B$ and $C$ have position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

**a** **i** $\overrightarrow{AB}$

**ii** $\overrightarrow{AC}$

**b** the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

**c** the shortest distance from $B$ to the line $AC$.

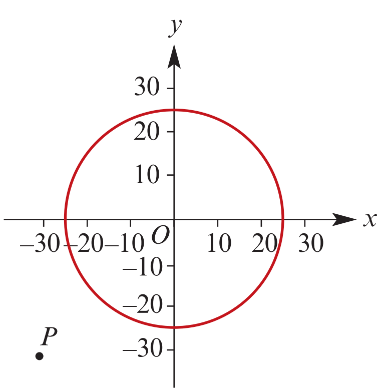
## Extended-response questions

**1** Let $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represent a displacement 1 km due east.

Let $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represent a displacement 1 km due north.

The diagram shows a circle of radius 25 km with centre at $O(0, 0)$. A lighthouse entirely surrounded by sea is located at $O$. The lighthouse is not visible from points outside the circle.

A ship is initially at point $P$, which 31 km west and 32 km south of the lighthouse.




**a** Write down the vector $\overrightarrow{OP}$.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Review

The ship is travelling in the direction of vector $\boldsymbol{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ with speed 20 km/h.

An hour after leaving $P$, the ship is at point $R$.

**b** Show that $\overrightarrow{PR} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$ and hence find the vector $\overrightarrow{OR}$.

**c** Show that the lighthouse first becomes visible when the ship reaches $R$.

**2** Given that $\boldsymbol{p} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{q} = -2\boldsymbol{i} + 4\boldsymbol{j}$, find:

**a** $|\boldsymbol{p} - \boldsymbol{q}|$

**b** $|\boldsymbol{p}| - |\boldsymbol{q}|$

**c** $\boldsymbol{r}$ such that $\boldsymbol{p} + 2\boldsymbol{q} + \boldsymbol{r} = \boldsymbol{0}$

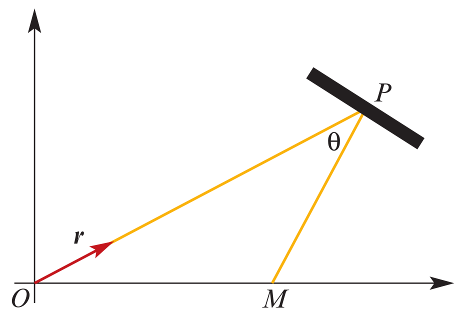
**3** The quadrilateral $PQRS$ is a parallelogram. The point $P$ has coordinates $(5, 8)$, the point $R$ has coordinates $(32, 17)$ and the vector $\overrightarrow{PQ}$ is given by $\overrightarrow{PQ} = \begin{bmatrix} 20 \\ -15 \end{bmatrix}$.

**a** Find the coordinates of $Q$ and write down the vector $\overrightarrow{QR}$.

**b** Write down the vector $\overrightarrow{RS}$ and show that the coordinates of $S$ are $(12, 32)$.

**4** The diagram shows the path of a light beam from its source at $O$ in the direction of the vector $\boldsymbol{r} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

At point $P$, the beam is reflected by an adjustable mirror and meets the $x$-axis at $M$. The position of $M$ varies, depending on the adjustment of the mirror at $P$.




**a** Given that $\overrightarrow{OP} = 4\boldsymbol{r}$, find the coordinates of $P$.

**b** The point $M$ has coordinates $(k, 0)$. Find an expression, in terms of $k$, for vector $\overrightarrow{PM}$.

**c** Find the magnitudes of vectors $\overrightarrow{OP}$, $\overrightarrow{OM}$ and $\overrightarrow{PM}$, and hence find the value of $k$ for which $\theta$ is equal to 90°.

**d** Find the value $\theta$ for which $M$ has coordinates $(9, 0)$.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# 4 Statics of a particle

## In this chapter



## Syllabus references

**Topic:** Algebra of vectors in the plane

**Subtopic:** 1.3.14

A force is a vector quantity, so it is recommended that you complete Chapter 3 before beginning this chapter.

A **force** is a measure of the strength of a **push** or **pull**. Forces can:

- start motion
- stop motion
- make objects move faster or slower
- change the direction of motion.

Force can be defined as the physical quantity that causes a change in motion.

In this chapter, we are only concerned with the situation where the forces 'cancel each other out'. This is the study of **statics**, which is part of the area of mathematical physics called **classical mechanics**.

For example, you may have heard of **Archimedes' principle**, which applies to objects in a fluid. If an inflatable raft is floating in a swimming pool, then the water is exerting an upwards force on the raft (called the buoyant force) that cancels out the downwards force of gravity.

Situations where the forces do not 'cancel out' are studied in Mathematics Specialist Units 3 & 4.


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# 4A Forces and triangle of forces

A force has both magnitude and direction – it may be represented by a vector.

When considering the forces that act on an object, it is convenient to treat the forces as acting on a single particle. The single particle may be thought of as a point at which the entire mass of the object is concentrated.

## Weight and units of force

Every object near the surface of Earth is subject to the force of gravity. We refer to this force as the **weight** of the object. Weight is a force that acts vertically downwards on an object (actually towards the centre of Earth).

The unit of force used in this chapter is the **kilogram weight** (kg wt). If an object has a **mass** of 1 kg, then the force due to gravity acting on the object is 1 kg wt.

This unit is convenient for objects near Earth's surface. An object with a mass of 1 kg would have a different weight on Earth's moon.

Note: The standard unit of force is the newton (N). At Earth's surface, a mass of $m$ kg has a force of $m$ kg wt $= mg$ N acting on it, where $g$ is the acceleration due to gravity.

## Resultant force and equilibrium

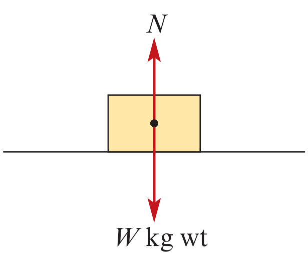
When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**. The resultant force is the vector sum of the forces acting on the particle.

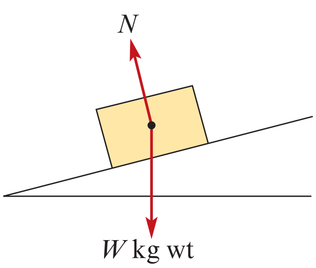
If the resultant force acting on an object is zero, the object will remain at rest or continue moving with constant velocity. The object is said to be in **equilibrium**.

Note: Planet Earth is moving and our galaxy is moving, but we use Earth as our frame of reference and so our observation of an object being at rest is determined in this way.

## Normal force

Any mass placed on a surface, either horizontal or inclined, experiences a force perpendicular to the surface. This force is referred to as a **normal force**.




For example, a book sitting on a table is obviously being subjected to a force due to gravity. But the fact that it does not fall to the ground indicates that there must be a second force on the book. The table is exerting a force on the book equal in magnitude to gravity, but in the opposite direction. Hence the book remains at rest; it is in **equilibrium**.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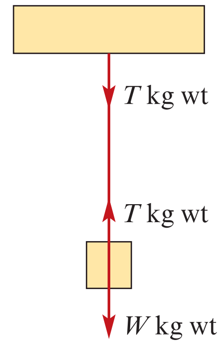
## Tension force



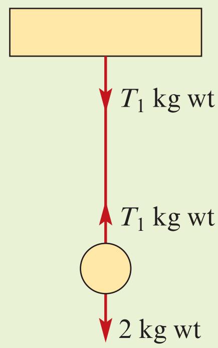

The diagram shows a string attached to the ceiling supporting a mass, which is at rest. The force of gravity, $W$ kg wt, acts downwards on the mass and the string exerts an equal force, $T$ kg wt, upwards on the mass. The force exerted by the string is called the **tension force**.

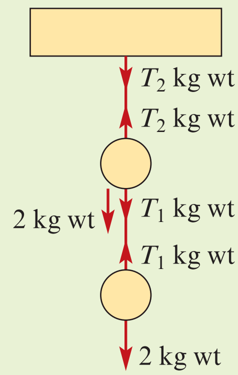
Note that there is a force, equal in magnitude but opposite in direction, acting on the ceiling at the point of contact.

### Example 1

**a** In the diagram on the left, one spherical mass of weight 2 kg wt is attached to the ceiling. Find $T_1$.

**b** In the diagram on the right, two equal spherical masses of weight 2 kg wt are attached to the ceiling as shown. Find $T_1$ and $T_2$.




**Solution**

The tensions in the one section of string have the same magnitude.

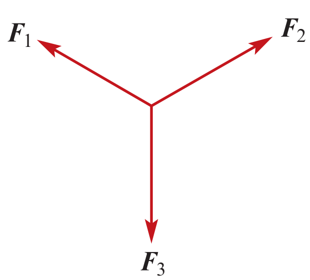
**a** $T_1 = 2$ kg wt

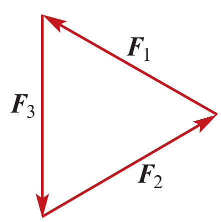
**b** The forces acting on the lower mass are the same as before, and so $T_1 = 2$ kg wt.
For the higher mass, we have $T_2 = 2 + T_1$ and so $T_2 = 4$ kg wt.

## Triangle of forces

If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

Suppose that three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on a particle in equilibrium, as shown in the diagram below on the left. Since the particle is in equilibrium, we must have $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$. Therefore the three forces can be rearranged into a triangle as shown below on the right.




The magnitudes of the forces and the angles between the forces can now be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.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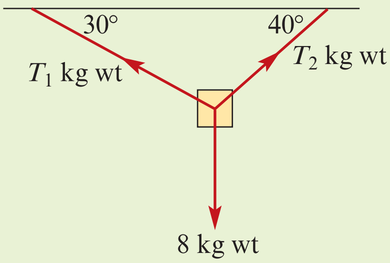
In the following examples and exercises, strings and ropes are considered to have negligible mass. A smooth light pulley is considered to have negligible mass and the friction between a rope and pulley is considered to be negligible.

## Example 2

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 40° to the horizontal, find the tension in each string.



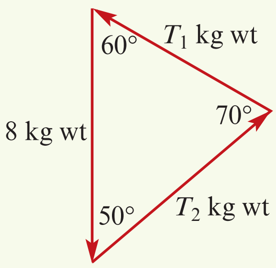

### Solution

Represent the forces in a triangle. The sine rule gives

$$\frac{T_1}{\sin 50^\circ} = \frac{T_2}{\sin 60^\circ} = \frac{8}{\sin 70^\circ}$$

$$T_1 = \frac{8}{\sin 70^\circ} \times \sin 50^\circ \approx 6.52 \text{ kg wt}$$

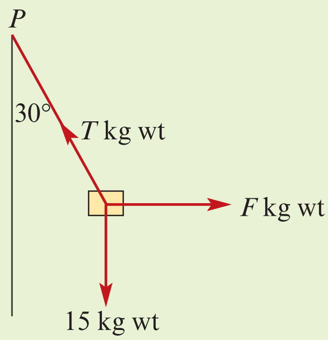
$$T_2 = \frac{8}{\sin 70^\circ} \times \sin 60^\circ \approx 7.37 \text{ kg wt}$$




## Example 3

A particle of mass 15 kg is suspended vertically from a point $P$ by a string. The particle is pulled horizontally by a force of $F$ kg wt so that the string makes an angle of 30° with the vertical.

Find the value of $F$ and the tension in the string.




### Solution

Representing the forces in a triangle gives

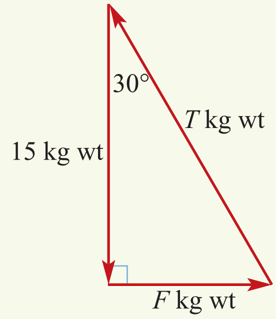
$$\frac{F}{15} = \tan 30^\circ$$

$$F = 15 \tan 30^\circ = 5\sqrt{3}$$

and $$\frac{15}{T} = \cos 30^\circ$$

$$T = \frac{15}{\cos 30^\circ} = 10\sqrt{3}$$

The tension in the string is $10\sqrt{3}$ kg wt.





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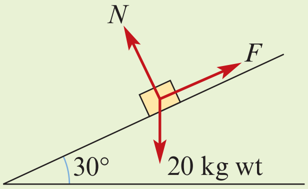
## Example 4

A body of mass 20 kg is placed on a smooth plane inclined at 30° to the horizontal. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.

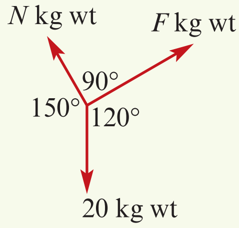


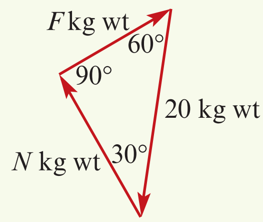

### Solution

The three forces form a triangle (as the body is in equilibrium). Therefore

$$F = 20 \sin 30^\circ = 10 \text{ kg wt}$$

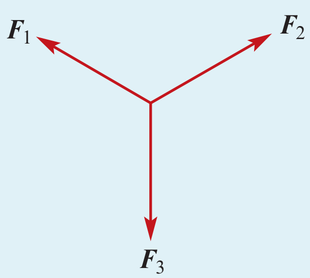
$$N = 20 \cos 30^\circ = 10\sqrt{3} \text{ kg wt}$$

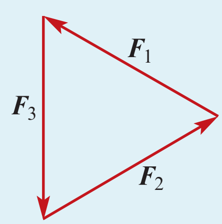



## Summary 4A

- **Force** is a vector quantity.
- The magnitude of a force can be measured using **kilogram weight** (kg wt). If an object near the surface of Earth has a mass of 1 kg, then the force due to gravity acting on the object is 1 kg wt.
- Triangle of forces
  If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.



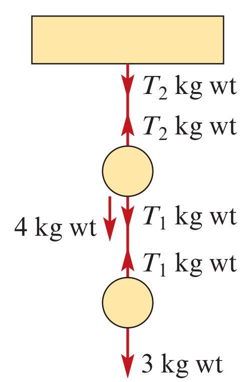




Skill-sheet

## Exercise 4A

Example 1

**1** Two spherical weights of mass 3 kg and 4 kg are attached to the ceiling as shown. Find $T_1$ and $T_2$.





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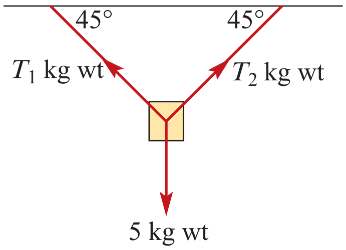
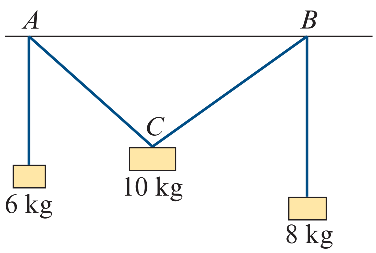
Example 2

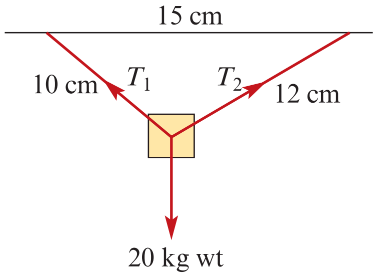
**2** A particle of mass 5 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 45° with the horizontal, find the tension in each string.




**3** Using strings and pulleys, three weights of mass 6 kg, 8 kg and 10 kg are suspended in equilibrium as shown. Calculate the magnitude of the angle $ACB$.




**4** A mass of 20 kg is suspended from two strings of length 10 cm and 12 cm, the ends of the strings being attached to two points in a horizontal line, 15 cm apart. Find the tension in each string.

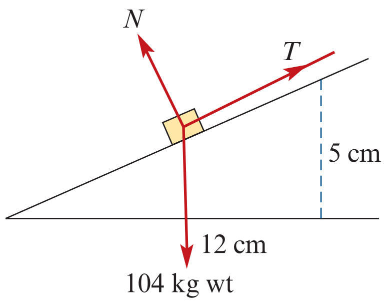



Example 3

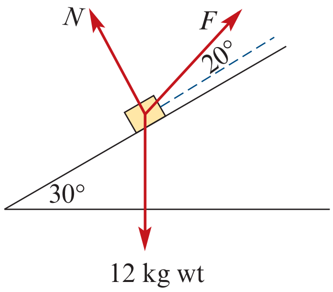
**5** A boat is being pulled by a force of 40 kg wt towards the east and by a force of 30 kg wt towards the north-west. What third force must be acting on the boat if it remains stationary? Give the magnitude and direction.

Example 4

**6** A body of mass 104 kg is placed on a smooth inclined plane which rises 5 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.




**7** A body of mass 12 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 20° to the plane. Find the magnitude of the force.





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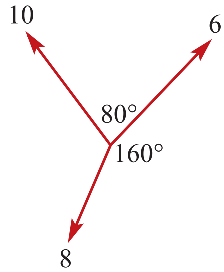
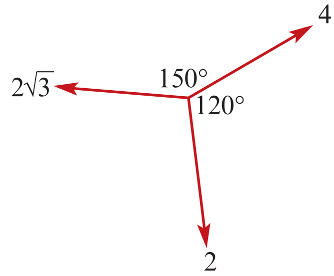
**8** In each of the following cases, determine whether the particle is in equilibrium:

**a**




**b**




**9** Three forces of magnitude 4 kg wt, 7 kg wt and 10 kg wt are in equilibrium. Determine the magnitudes of the angles between the forces.

**10** A mass of 15 kg is maintained at rest on a smooth inclined plane by a string that is parallel to the plane. Determine the tension in the string if:

**a** the plane is at 30° to the horizontal

**b** the plane is at 40° to the horizontal

**c** the plane is at 30° to the horizontal, but the string is held at an angle of 10° to the plane.

**11** A string is connected to two points $A$ and $D$ in a horizontal line and masses of 12 kg and $W$ kg are attached at points $B$ and $C$. If $AB$, $BC$ and $CD$ make angles of 40°, 20° and 50° respectively with the horizontal, calculate the tensions in the string and the value of $W$.

## 4B Resolution of forces

Obviously there are many situations where more than three forces (or in fact only two forces) will be acting on a body. An alternative method is required to solve such problems.

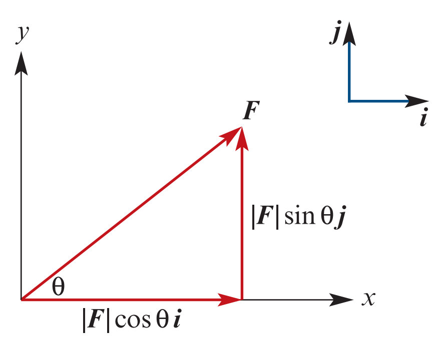
If all forces under consideration are acting in the same plane, then these forces and the resultant force can each be expressed as a sum of its $\boldsymbol{i}$- and $\boldsymbol{j}$-components.

If a force $\boldsymbol{F}$ acts at an angle of $\theta$ to the $x$-axis, then $\boldsymbol{F}$ can be written as the sum of two forces, one 'horizontal' and the other 'vertical':

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\,\boldsymbol{j}$$

The force $\boldsymbol{F}$ is **resolved** into two components:

- the $\boldsymbol{i}$-component is parallel to the $x$-axis
- the $\boldsymbol{j}$-component is parallel to the $y$-axis.





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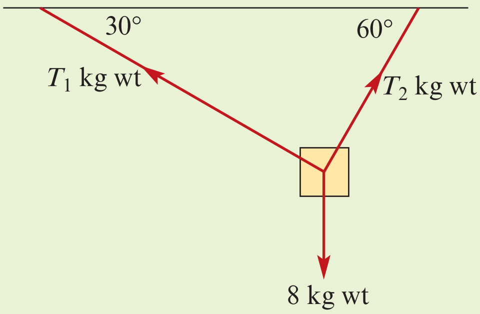
For a particle that is in equilibrium, if all the forces acting on the particle are resolved into their $\boldsymbol{i}$- and $\boldsymbol{j}$-components, then:

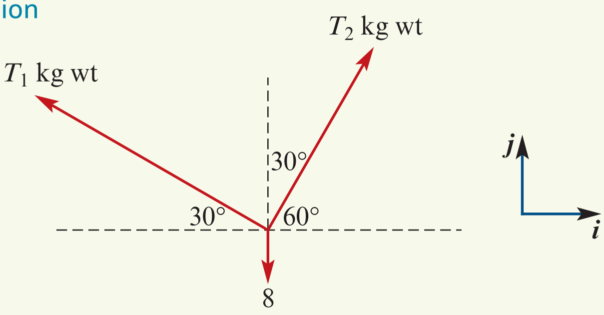
- the sum of all the $\boldsymbol{i}$-components is zero
- the sum of all the $\boldsymbol{j}$-components is zero.

## Example 5

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 60° to the horizontal, find the tension in each string.




### Solution




Resolution in the $\boldsymbol{j}$-direction:

$$T_1 \sin 30^\circ + T_2 \sin 60^\circ - 8 = 0$$

$$T_1\left(\frac{1}{2}\right) + T_2\left(\frac{\sqrt{3}}{2}\right) - 8 = 0 \qquad (1)$$

Resolution in the $\boldsymbol{i}$-direction:

$$-T_1 \cos 30^\circ + T_2 \cos 60^\circ = 0$$

$$-T_1\left(\frac{\sqrt{3}}{2}\right) + T_2\left(\frac{1}{2}\right) = 0 \qquad (2)$$

From (2): $\sqrt{3}\,T_1 = T_2$

Substituting in (1) gives

$$T_1\left(\frac{1}{2}\right) + \sqrt{3}\,T_1\left(\frac{\sqrt{3}}{2}\right) - 8 = 0$$

$$4T_1 = 16$$

$$\therefore \quad T_1 = 4$$

Hence $T_1 = 4$ and $T_2 = 4\sqrt{3}$. The tensions in the strings are 4 kg wt and $4\sqrt{3}$ kg wt.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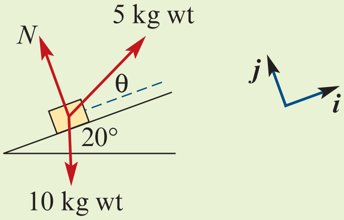
## Example 6

A body of mass 10 kg is held at rest on a smooth plane inclined at 20° by a string with tension 5 kg wt as shown.

Find the angle between the string and the inclined plane.




### Solution

We resolve the forces parallel and perpendicular to the plane. Then $N$ has no parallel component, since $N$ is perpendicular to the plane.

Resolving in the $\boldsymbol{i}$-direction:

$$5\cos\theta^\circ - 10\sin 20^\circ = 0$$

$$\cos\theta^\circ = \frac{10\sin 20^\circ}{5}$$

$$\therefore \quad \theta = \cos^{-1}(0.684)$$

$$= 46.84^\circ$$

## Summary 4B

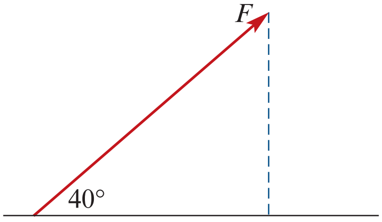
- A force $\boldsymbol{F}$ is **resolved** into components by writing it in the form $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$.
- If forces are acting on a particle that is in equilibrium, then:
  - the sum of the $\boldsymbol{i}$-components of all the forces is zero
  - the sum of the $\boldsymbol{j}$-components of all the forces is zero.

Skill-sheet

## Exercise 4B

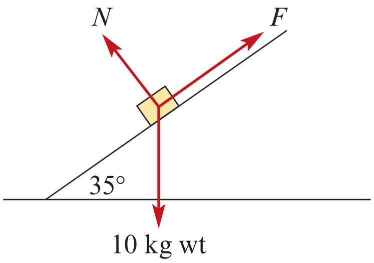
*For the following questions, give answers correct to two decimal places.*

**1** A force $F$ kg wt makes an angle of 40° with the horizontal. If its horizontal component is a force of 10 kg wt, find the magnitude of $F$.




Example 5

**2** Find the magnitude of the force, acting on a smooth inclined plane of angle 35°, required to support a mass of 10 kg resting on the plane.





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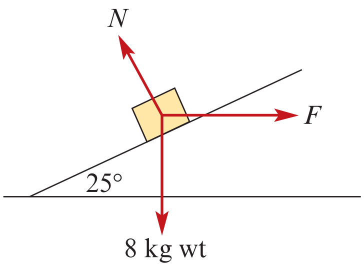
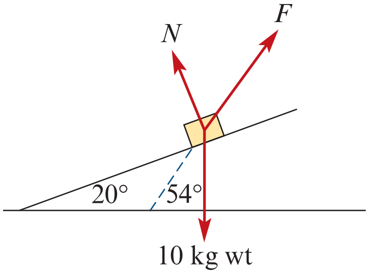
**3** A body of mass 8 kg rests on a smooth inclined plane of angle 25° under the action of a horizontal force. Find the magnitude of the force and the reaction of the plane on the body.




Example 6

**4** A body of mass 10 kg rests on a smooth inclined plane of angle 20°. Find the force that will keep it in equilibrium when it acts at an angle of 54° with the horizontal.

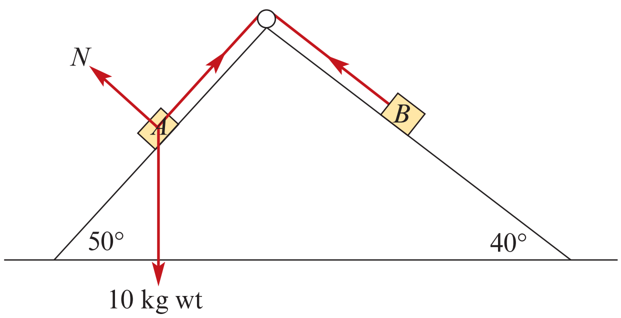



**5** If a body of mass 12 kg is suspended by a string, find the horizontal force required to hold it at an angle of 30° from the vertical.

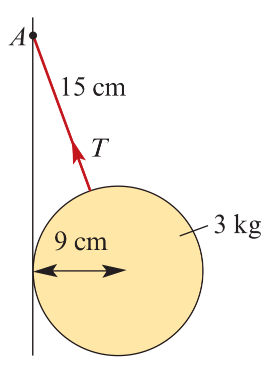
**6** A force of 20 kg wt acting directly up a smooth plane inclined at an angle of 40° maintains a body in equilibrium on the plane. Calculate the weight of the body and the pressure it exerts on the plane.

**7** Two men are supporting a block by ropes. One exerts a force of 20 kg wt, his rope making an angle of 35° with the vertical, and the other exerts a force of 30 kg wt. Determine the weight of the block and the angle of direction of the second rope.

**8** A body *A* of mass 10 kg is supported against a smooth plane of angle 50°. Find the pressure of the body on the plane and the tension in the string, which is parallel to the slope. A body *B* on a plane of angle 40° is connected to *A* by a string passing over a smooth pulley on the ridge. If the system is in equilibrium, what is the mass of *B*?




**9** A sphere of radius 9 cm is attached to a point *A* on a vertical wall by a string of length 15 cm. If the mass of the sphere is 3 kg, find the tension in the string.





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Review

## Chapter summary

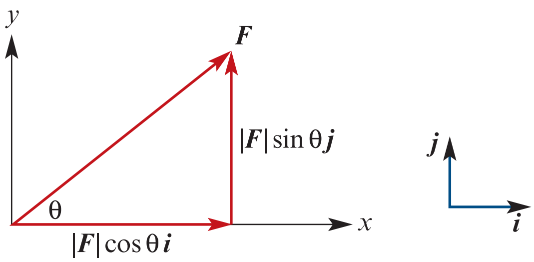
Assign-ment

Nrich

- **Resultant force** When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**.
- **Equilibrium** If the resultant force acting on an object is zero, then the object is said to be in **equilibrium**; it will remain at rest or continue moving with constant velocity.
- **Triangle of forces** If three forces are acting on a particle in equilibrium, then the vectors representing the forces may be arranged to form a triangle. The magnitudes of the forces and the angles between them can be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.
- **Resolution of forces**

  If all forces on a particle are acting in two dimensions, then each force can be expressed in terms of its components in the $\boldsymbol{i}$- and $\boldsymbol{j}$-directions:

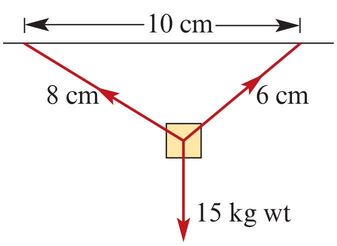
  $$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\,\boldsymbol{j}$$



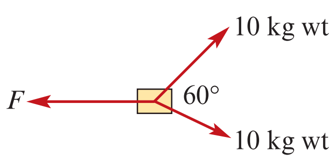

  For the particle to be in equilibrium, the sum of all the $\boldsymbol{i}$-components must be zero and the sum of all the $\boldsymbol{j}$-components must be zero.

## Short-answer questions

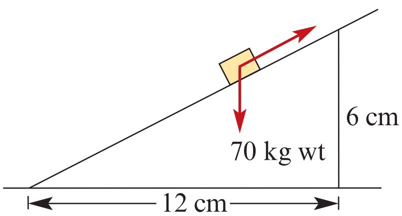
**1** A mass of 15 kg is suspended from two strings of length 6 cm and 8 cm, the ends of the strings being attached to two points in a horizontal line, 10 cm apart. Find the tension in each string.




**2** An object is being pulled by two forces of 10 kg wt as shown in the diagram. What is the magnitude and direction of the third force acting on the object if it remains stationary?




**3** A body of mass 70 kg is placed on a smooth inclined plane which rises 6 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.





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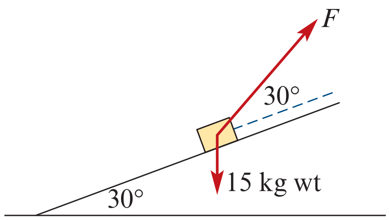
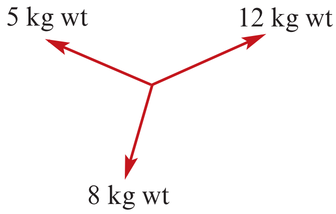
Review

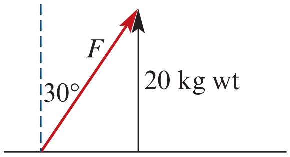
**4** A body of mass 15 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 30° to the plane. Find the magnitude of the force.



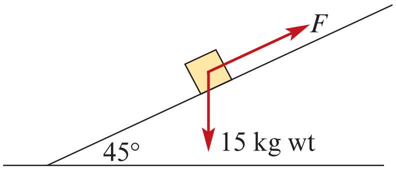

**5** Three forces of magnitude 5 kg wt, 8 kg wt and 12 kg wt are in equilibrium. Determine the cosine of the angle between the 5 kg wt and 12 kg wt forces.




**6** A force of $F$ kg wt makes an angle of 30° with the vertical. If its vertical component is a force of 20 kg wt, find the magnitude of $F$.




**7** Find the magnitude of the force, acting up a smooth inclined plane of angle 45°, required to support a mass of 15 kg resting on the plane.




**8** A force of 14 kg wt acting directly up a smooth plane inclined at an angle of 30° maintains a body in equilibrium on the plane. Calculate the weight of the body and the force it exerts on the plane.

**9** A body of mass 12 kg is kept at rest on a smooth incline of 30° by a horizontal force. Find the magnitude of the force.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# 5 Revision of Chapters 1–4

Revision

## 5A Short-answer questions

**1** How many ways can four different books be arranged on a shelf?

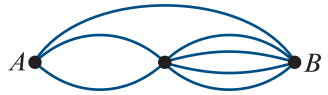
**2** How many ways can three teachers and three students be arranged in a row if a teacher must be at the start of the row?

**3** How many different three-digit numbers can be formed using the digits 1, 3, 5, 7 and 9:

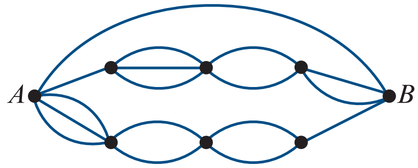
**a** as many times as you would like  **b** at most once?

**4** Travelling from left to right, how many paths are there from point $A$ to point $B$ in each of the following diagrams?

**a**




**b**




**5** Evaluate each of the following:

**a** $4!$  **b** $\dfrac{6!}{4!}$  **c** $\dfrac{8!}{6!\,2!}$  **d** ${}^{10}C_2$

**6** How many ways can five children be arranged on a bench with space for:

**a** four children  **b** five children?

**7** A bookshelf has three different mathematics books and two different physics books. How many ways can these books be arranged:

**a** without restriction

**b** if the mathematics books are kept together?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**8** Using the digits 0, 1, 2, 3 and 4 without repetition, how many five-digit numbers can you form:

**a** without restriction

**b** that are divisible by 10

**c** that are greater than 20 000

**d** that are even?

**9** Asha has three identical 20 cent pieces and two identical 10 cent pieces. How many ways can she arrange these coins in a row?

**10** How many ways can you select:

**a** three children from a group of six

**b** two letters from the alphabet

**c** four numbers from the set $\{1, 2, \dots, 10\}$

**d** three sides of an octagon?

**11** Consider the set of numbers $X = \{1, 2, \dots, 8\}$.

**a** How many subsets of $X$ have exactly two elements?

**b** How many subsets of $X$ have exactly three elements, one of which is the number 8?

**c** Find the total number of subsets of $X$.

**12** How many ways can you select three boys and two girls from a group of five boys and four girls?

**13** There are four Labor and five Liberal parliamentarians, from which four are to be selected to form a committee. If the committee must include at least one member from each party, how many ways can this be done?

**14** There are 10 blue, 11 green and 12 red balls in a bag. How many balls must be chosen at random to be sure that at least three will have the same colour?

**15** A group of 50 students were interviewed about the types of movies that they watch: 25 of the students like action movies, 26 like comedy, 17 like drama, 11 like action and comedy, 5 like action and drama, 8 like comedy and drama, and 3 like all three. How many of these students:

**a** like none of these types of movies

**b** like action movies only

**c** like action and comedy but not drama?


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**16** Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -3\boldsymbol{i} - 2\boldsymbol{j}$. Find:

**a** $\boldsymbol{a} \cdot \boldsymbol{a}$

**b** $\boldsymbol{b} \cdot \boldsymbol{b}$

**c** $\boldsymbol{c} \cdot \boldsymbol{c}$

**d** $\boldsymbol{a} \cdot \boldsymbol{b}$

**e** $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$

**f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

**g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

**17** The points $A$, $B$, $C$ and $D$ have position vectors $\overrightarrow{OA} = 4\boldsymbol{i} + 2\boldsymbol{j}$, $\overrightarrow{OB} = -\boldsymbol{i} + 7\boldsymbol{j}$, $\overrightarrow{OC} = 8\boldsymbol{i} + 6\boldsymbol{j}$ and $\overrightarrow{OD} = p\boldsymbol{i} - 2\boldsymbol{j}$.

**a** Find the values of $m$ and $n$ such that $m\overrightarrow{OA} + n\overrightarrow{BC} = 2\boldsymbol{i} + 10\boldsymbol{j}$.

**b** Find the value of $p$ such that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CD}$.

**c** Find the value of $p$ such that $|\overrightarrow{AD}| = \sqrt{17}$.

**18** Two forces of equal magnitude $F$ kg wt act on a particle and they have a resultant force of magnitude 6 kg wt. When one of the forces is doubled in magnitude, the resultant force is 11 kg wt. Find the value of $F$ and the cosine of the angle between the two forces.

**19** Two forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ of magnitudes $P = 6$ kg wt and $Q = 2$ kg wt have a resultant force of magnitude 5 kg wt. Find the cosine of the angle between forces $\boldsymbol{P}$ and $\boldsymbol{Q}$.

**20** The following forces are acting in the same plane on a particle:

- 6 kg wt in a direction of 045°
- 8 kg wt in a direction of 180°
- $Q$ kg wt

If the particle is in equilibrium, find the square of the magnitude of $Q$.

**21** A block of mass 10 kg is maintained at rest on a smooth plane inclined at 30° to the horizontal by a string. Calculate the tension in the string and the reaction of the plane if:

**a** the string is parallel to the plane

**b** the string is horizontal.

**22** A mass of 8 kg is supported by a string attached to a fixed point and is pulled from the vertical by a horizontal force of 6 kg wt. Find the tension in the string and the tangent of the angle that the string makes with the vertical.

**23** A mass of 10 kg is suspended by two strings of lengths 5 cm and 12 cm that are attached to fixed points on the same horizontal level 13 cm apart. Find the tensions in the strings.


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## 5B Extended-response questions

**1** A five-digit number is formed using the digits 0, 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done:

**a** without restriction

**b** if the number is divisible by 10

**c** if the number is odd

**d** if the number is even?

**2** Mike and Sonia belong to a group of eight coworkers. There are three men and five women in this group. A team of four workers is required to complete a project. How many ways can the team be selected:

**a** without restriction

**b** if it must contain two men and two women

**c** if it must contain both Mike and Sonia

**d** if it must not contain both Mike and Sonia?

**3** A sailing boat has three identical black flags and three identical red flags. The boat can send signals to nearby boats by arranging flags along its mast.

**a** How many ways can all six flags be arranged in a row?

**b** How many ways can all six flags be arranged in a row if no two black flags are adjacent?

**c** Using at least one flag, how many arrangements in a row are possible?

**4** Consider the letters in the word BAGGAGE.

**a** How many arrangements of these letters are there?

**b** How many arrangements begin and end with a vowel?

**c** How many arrangements begin and end with a consonant?

**d** How many arrangements have all vowels together and all consonants together?

**5** There are 25 people at a party.

**a** If every person shakes hands with every other person, what is the total number of handshakes?

**b** In fact, there are two rival groups at the party, so everyone only shakes hands with every other person in their group. If there are 150 handshakes, how many people are in each of the rival groups?

**c** At another party, there are 23 guests. Explain why it is not possible for each person to shake hands with exactly three other guests.

**6** Seventy-six photographers submitted work for a photographic exhibition in which they were permitted to enter not more than one photograph in each of three categories: black and white ($B$), colour prints ($C$), transparencies ($T$). Eighteen entrants had all their work rejected, while 30 $B$, 30 $T$ and 20 $C$ were accepted.


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- From the exhibitors, as many showed $T$ only as showed $T$ and $C$.
- There were three times as many exhibitors showing $B$ only as showing $C$ only.
- Four exhibitors showed $B$ and $T$ but not $C$.

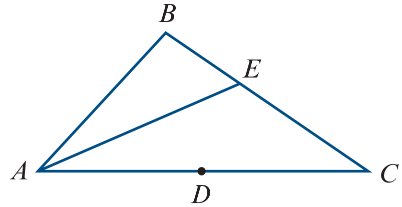
**a** Write the last three sentences in symbolic form.

**b** Draw a Venn diagram representing the information.

**c** **i** Find $|B \cap C \cap T|$.

**ii** Find $|B \cap C \cap T'|$.

**7** In the diagram, $D$ is the midpoint of $AC$ and $E$ is the point on $BC$ such that $BE : EC = 1 : t$, where $t > 0$. Suppose that $DE$ is extended to a point $F$ such that $DE : EF = 1 : 7$.




Let $\boldsymbol{a} = \overrightarrow{AD}$ and $\boldsymbol{b} = \overrightarrow{AB}$.

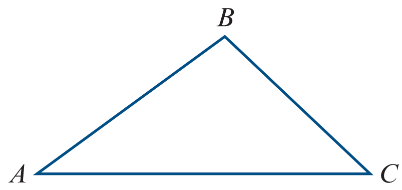
**a** Express $\overrightarrow{AE}$ in terms of $t$, $\boldsymbol{a}$ and $\boldsymbol{b}$.

**b** Express $\overrightarrow{AE}$ in terms of $\boldsymbol{a}$ and $\overrightarrow{AF}$.

**c** Show that $\overrightarrow{AF} = \dfrac{9 - 7t}{1 + t}\boldsymbol{a} + \dfrac{8t}{1 + t}\boldsymbol{b}$.

**d** If $A$, $B$ and $F$ are collinear, find the value of $t$.

**8** The vertices $A$, $B$ and $C$ of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin in the plane $ABC$.




**a** Let $P$ be an arbitrary point on the line segment $AB$. Show that the position vector of $P$ can be written in the form

$$m\boldsymbol{a} + n\boldsymbol{b}, \quad \text{where } m \geq 0,\ n \geq 0 \text{ and } m + n = 1$$

Hint: Assume that $P$ divides $AB$ in the ratio $x : y$.

**b** Find $\overrightarrow{PC}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

**c** Let $Q$ be an arbitrary point on the line segment $PC$. Show that the position vector of $Q$ can be written in the form

$$\lambda\boldsymbol{a} + \mu\boldsymbol{b} + \gamma\boldsymbol{c}, \quad \text{where } \lambda \geq 0,\ \mu \geq 0,\ \gamma \geq 0 \text{ and } \lambda + \mu + \gamma = 1$$

Note: The triple of numbers $(\lambda, \mu, \gamma)$ are known as the **barycentric coordinates** of the point $Q$ in the triangle $ABC$.

**9** $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OP} = \dfrac{4}{5}\overrightarrow{OA}$ and $Q$ is the midpoint of $AB$.

**a** Express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

**b** The line segments $PQ$ and $OB$ are extended to meet at $R$, with $\overrightarrow{QR} = n\overrightarrow{PQ}$ and $\overrightarrow{BR} = k\boldsymbol{b}$. Express the vector $\overrightarrow{QR}$ in terms of:

**i** $n$, $\boldsymbol{a}$ and $\boldsymbol{b}$

**ii** $k$, $\boldsymbol{a}$ and $\boldsymbol{b}$

**c** Find the values of $n$ and $k$.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Revision

**10** **a** A man walks north at a rate of 4 km/h and notices that the wind *appears* to blow from the west. He doubles his speed and now the wind appears to blow from the north-west. What is the velocity of the wind?

Note: Both the direction and the magnitude must be given.

**b** A river 400 m wide flows from east to west at a steady speed of 1 km/h. A swimmer, whose speed in still water is 2 km/h, starts from the south bank and heads north across the river. Find the swimmer's speed over the river bed and how far downstream he is when he reaches the north bank.

**c** To a motorcyclist travelling due north at 50 km/h, the wind appears to come from the north-west at 60 km/h. What is the true velocity of the wind?

**d** A dinghy in distress is 6 km on a bearing of 230° from a lifeboat and is drifting in a direction of 150° at 5 km/h. In what direction should the lifeboat travel to reach the dinghy as quickly as possible if the maximum speed of the lifeboat is 35 km/h?

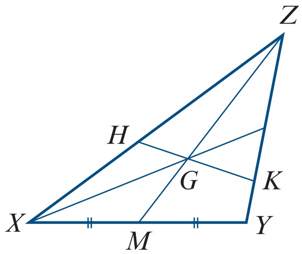
**11** **a** Let points $O$, $A$, $B$ and $C$ be coplanar and let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel. If points $A$, $B$ and $C$ are collinear with

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} \quad \text{where } \alpha, \beta \in \mathbb{R}$$

show that $\alpha + \beta = 1$.

**b** In the figure, the point $G$ is the centroid of a triangle (i.e. the point where the lines joining each vertex to the midpoint of the opposite side meet).

A line passing through $G$ meets $ZX$ and $ZY$ at points $H$ and $K$ respectively, with $ZH = hZX$ and $ZK = kZY$.




**i** Prove that $\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM}$.

**ii** Express $\overrightarrow{ZG}$ in terms of $h$, $k$, $\overrightarrow{ZH}$ and $\overrightarrow{ZK}$.

**iii** Find the value of $\frac{1}{h} + \frac{1}{k}$. (Use the result from part **a**.)

**iv** If $h = k$, find the value of $h$ and describe geometrically what this implies.

**v** If the area of triangle $XYZ$ is 1 cm$^2$, what is the area of triangle $HKZ$ when $h = k$?

**vi** If $k = 2h$, find the value of $h$ and describe geometrically what this implies.

**vii** Describe the restrictions on $h$ and $k$, and sketch the graph of $h$ against $k$ for suitable values of $k$.

**viii** Investigate the area, $A$ cm$^2$, of triangle $HKZ$ as a ratio with respect to the area of triangle $XYZ$, as $k$ varies. Sketch the graph of $A$ against $k$. Be careful with the domain.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# 6 Number and proof 1

## In this chapter



## Syllabus references

**Topics:** The nature of proof

**Subtopics:** 1.1.1 – 1.1.5

A **mathematical proof** is an argument that demonstrates the absolute truth of a statement.

It is certainty that makes mathematics different from other sciences. In science, a theory is never proved true. Instead, one aims to prove that a theory is not true. And if such evidence is hard to come by, then this increases the likelihood that a theory is correct, but never provides a guarantee. The possibility of absolute certainty is reserved for mathematics alone.

When writing a proof you should always aim for three things:

- correctness
- clarity
- simplicity.

Perhaps the following proof of Pythagoras' theorem exemplifies these three aims.



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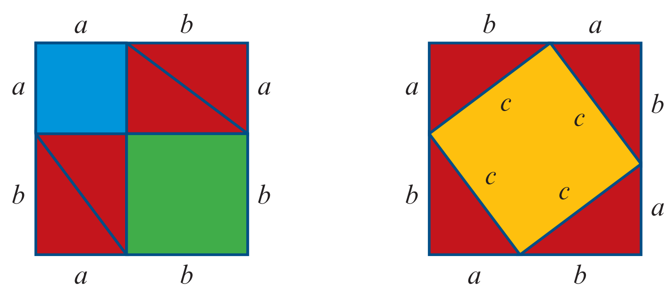
**Pythagoras' theorem**

Take any triangle with side lengths $a$, $b$ and $c$. If the angle between $a$ and $b$ is 90°, then

$$a^2 + b^2 = c^2$$

**Proof** Consider the two squares shown below.




The two squares each have the same total area. So subtracting four red triangles from each figure will leave the same area. Therefore $a^2 + b^2 = c^2$.

The ideas introduced in this chapter will be used in proofs throughout the rest of this book.

# 6A Direct proof

## Conditional statements

Consider the following sentence:

| | | | | |
|---|---|---|---|---|
| Statement | If | it is raining | then | the grass is wet. |

This is called a **conditional statement** and has the form:

| | | | | |
|---|---|---|---|---|
| Statement | If | $P$ is true | then | $Q$ is true. |

This can be abbreviated as

$$P \Rightarrow Q$$

which is read '$P$ **implies** $Q$'. We call $P$ the **hypothesis** and $Q$ the **conclusion**.

Not all conditional statements will be true. For example, switching the hypothesis and the conclusion above gives:

| | | | | |
|---|---|---|---|---|
| Statement | If | the grass is wet | then | it is raining. |

Anyone who has seen dewy grass on a cloudless day knows this to be false. In this chapter we will learn how to prove (and disprove) mathematical statements.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## Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis $P$ is true, and then show that the conclusion $Q$ follows.

### Example 1

Prove the following statements:

**a** If $a$ is odd and $b$ is even, then $a + b$ is odd.

**b** If $a$ is odd and $b$ is odd, then $ab$ is odd.

#### Solution

**a** Assume that $a$ is odd and $b$ is even.

Since $a$ is odd, we have $a = 2m + 1$ for some $m \in \mathbb{Z}$. Since $b$ is even, we have $b = 2n$ for some $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} a + b &= (2m + 1) + 2n \\ &= 2m + 2n + 1 \\ &= 2(m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = m + n \in \mathbb{Z} \end{aligned}$$

Hence $a + b$ is odd.

Note: We must use two different pronumerals $m$ and $n$ here, because these two numbers may be different.

**b** Assume that both $a$ and $b$ are odd. Then $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Therefore

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2mn + m + n \in \mathbb{Z} \end{aligned}$$

Hence $ab$ is odd.

### Example 2

Let $p, q \in \mathbb{Z}$ such that $p$ is divisible by 5 and $q$ is divisible by 3. Prove that $pq$ is divisible by 15.

#### Solution

Since $p$ is divisible by 5, we have $p = 5m$ for some $m \in \mathbb{Z}$. Since $q$ is divisible by 3, we have $q = 3n$ for some $n \in \mathbb{Z}$. Thus

$$\begin{aligned} pq &= (5m)(3n) \\ &= 15mn \end{aligned}$$

and so $pq$ is divisible by 15.


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### Example 3

Let $x$ and $y$ be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

**Solution**

Assume that $x > y$. Then $x - y > 0$.

Since $x$ and $y$ are positive, we also know that $x + y > 0$.

Therefore

$$x^2 - y^2 = \overbrace{(x-y)}^{\text{positive}}\overbrace{(x+y)}^{\text{positive}} > 0$$

Hence $x^2 > y^2$.

**Explanation**

When trying to prove that $x^2 > y^2$, it is easier to first prove that $x^2 - y^2 > 0$.

Also, note that the product of two positive numbers is positive.

### Example 4

Let $x$ and $y$ be any two positive real numbers. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

**Solution**

A **false proof** might begin with the statement that we are trying to prove.

$$\begin{aligned}
& & \frac{x+y}{2} &\geq \sqrt{xy} \\
&\Rightarrow & x+y &\geq 2\sqrt{xy} \\
&\Rightarrow & (x+y)^2 &\geq 4xy \qquad \text{(using Example 3)} \\
&\Rightarrow & x^2 + 2xy + y^2 &\geq 4xy \\
&\Rightarrow & x^2 - 2xy + y^2 &\geq 0 \\
&\Rightarrow & (x-y)^2 &\geq 0
\end{aligned}$$

Although it is true that $(x - y)^2 \geq 0$, the argument is faulty. We cannot prove that the result is true by assuming that the result is true! However, the above work is not a waste of time. We can correct the proof by reversing the order of the steps shown above.

**Note:** In the corrected proof, we need to use the fact that $a > b$ implies $\sqrt{a} > \sqrt{b}$ for all positive numbers $a$ and $b$. This is shown in Question 8 of Exercise 6B.

## Breaking a proof into cases

Sometimes it helps to break a problem up into different cases.

### Example 5

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'We are both knaves.' What are Alice and Bob?


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**Solution**

We will prove that Alice is a knave and Bob is a knight.

**Case 1**

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice and Bob are both knaves.

$\Rightarrow$ Alice is a knave and a knight.

This is impossible.

**Case 2**

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice and Bob are not both knaves.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knave and Bob must be a knight.

## Summary 6A

- A **mathematical proof** establishes the truth of a statement.
- A **conditional statement** has the form: If $P$ is true, then $Q$ is true. This can be abbreviated as $P \Rightarrow Q$, which is read '$P$ **implies** $Q$'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that $P$ is true and show that $Q$ follows.

Skill-sheet

## Exercise 6A

Example 1

**1** Assume that $m$ is even and $n$ is even. Prove that:

**a** $m + n$ is even **b** $mn$ is even.

**2** Assume that $m$ is odd and $n$ is odd. Prove that $m + n$ is even.

**3** Assume that $m$ is even and $n$ is odd. Prove that $mn$ is even.

Example 2

**4** Suppose that $m$ is divisible by 3 and $n$ is divisible by 7. Prove that:

**a** $mn$ is divisible by 21 **b** $m^2n$ is divisible by 63.

**5** Suppose that $m$ and $n$ are perfect squares. Show that $mn$ is a perfect square.

**6** Let $m$ and $n$ be integers. Prove that $(m + n)^2 - (m - n)^2$ is divisible by 4.

**7** Suppose that $n$ is an even integer. Prove that $n^2 - 6n + 5$ is odd.

**8** Suppose that $n$ is an odd integer. Prove that $n^2 + 8n + 3$ is even.

**9** Let $n \in \mathbb{Z}$. Prove that $5n^2 + 3n + 7$ is odd.
Hint: Consider the cases when $n$ is odd and $n$ is even.

Example 3

**10** Let $x$ and $y$ be positive real numbers. Show that if $x > y$, then $x^4 > y^4$.

Example 4

**11** Let $x, y \in \mathbb{R}$. Show that $x^2 + y^2 \geq 2xy$.


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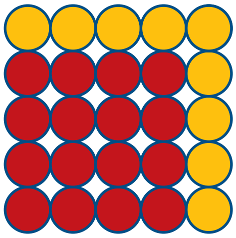
Example 5 **12** Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Determine whether Alice and Bob are knights or knaves in each of the following separate instances:

**a** Alice says: 'We are both knaves.'

**b** Alice says: 'We are both of the same kind.' Bob says: 'We are of a different kind.'

**c** Alice says: 'Bob is a knave.' Bob says: 'Neither of us is a knave.'

**13** The diagram shows that 9 can be written as the difference of two squares: $9 = 5^2 - 4^2$.



**a** Draw another diagram to show that 11 can be written as the difference of two squares.

**b** Prove that every odd number can be written as the difference of two squares.

**c** Hence, express 101 as the difference of two squares.

**14** **a** Consider the numbers $\frac{9}{10}$ and $\frac{10}{11}$. Which is larger?

**b** Let $n$ be a natural number. Prove that $\frac{n}{n+1} > \frac{n-1}{n}$.

**15** **a** Prove that

$$\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$$

**b** Let $n > 0$. Prove that

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

**16** Let $a, b \in \mathbb{R}$. Prove that $\frac{a^2 + b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$.

**17** **a** Expand $(x - y)(x^2 + xy + y^2)$.

**b** Prove that $x^2 + yx + y^2 \geq 0$ for all $x, y \in \mathbb{R}$.

Hint: Complete the square by thinking of $y$ as a constant.

**c** Hence, prove that if $x \geq y$, then $x^3 \geq y^3$.

**18** Sally travels from home to work at a speed of 12 km/h and immediately returns home at a speed of 24 km/h.

**a** Show that her average speed is 16 km/h.

**b** Now suppose that Sally travels to work at a speed of $a$ km/h and immediately returns home at a speed of $b$ km/h. Show that her average speed is $\frac{2ab}{a+b}$ km/h.

**c** Let $a$ and $b$ be any two positive real numbers. Prove that

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

Note: This proves that Sally's average speed for the whole journey can be no greater than the average of her speeds for the two individual legs of the journey.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# 6B Proof by contrapositive

## The negation of a statement

To **negate** a statement $P$ we write its very opposite, which we call '**not** $P$'. For example, consider the following four statements and their negations.

| $P$ | | not $P$ | |
|---|---|---|---|
| The sky is green. | (false) | The sky is not green. | (true) |
| $1 + 1 = 2$ | (true) | $1 + 1 \neq 2$ | (false) |
| All prime numbers are odd. | (false) | There exists an even prime number. | (true) |
| All triangles have three sides. | (true) | Some triangle does not have three sides. | (false) |

Notice that negation turns a true statement into a false statement, and a false statement into a true statement.

### Example 6

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

**a** $2 > 1$
**b** 5 is divisible by 3
**c** The sum of any two odd numbers is even.
**d** There are two primes whose product is 12.

#### Solution

**a** $P$: $2 > 1$ (true)
not $P$: $2 \leq 1$ (false)

**b** $P$: 5 is divisible by 3 (false)
not $P$: 5 is not divisible by 3 (true)

**c** $P$: The sum of any two odd numbers is even. (true)
not $P$: There are two odd numbers whose sum is odd. (false)

**d** $P$: There are two primes whose product is 12. (false)
not $P$: There are no two primes whose product is 12. (true)

## De Morgan's laws

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

### De Morgan's laws

not ($P$ *and* $Q$) is the same as (not $P$) *or* (not $Q$)

not ($P$ *or* $Q$) is the same as (not $P$) *and* (not $Q$)


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**Example 7**

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

**a** 6 is divisible by 2 and 3

**b** 10 is divisible by 2 or 7

**Solution**

**a** $P$: 6 is divisible by 2 and 6 is divisible by 3 (true)

not $P$: 6 is not divisible by 2 or 6 is not divisible by 3 (false)

**b** $P$: 10 is divisible by 2 or 10 is divisible by 7 (true)

not $P$: 10 is not divisible by 2 and 10 is not divisible by 7 (false)

**Example 8**

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave or Bob is a knight.' What are Alice and Bob?

**Solution**

We will prove that Alice is a knight and Bob is a knight.

**Case 1**

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice is a knight AND Bob is a knave.

$\Rightarrow$ Alice is a knight and a knave.

This is impossible.

**Case 2**

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice is a knave OR Bob is a knight.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knight and Bob must be a knight.

## Proof by contrapositive

Consider this statement:

| | | |
|---|---|---|
| Statement | If it is the end of term | then the students are happy. |

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

| | | |
|---|---|---|
| Contrapositive | If the students are *not* happy | then it is *not* the end of term. |

Note that the original statement and its contrapositive are logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.


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- The **contrapositive** of $P \Rightarrow Q$ is the statement (not $Q$) $\Rightarrow$ (not $P$).
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

## Example 9

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2$ is even, then $n$ is even.

**a** Write down the contrapositive. **b** Prove the contrapositive.

### Solution

**a** If $n$ is odd, then $n^2$ is odd.

**b** Assume that $n$ is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$. Squaring $n$ gives

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 2m \in \mathbb{Z} \end{aligned}$$

Therefore $n^2$ is odd.

Note: Although we proved the contrapositive, remember that we have actually proved that if $n^2$ is even, then $n$ is even.

## Example 10

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then $n$ is odd.

**a** Write down the contrapositive. **b** Prove the contrapositive.

### Solution

**a** If $n$ is even, then $n^2 + 4n + 1$ is odd.

**b** Assume that $n$ is even. Then $n = 2m$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 4n + 1 &= (2m)^2 + 4(2m) + 1 \\ &= 4m^2 + 8m + 1 \\ &= 2(2m^2 + 4m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 4m \in \mathbb{Z} \end{aligned}$$

Hence $n^2 + 4n + 1$ is odd.

## Example 11

Let $x$ and $y$ be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

**a** Write down the contrapositive. **b** Prove the contrapositive.

### Solution

**a** If $\sqrt{x} \geq \sqrt{y}$, then $x \geq y$.

**b** Assume that $\sqrt{x} \geq \sqrt{y}$. Then $x \geq y$ by Example 3, since $\sqrt{x}$ and $\sqrt{y}$ are positive.


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## Summary 6B

- To **negate** a statement we write its opposite.
- For a statement $P \Rightarrow Q$, the **contrapositive** is the statement (not $Q$) $\Rightarrow$ (not $P$). That is, we switch the hypothesis and the conclusion and negate both.
- A statement and its contrapositive are logically equivalent.
- Proving the contrapositive of a statement may be easier than giving a direct proof.

Skill-sheet

## Exercise 6B

Example 6

**1** Write down each statement and its negation. Which of the statement and its negation is true and which is false?

**a** $1 > 0$ **b** 4 is divisible by 8

**c** Each pair of primes has an even sum.

**d** Some rectangle has four sides of equal length.

Example 7

**2** Write down each statement and its negation. Which of the statement and its negation is true and which is false?

**a** 14 is divisible by 7 and 2 **b** 12 is divisible by 3 or 4

**c** 15 is divisible by 3 and 6 **d** 10 is divisible by 2 or 3

Example 8

**3** Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave and Bob is a knight.' What are Alice and Bob?

Example 9

**4** Write down the contrapositive version of each of these statements:

**a** If it is raining, then there are clouds in the sky.

**b** If you are smiling, then you are happy.

**c** If $x = 1$, then $2x = 2$.

**d** If $x > y$, then $x^5 > y^5$.

**e** Let $n \in \mathbb{Z}$. If $n^2$ is odd, then $n$ is odd.

**f** Let $m, n \in \mathbb{Z}$. If $m$ and $n$ are odd, then $mn$ is odd.

**g** Let $m, n \in \mathbb{Z}$. If $m + n$ is even, then $m$ and $n$ are either both even or both odd.

Example 10

**5** Let $m, n \in \mathbb{Z}$. For each of the following statements, write down and prove the contrapositive statement:

**a** If $3n + 5$ is even, then $n$ is odd. **b** If $n^2$ is odd, then $n$ is odd.

**c** If $n^2 - 8n + 3$ is even, then $n$ is odd.

**d** If $n^2$ is not divisible by 3, then $n$ is not divisible by 3.

**e** If $n^3 + 1$ is even, then $n$ is odd.

**f** If $mn$ is not divisible by 3, then $m$ is not divisible by 3 and $n$ is not divisible by 3.

**g** If $m + n$ is odd, then $m \neq n$.


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**6** Let $x, y \in \mathbb{R}$. For each of the following statements, write down and prove the contrapositive statement:

**a** If $x^2 + 3x < 0$, then $x < 0$.

**b** If $x^3 - x > 0$, then $x > -1$.

**c** If $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

**d** If $2x + 3y \geq 12$, then $x \geq 3$ or $y \geq 2$.

**7** Let $m, n \in \mathbb{Z}$ and consider this statement: If $mn$ and $m + n$ are even, then $m$ and $n$ are even.

**a** Write down the contrapositive.

**b** Prove the contrapositive. You will have to consider cases.

Example 11

**8** Let $x$ and $y$ be positive real numbers.

**a** Prove that

$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

**b** Hence, prove that if $x > y$, then $\sqrt{x} > \sqrt{y}$.

**c** Give a simpler proof by considering the contrapositive.

# 6C Equivalent statements

## The converse of a statement

At the beginning of this chapter, we proved Pythagoras' theorem. Consider any triangle with side lengths $a$, $b$ and $c$.

| | | | | |
|---|---|---|---|---|
| Statement | If | the angle between $a$ and $b$ is 90° | then | $a^2 + b^2 = c^2$. |

By switching the hypothesis and the conclusion, we obtain the **converse** statement:

| | | | | |
|---|---|---|---|---|
| Converse | If | $a^2 + b^2 = c^2$ | then | the angle between $a$ and $b$ is 90°. |

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

Note: The converse of a true statement may not be true. For example:

| | | |
|---|---|---|
| Statement | If it is raining, then there are clouds in the sky. | (true) |
| Converse | If there are clouds in the sky, then it is raining. | (false) |


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**Example 12**

Let $x$ and $y$ be positive real numbers. Consider the statement: If $x < y$, then $x^2 < y^2$.

**a** Write down the converse of this statement.

**b** Prove the converse.

**Solution**

**a** If $x^2 < y^2$, then $x < y$.

**b** Assume that $x^2 < y^2$. Then, since both $x$ and $y$ are positive,

$$\begin{aligned} & x^2 - y^2 < 0 && \text{(subtract } y^2\text{)} \\ \Rightarrow\quad & (x - y)(x + y) < 0 && \text{(factorising)} \\ \Rightarrow\quad & x - y < 0 && \text{(divide both sides by } x + y > 0\text{)} \\ \Rightarrow\quad & x < y \end{aligned}$$

as required.

**Example 13**

Let $m$ and $n$ be integers. Consider the statement: If $m$ and $n$ are even, then $m + n$ is even.

**a** Write down the converse of this statement.

**b** Show that the converse is not true.

**Solution**

**a** If $m + n$ is even, then $m$ is even and $n$ is even.

**b** Clearly $1 + 3 = 4$ is even, although 1 and 3 are not.

## Equivalent statements

Now consider the following two statements:

$P$: your heart is beating

$Q$: you are alive

Notice that both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that $P$ and $Q$ are **equivalent** statements and we write

$P \Leftrightarrow Q$

We will also say that $P$ is true **if and only if** $Q$ is true. So in the above example, we can say

Your heart is beating if and only if you are alive.

To prove that two statements $P$ and $Q$ are equivalent, you have to prove two things:

$P \Rightarrow Q$ and $Q \Rightarrow P$


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## Example 14

Let $n \in \mathbb{Z}$. Prove that $n$ is even if and only if $n + 1$ is odd.

### Solution

($\Rightarrow$) Assume that $n$ is even. Then $n = 2m$ for some $m \in \mathbb{Z}$.
Therefore $n + 1 = 2m + 1$, and so $n + 1$ is odd.

($\Leftarrow$) Assume that $n + 1$ is odd. Then $n + 1 = 2m + 1$ for some $m \in \mathbb{Z}$.
Subtracting 1 from both sides gives $n = 2m$. Therefore $n$ is even.

Note: To prove that $P \Leftrightarrow Q$, we have to show that $P \Rightarrow Q$ and $P \Leftarrow Q$. When we are about to prove $P \Rightarrow Q$, we write ($\Rightarrow$). When we are about to prove $P \Leftarrow Q$, we write ($\Leftarrow$).

## Summary 6C

- For a statement $P \Rightarrow Q$, the **converse** is the statement $Q \Rightarrow P$.
  That is, we switch the hypothesis and the conclusion.
- If $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is true, then we say that $P$ is **equivalent** to $Q$, or that $P$ is true **if and only if** $Q$ is true.
- If $P$ and $Q$ are equivalent, we write $P \Leftrightarrow Q$.

Skill-sheet

## Exercise 6C

Example 12

**1** Write down and prove the converse of each of these statements:

**a** Let $x \in \mathbb{R}$. If $2x + 3 = 5$, then $x = 1$.

**b** Let $n \in \mathbb{Z}$. If $n$ is odd, then $n - 3$ is even.

**c** Let $m \in \mathbb{Z}$. If $m^2 + 2m + 1$ is even, then $m$ is odd.

**d** Let $n \in \mathbb{Z}$. If $n^2$ is divisible by 5, then $n$ is divisible by 5.

Example 13

**2** Let $m$ and $n$ be integers. Consider the statement: If $m$ and $n$ are even, then $mn$ is a multiple of 4.

**a** Write down the converse of this statement.

**b** Show that the converse is not true.

**3** Which of these pairs of statements are equivalent?

**a** $P$: Vivian is in China.
$Q$: Vivian is in Asia.

**b** $P$: $2x = 4$
$Q$: $x = 2$


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**c** $P$: $x > 0$ and $y > 0$
$Q$: $xy > 0$

**d** $P$: $m$ is even or $n$ is even, where $m, n \in \mathbb{Z}$
$Q$: $mn$ is even, where $m, n \in \mathbb{Z}$

Example 14

**4** Let $n$ be an integer. Prove that $n + 1$ is odd if and only if $n + 2$ is even.

**5** Let $n \in \mathbb{N}$. Prove that $n^2 - 4$ is a prime number if and only if $n = 3$.

**6** Let $n$ be an integer. Prove that $n^3$ is even if and only if $n$ is even.

**7** Let $n$ be an integer. Prove that $n$ is odd if and only if $n = 4k \pm 1$ for some $k \in \mathbb{Z}$.

**8** Let $x, y \in \mathbb{R}$. Prove that $(x + y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

**9** Let $m$ and $n$ be integers.

**a** By expanding the right-hand side, prove that $m^3 - n^3 = (m - n)(m^2 + mn + n^2)$.
**b** Hence, prove that $m - n$ is even if and only if $m^3 - n^3$ is even.

**10** Prove that an integer is divisible by 4 if and only if the number formed by its last two digits is divisible by 4. Hint: 100 is divisible by 4.

# 6D Disproving statements

## Quantification using 'for all' and 'there exists'

### For all

**Universal quantification** claims that a property holds for *all* members of a given set. For example, consider this statement:

| | |
|---|---|
| Statement | For all natural numbers $n$, we have $2n \geq n + 1$. |

To prove that this statement is true, we need to give a general argument that applies to every natural number $n$.

### There exists

**Existential quantification** claims that a property holds for *at least one* member of a given set. For example, consider this statement:

| | |
|---|---|
| Statement | There exists an integer $m$ such that $m^2 = 25$. |

To prove that this statement is true, we just need to give an example: $5 \in \mathbb{Z}$ with $5^2 = 25$.


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## Example 15

Rewrite each statement using either 'for all' or 'there exists':

**a** Some real numbers are irrational.

**b** Every integer that is divisible by 4 is also divisible by 2.

### Solution

**a** There exists $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$.

**b** For all $m \in \mathbb{Z}$, if $m$ is divisible by 4, then $m$ is divisible by 2.

### Negating 'for all' and 'there exists'

To negate a statement involving a quantifier, we interchange 'for all' with 'there exists' and then negate the rest of the statement.

## Example 16

Write down the negation of each of the following statements:

**a** For all natural numbers $n$, we have $2n \geq n + 1$.

**b** There exists an integer $m$ such that $m^2 = 4$ and $m^3 = -8$.

### Solution

**a** There exists a natural number $n$ such that $2n < n + 1$.

**b** For all integers $m$, we have $m^2 \neq 4$ or $m^3 \neq -8$.

Note: For part **b**, we used one of De Morgan's laws.

### Notation

The words 'for all' can be abbreviated using the *turned A* symbol, $\forall$. The words 'there exists' can be abbreviated using the *turned E* symbol, $\exists$. For example:

- 'For all natural numbers $n$, we have $2n \geq n + 1$' can be written as $(\forall n \in \mathbb{N})\ 2n \geq n + 1$.
- 'There exists an integer $m$ such that $m^2 = 25$' can be written as $(\exists m \in \mathbb{Z})\ m^2 = 25$.

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.


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## Counterexamples

Consider the quadratic function $f(n) = n^2 - n + 11$. Notice how $f(n)$ is a prime number for small natural numbers $n$:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f(n)$ | 11 | 13 | 17 | 23 | 31 | 41 | 53 | 67 | 83 | 101 |

From this, we might be led to believe that the following statement is true:

| Statement | For all natural numbers $n$, the number $f(n)$ is prime. |
|---|---|

We call this a **universal statement**, because it asserts the truth of a statement without exception. So to disprove a universal statement, we need only show that it is not true in some particular instance. For our example, we need to find $n \in \mathbb{N}$ such that $f(n)$ is not prime. Luckily, we do not have to look very hard.

### Example 17

Let $f(n) = n^2 - n + 11$. Disprove this statement: For all $n \in \mathbb{N}$, the number $f(n)$ is prime.

**Solution**

When $n = 11$, we obtain

$$f(11) = 11^2 - 11 + 11 = 11^2$$

Therefore $f(11)$ is not prime.

To disprove a statement of the form $P \Rightarrow Q$, we simply need to give one example for which $P$ is true and $Q$ is not true. Such an example is called a **counterexample**.

### Example 18

Find a counterexample to disprove this statement: For all $x, y \in \mathbb{R}$, if $x > y$, then $x^2 > y^2$.

**Solution**

Let $x = 1$ and $y = -2$. Clearly $1 > -2$, but $1^2 = 1 \le 4 = (-2)^2$.


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## Disproving existence statements

Consider this statement:

| Statement | There exists $n \in \mathbb{N}$ such that $n^2 + 3n + 2$ is a prime number. |
|---|---|

We call this an **existence statement**, because it claims the existence of an object possessing a particular property. To show that such a statement is false, we prove that its negation is true:

| Negation | For all $n \in \mathbb{N}$, the number $n^2 + 3n + 2$ is not a prime number. |
|---|---|

This is easy to prove, as

$$n^2 + 3n + 2 = (n + 1)(n + 2)$$

is clearly a composite number for each $n \in \mathbb{N}$.

### Example 19

Disprove this statement: There exists $n \in \mathbb{N}$ such that $n^2 + 13n + 42$ is a prime number.

**Solution**

We need to prove that, for all $n \in \mathbb{N}$, the number $n^2 + 13n + 42$ is not prime.

This is true, since

$$n^2 + 13n + 42 = (n + 6)(n + 7)$$

is clearly a composite number for each $n \in \mathbb{N}$.

### Example 20

Show that this statement is false: There exists some real number $x$ such that $x^2 = -1$.

**Solution**

We have to prove that the negation is true: For *all* real numbers $x$, we have $x^2 \neq -1$.

This is easy to prove since, for any real number $x$, we have $x^2 \geq 0$ and so $x^2 \neq -1$.

### Summary 6D

- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- A universal statement of the form $P \Rightarrow Q$ can be disproved by giving one example of an instance when $P$ is true but $Q$ is not.
- Such an example is called a **counterexample**.
- To disprove an existence statement, we prove that its negation is true.


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Skill-sheet

## Exercise 6D

Example 15

**1** Which of the following are universal statements ('for all') and which are existence statements ('there exists')?

**a** For each $n \in \mathbb{N}$, the number $5n^2 + 3n + 7$ is odd.

**b** There is an even prime number.

**c** Every natural number greater than 1 has a prime factorisation.

**d** All triangles have three sides.

**e** Some natural numbers are primes.

**f** At least one real number $x$ satisfies the equation $x^2 - x - 1 = 0$.

**g** Any positive real number has a square root.

**h** The angle sum of a triangle is 180°.

**2** Which of the following statements are true and which are false?

**a** There exists a real number $x$ such that $x^2 = 2$.

**b** There exists a real number $x$ such that $x^2 < 0$.

**c** For all natural numbers $n$, the number $2n - 1$ is odd.

**d** There exists $n \in \mathbb{N}$ such that $2n$ is odd.

**e** For all $x \in \mathbb{R}$, we have $x^3 \geq 0$.

Example 16

**3** Write down the negation of each of the following statements:

**a** For every natural number $n$, the number $2n^2 - 4n + 31$ is prime.

**b** For all $x \in \mathbb{R}$, we have $x^2 > x$.

**c** There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.

**d** For all $x, y \in \mathbb{R}$, we have $(x + y)^2 = x^2 + y^2$.

**e** There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 > y^2$.

Example 17
Example 18

**4** Prove that each of the following statements is false by finding a counterexample:

**a** For every natural number $n$, the number $2n^2 - 4n + 31$ is prime.

**b** If $x, y \in \mathbb{R}$, then $(x + y)^2 = x^2 + y^2$.

**c** For all $x \in \mathbb{R}$, we have $x^2 > x$.

**d** Let $n \in \mathbb{Z}$. If $n^3 - n$ is even, then $n$ is even.

**e** If $m, n \in \mathbb{N}$, then $m + n \leq mn$.

**f** Let $m, n \in \mathbb{Z}$. If 6 divides $mn$, then 6 divides $m$ or 6 divides $n$.

Example 19
Example 20

**5** Show that each of the following existence statements is false:

**a** There exists $n \in \mathbb{N}$ such that $9n^2 - 1$ is a prime number.

**b** There exists $n \in \mathbb{N}$ such that $n^2 + 5n + 6$ is a prime number.

**c** There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.


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**6** Provide a counterexample to disprove each of the following statements.
Hint: $\sqrt{2}$ might come in handy.

**a** If $a$ is irrational and $b$ is irrational, then $ab$ is irrational.

**b** If $a$ is irrational and $b$ is irrational, then $a + b$ is irrational.

**c** If $a$ is irrational and $b$ is irrational, then $\frac{a}{b}$ is irrational.

**7** Let $a \in \mathbb{Z}$.

**a** Prove that if $a$ is divisible by 4, then $a^2$ is divisible by 4.

**b** Prove that the converse is not true.

**8** Let $a, b \in \mathbb{Z}$.

**a** Prove that if $a - b$ is divisible by 3, then $a^2 - b^2$ is divisible by 3.

**b** Prove that the converse is not true.

**9** Prove that each of the following statements is false:

**a** There exist real numbers $a$ and $b$ such that $a^2 - 2ab + b^2 = -1$.

**b** There exists some real number $x$ such that $x^2 - 4x + 5 = \frac{3}{4}$.

**10** The numbers $\{1, 2, \ldots, 8\}$ can be paired so that the sum of each pair is a square number:

$$1 + 8 = 9, \quad 2 + 7 = 9, \quad 3 + 6 = 9, \quad 4 + 5 = 9$$

**a** Prove that you can also do this with the numbers $\{1, 2, \ldots, 16\}$.

**b** Prove that you cannot do this with the numbers $\{1, 2, \ldots, 12\}$.

**11** Let $f(n) = an^2 + bn + c$ be a quadratic function, where $a, b, c$ are natural numbers and $c \geq 2$. Show that there is an $n \in \mathbb{N}$ such that $f(n)$ is not a prime number.


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Review

## Chapter summary

Assignment

Nrich

- A **conditional statement** has the form: If $P$ is true, then $Q$ is true. This can be abbreviated as $P \Rightarrow Q$, which is read '$P$ implies $Q$'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that $P$ is true and show that $Q$ follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements $P$ and $Q$ are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- The **contrapositive** of $P \Rightarrow Q$ is (not $Q$) $\Rightarrow$ (not $P$).
- Proving the contrapositive of a statement may be easier than giving a direct proof.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- **Counterexamples** can be used to demonstrate that a universal statement is false.

## Short-answer questions

**1** For each of the following statements, if the statement is true, then prove it, and otherwise give a counterexample to show that it is false:

**a** The sum of any three consecutive integers is divisible by 3.

**b** The sum of any four consecutive integers is divisible by 4.

**2** **a** Show that one of three consecutive integers is always divisible by 3.

**b** Hence, prove that $n^3 + 3n^2 + 2n$ is divisible by 3 for all $n \in \mathbb{Z}$.

**3** **a** Suppose that both $m$ and $n$ are divisible by $d$. Prove that $m - n$ is divisible by $d$.

**b** Hence, prove that the highest common factor of two consecutive integers is 1.

**c** Find the highest common factor of 1002 and 999.

**4** A student claims that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, for all $x \geq 0$ and $y \geq 0$.

**a** Using a counterexample, prove that the equation is not always true.

**b** Prove that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ if and only if $x = 0$ or $y = 0$.

**5** Let $n \in \mathbb{Z}$. Prove that $n^2 + 3n + 4$ is even.

Hint: Consider the cases when $n$ is odd and $n$ is even.

**6** Suppose that $a$, $b$, $c$ and $d$ are positive integers.

**a** Provide a counterexample to disprove the equation

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

**b** Now suppose that $\frac{c}{d} > \frac{a}{b}$. Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$


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## Extended-response questions

**1** Define $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

**a** Prove that $10! + 2, 10! + 3, \dots, 10! + 10$ are nine consecutive composite numbers.
Hint: The first number is divisible by 2.

**b** Find a sequence of ten consecutive composite numbers.

**2** Let $n \in \mathbb{Z}$ and consider the statement: If $n$ is divisible by 8, then $n^2$ is divisible by 8.

**a** Prove the statement.

**b** Write down the converse of the statement.

**c** If the converse is true, prove it. Otherwise, give a counterexample.

**3** **a** Let $x \geq 0$ and $y \geq 0$. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

by substituting $x = a^2$ and $y = b^2$ into $\frac{x+y}{2} - \sqrt{xy}$.

**b** Using the above inequality, or otherwise, prove each of the following:

**i** If $a > 0$, then $a + \frac{1}{a} \geq 2$.

**ii** If $a, b$ and $c$ are positive real numbers, then $(a+b)(b+c)(c+a) \geq 8abc$.

**iii** If $a, b$ and $c$ are positive real numbers, then $a^2 + b^2 + c^2 \geq ab + bc + ca$.

**c** Take any rectangle of length $x$ and width $y$. Prove that a square with the same perimeter has an area greater than or equal to that of the rectangle.

**4** Exactly one of the following three people is lying. Who is the liar?

- Jay says: ‘Kaye is lying.’
- Kaye say: ‘Elle is lying.’
- Elle says: ‘I am not lying.’

**5** There are four sentences written below. Which of them is true?

- Exactly one of these statements is false.
- Exactly two of these statements are false.
- Exactly three of these statements are false.
- Exactly four of these statements are false.

**6** We will say that a set of numbers can be **split** if it can be divided into two groups so that no two numbers appear in the same group as their sum. For example, the set $\{1, 2, 3, 4, 5, 6\}$ can be split into the two groups $\{1, 2, 4\}$ and $\{3, 5, 6\}$.

**a** Prove that the set $\{1, 2, \dots, 8\}$ can be split.

**b** Hence, explain why the set $\{1, 2, \dots, n\}$ can be split, where $1 \leq n \leq 8$.

**c** Prove that it is impossible to split the set $\{1, 2, \dots, 9\}$.

**d** Hence, prove that it is impossible to split the set $\{1, 2, \dots, n\}$, where $n \geq 9$.


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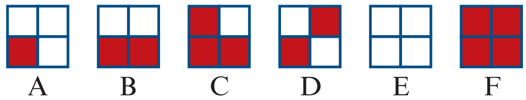
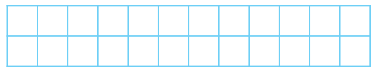
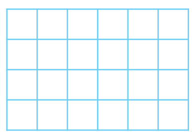
**7** Consider the set of six $2 \times 2$ square tiles shown below.




**a** Tile the $2 \times 12$ grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



**b** Prove that there are only four ways to tile the $4 \times 6$ grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.

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# 7 Circle geometry

## In this chapter



## Syllabus references

**Topics:** Circle properties, including proof and use

**Subtopics:** 1.1.6 – 1.1.15

A **circle** is the set of all points in the plane at a fixed distance $r$ from a point $O$. Circles with the same radius are congruent to each other (and are said to be equal circles). We have seen in the previous chapter that all circles are similar to each other.

You may have come across the Cartesian equation of the circle in Mathematics Methods Units 1 & 2. For example, the circle with radius 1 and centre the origin has equation $x^2 + y^2 = 1$. In this chapter we take a different approach to the study of circles.

The theorems and related results in this chapter can be investigated through a geometry package such as GeoGebra or Cabri Geometry.


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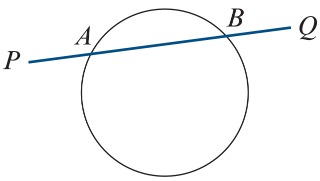
# 7A Angle properties of the circle

A line segment joining two points on a circle is called a **chord**. A line that cuts a circle at two distinct points is called a **secant**.

For example, in the diagram, the line $PQ$ is a secant and the line segment $AB$ is a chord.



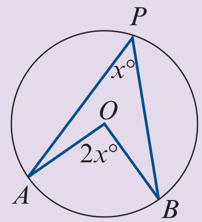

Suppose that we have a line segment or an **arc** $AB$ and a point $P$ not on $AB$. Then $\angle APB$ is the angle **subtended** by $AB$ at the point $P$.

You should prove the following two results. The first proof uses the SSS congruence test and the second uses the SAS congruence test.

- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.

**Theorem 1**

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



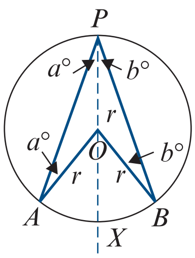

**Proof** Join points $P$ and $O$ and extend the line through $O$, as shown in the diagram on the right.

Note that $AO = BO = PO = r$, the radius of the circle. Therefore triangles $PAO$ and $PBO$ are isosceles.

Let $\angle APO = \angle PAO = a°$ and $\angle BPO = \angle PBO = b°$.

Then angle $AOX$ is $2a°$ (exterior angle of a triangle) and angle $BOX$ is $2b°$ (exterior angle of a triangle).

Hence $\angle AOB = 2a° + 2b° = 2(a + b)° = 2\angle APB$.



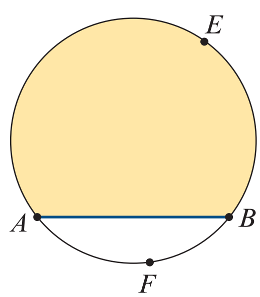

Note: In this proof, the centre $O$ and point $P$ are on the same side of chord $AB$. Slight variations of this proof can be used for other cases. The result is always true.

**Converse of Theorem 1** Let $A$ and $B$ be points on a circle, centre $O$, and let $P$ be a point on the same side of $AB$ as $O$. If the angle $APB$ is half the angle $AOB$, then $P$ lies on the circle.

## Segments

A **segment** of a circle is the part of the plane bounded by an arc and its chord. For example, in the diagram:

- Arc $AEB$ and chord $AB$ define a **major segment** (shaded).
- Arc $AFB$ and chord $AB$ define a **minor segment** (unshaded).





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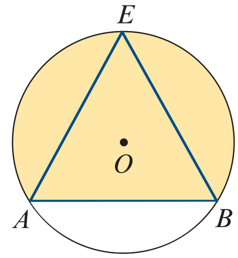
## Angles in a segment

$\angle AEB$ is said to be an angle in segment $AEB$.



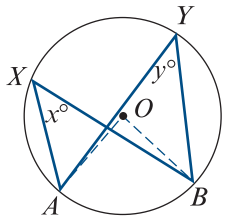

**Theorem 2:** Angles in the same segment

Angles in the same segment of a circle are equal.

**Proof** Let $\angle AXB = x^\circ$ and $\angle AYB = y^\circ$.

Then, by Theorem 1, $\angle AOB = 2x^\circ = 2y^\circ$.

Therefore $x = y$.

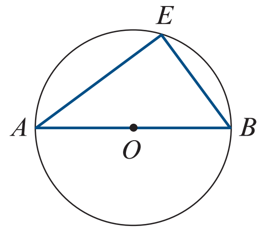



Note: A converse of this result is proved later in this section.

**Theorem 3:** Angle subtended by a diameter

The angle subtended by a **diameter** at the circumference is equal to a right angle (90°).

**Proof** The angle subtended at the centre is 180°, and so the result follows from Theorem 1.



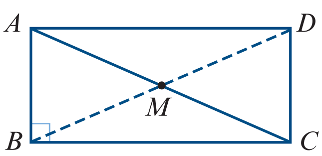

We can give a straightforward proof of a converse of this result.

**Converse of Theorem 3** The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.

**Proof** In the diagram, triangle $ABC$ has a right angle at $B$. Let $M$ be the midpoint of the hypotenuse $AC$.

We need to prove that $MA = MB = MC$.

Complete the rectangle $ABCD$.




The diagonals of a rectangle are equal and bisect each other.

Hence $AC = BD$ and $M$ is also the midpoint of $BD$. It follows that $MA = MB = MC$.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## Cyclic polygons

- A set of points is said to be **concyclic** if they all lie on a common circle.
- A polygon is said to be **inscribed in a circle** if all its vertices lie on the circle. This implies that no part of the polygon lies outside the circle.
- A quadrilateral that can be inscribed in a circle is called a **cyclic quadrilateral**.

**Theorem 4**

The opposite angles of a quadrilateral inscribed in a circle sum to 180°.

That is, the opposite angles of a cyclic quadrilateral are supplementary.

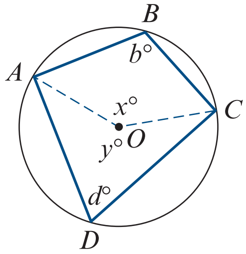
**Proof** In the diagram, the quadrilateral $ABCD$ is inscribed in a circle with centre $O$.

By Theorem 1, we have $x = 2d$ and $y = 2b$.

Now $\quad x + y = 360$

and so $\quad 2b + 2d = 360$

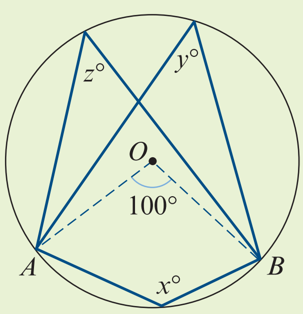
Hence $\quad b + d = 180$




**Converse of Theorem 4** If opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

### Example 1

Find the value of each of the pronumerals in the diagram, where $O$ is the centre of the circle and $\angle AOB = 100°$.




**Solution**

Theorem 1 gives $y = z = 50$.

The value of $x$ can be found by observing either of the following:

**1** Reflex angle $AOB$ is 260°.
Therefore $x = 130$ (by Theorem 1).

**2** We have $x + y = 180$ (by Theorem 4).
Therefore $x = 180 - 50 = 130$.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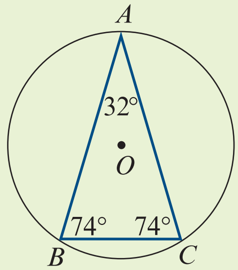
**Example 2**

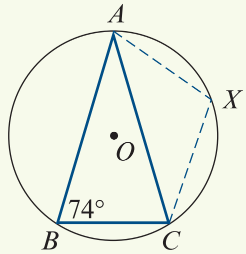
An isosceles triangle is inscribed in a circle as shown. Find the angles in the three minor segments of the circle cut off by the sides of this triangle.




**Solution**

To find $\angle AXC$, form the cyclic quadrilateral $AXCB$. Then $\angle AXC$ and $\angle ABC$ are supplementary. Therefore $\angle AXC = 106°$, and so all angles in the minor segment formed by $AC$ have magnitude 106°.

Similarly, it can be shown that all angles in the minor segment formed by $AB$ have magnitude 106°, and that all angles in the minor segment formed by $BC$ have magnitude 148°.




**Example 3**

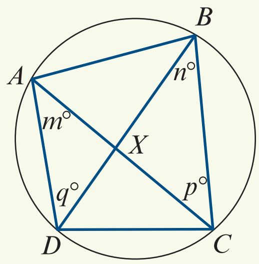
$A$, $B$, $C$ and $D$ are points on a circle. The diagonals of quadrilateral $ABCD$ meet at $X$. Prove that triangles $ADX$ and $BCX$ are similar.

**Solution**

$\angle DAC$ and $\angle DBC$ are in the same segment. Therefore $m = n$.

$\angle ADB$ and $\angle ACB$ are in the same segment. Therefore $q = p$.

Hence triangles $ADX$ and $BCX$ are similar (AAA).



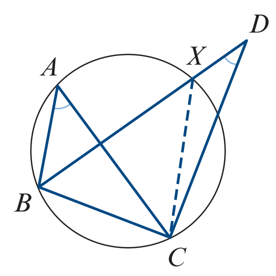

## The converse theorems

We only prove a converse of Theorem 2 here, but the proofs of the converses of Theorems 1 and 4 use similar techniques. Try them for yourself.

**Converse of Theorem 2** If a line segment subtends equal angles at two points on the same side of the line segment, then the two points and the endpoints of the line segment are concyclic.

**Proof** A circle is drawn through points $A$, $B$ and $C$. (This can be done with any three non-collinear points.)

Assume that $\angle BAC = \angle BDC$ and that $D$ lies outside the circle. (There is another case to consider when $D$ is inside, but the proof is similar. If $D$ lies on the circle, then we are finished.)





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Let $X$ be the point of intersection of line $BD$ with the circle. Then, by Theorem 2, $\angle BAC = \angle BXC$ and so $\angle BDC = \angle BXC$. But this is impossible. (You can use the equality of the angle sums of $\triangle BXC$ and $\triangle BDC$ to show this.)

Hence $D$ lies on the same circle as $A$, $B$ and $C$.

## Summary 7A

- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.
- **Theorem 1** The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
- **Theorem 2** Angles in the same segment of a circle are equal.
- **Theorem 3** The angle subtended by a diameter at the circumference is 90°.
- **Theorem 4** Opposite angles of a cyclic quadrilateral sum to 180°.

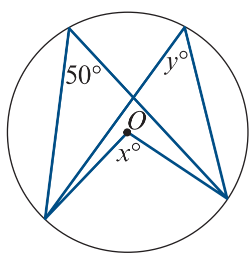
Skill-sheet

## Exercise 7A

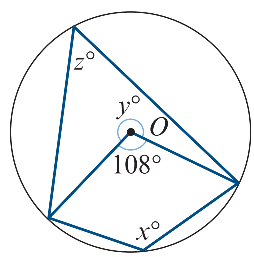
Example 1

**1** Find the values of the pronumerals for each of the following, where $O$ denotes the centre of the given circle:

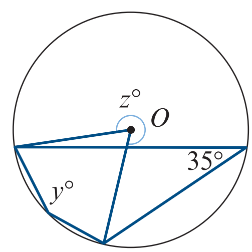
**a**



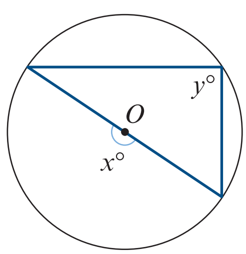

**b**



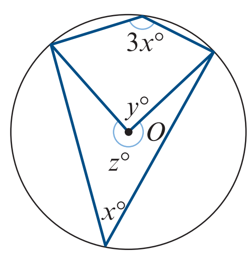

**c**




**d**

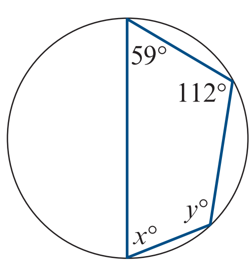



**e**

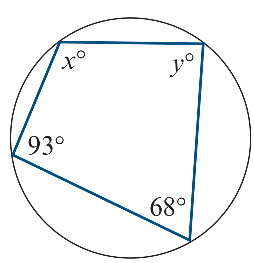



**2** Find the values of the pronumerals for each of the following:

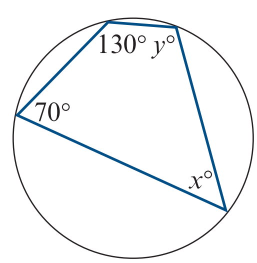
**a**




**b**




**c**





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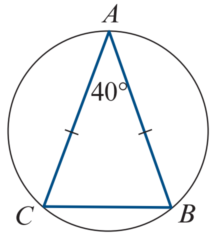
Example 2

**3** An isosceles triangle $ABC$ is inscribed in a circle. (Inscribed means that all the vertices of the triangle lie on the circle.) What are the angles in the three minor segments cut off by the sides of this triangle?




**4** $ABCDE$ is a pentagon inscribed in a circle. If $AE = DE$, $\angle BDC = 20°$, $\angle CAD = 28°$ and $\angle ABD = 70°$, find all the interior angles of the pentagon.

Example 3

**5** Prove that if two opposite sides of a cyclic quadrilateral are equal, then the other two sides are parallel.

**6** $ABCD$ is a parallelogram. The circle through the points $A$, $B$ and $C$ cuts $CD$ (extended if necessary) at $E$. Prove that $AE = AD$.

**7** $ABCD$ is a cyclic quadrilateral and $O$ is the centre of the circle through $A$, $B$, $C$ and $D$. If $\angle AOC = 120°$, find the magnitude of $\angle ADC$.

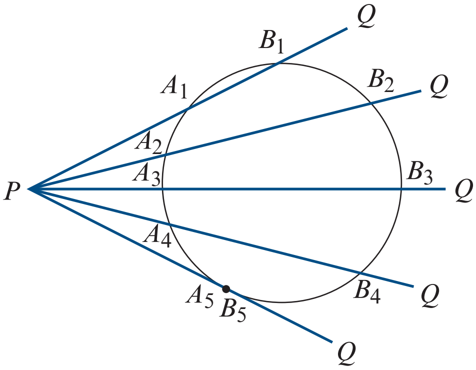
**8** $PQRS$ is a cyclic quadrilateral with $\angle SQR = 36°$, $\angle PSQ = 64°$ and $\angle RSQ = 42°$. Find the interior angles of the quadrilateral.

**9** Prove that if a parallelogram can be inscribed in a circle, then it must be a rectangle.

**10** Prove that the bisectors of the four interior angles of a quadrilateral form a cyclic quadrilateral.

# 7B Tangents

Consider a point $P$ outside a circle, as shown in the diagram. By rotating the secant $PQ$, with $P$ as the pivot point, we obtain a sequence of pairs of points on the circle. As $PQ$ moves towards the edge of the circle, the pairs of points become closer together, until they eventually coincide.




When $PQ$ is in this final position (i.e. when the intersection points $A$ and $B$ coincide), it is called a **tangent** to the circle.

A tangent touches the circle at only one point, and this point is called the **point of contact**.

The **length of a tangent** from a point $P$ outside the circle is the distance between $P$ and the point of contact.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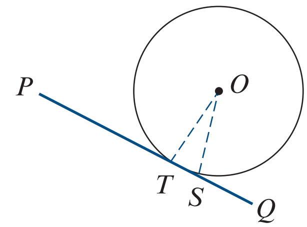
**Theorem 5:** Tangent is perpendicular to radius

A tangent to a circle is perpendicular to the radius drawn from the point of contact.

**Proof** This will be a proof by contradiction.




Let $T$ be the point of contact of tangent $PQ$ and suppose that $\angle OTP$ is not a right angle.

Let $S$ be the point on $PQ$, not $T$, such that $OSP$ is a right angle. Then triangle $OST$ has a right angle at $S$.

Therefore $OT > OS$, as $OT$ is the hypotenuse of triangle $OST$.

This implies that $S$ is inside the circle, as $OT$ is a radius.

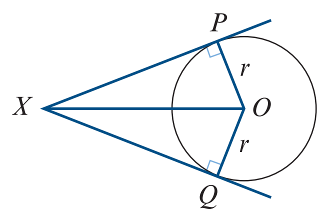
Thus the line through $T$ and $S$ must cut the circle again. But $PQ$ is a tangent, and so this is a contradiction. Hence we have shown that $\angle OTP$ is a right angle.

**Theorem 6:** Two tangents from the same point

The two tangents drawn from an external point to a circle are the same length.

**Proof** We can see that $\triangle XPO$ is congruent to $\triangle XQO$ using the RHS test, as $\angle XPO = \angle XQO = 90°$, the side $XO$ is common and $OP = OQ$ (radii).

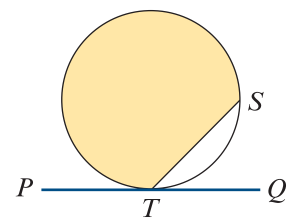
Therefore $XP = XQ$.




## The alternate segment theorem

In the diagram:

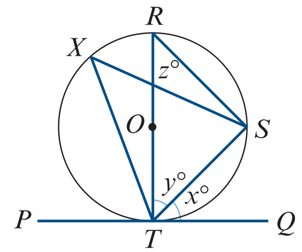
- The shaded segment is called the **alternate segment** in relation to $\angle STQ$.
- The unshaded segment is alternate to $\angle STP$.




**Theorem 7:** Alternate segment theorem

The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

**Proof** Let $\angle STQ = x°$, $\angle RTS = y°$ and $\angle TRS = z°$, where $RT$ is a diameter.




Then $\angle RST = 90°$ (Theorem 3, angle subtended by a diameter), and therefore $y + z = 90$.

Also $\angle RTQ = 90°$ (Theorem 5, tangent is perpendicular to radius), and therefore $x + y = 90$.

Thus $x = z$. But $\angle TXS$ is in the same segment as $\angle TRS$ and so $\angle TXS = x°$.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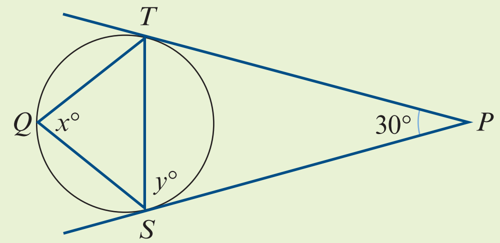
## Example 4

Find the magnitudes of the angles $x$ and $y$ in the diagram.




**Solution**

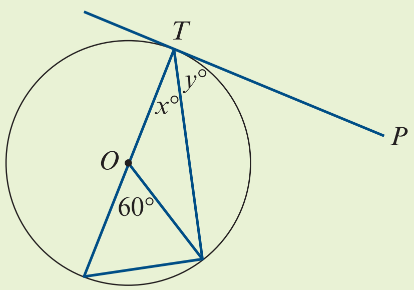
Triangle $PST$ is isosceles (Theorem 6, two tangents from the same point).

Therefore $\angle PST = \angle PTS$ and so $y = 75$.

The alternate segment theorem gives $x = y = 75$.

## Example 5

Find the values of $x$ and $y$, where $PT$ is tangent to the circle centre $O$.

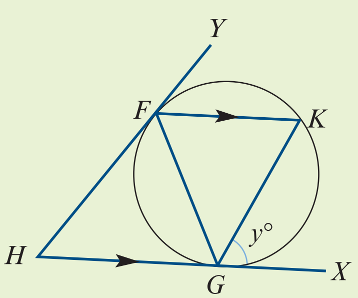



**Solution**

$x = 30$ as the angle at the circumference is half the angle subtended at the centre, and so $y = 60$ as $\angle OTP$ is a right angle.

## Example 6

The tangents to a circle at $F$ and $G$ meet at $H$. A chord $FK$ is drawn parallel to $HG$. Prove that triangle $FGK$ is isosceles.




**Solution**

Let $\angle XGK = y°$.

Then $\angle GFK = y°$ (alternate segment theorem) and $\angle GKF = y°$ (alternate angles).

Therefore triangle $FGK$ is isosceles with $FG = KG$.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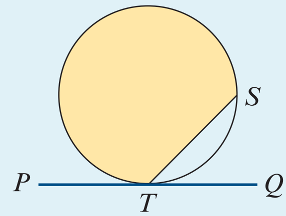
## Summary 7B

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.
- The two tangents drawn from an external point to a circle are the same length.
- In the diagram, the **alternate segment** to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.
- Alternate segment theorem The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



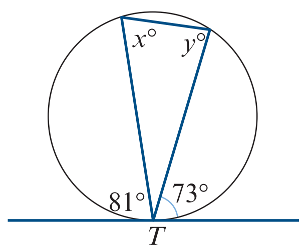

Skill-sheet

## Exercise 7B

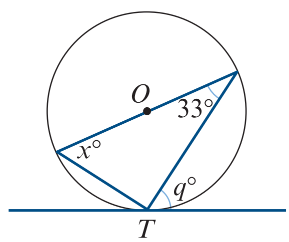
Example 4

**1** Find the values of the pronumerals for each of the following, where $T$ is the point of contact of the tangent and $O$ is the centre of the circle:

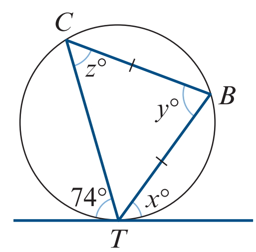
**a**




**b**

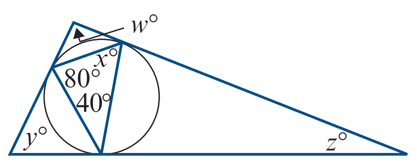



**c**

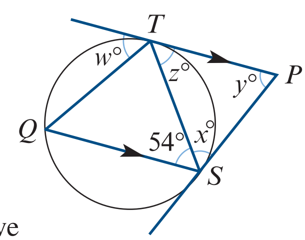



$BC = BT$

**d**




**e**




Note: In the diagram for part **e**, the two tangents from $P$ have points of contact at $S$ and $T$, and $TP$ is parallel to $QS$.

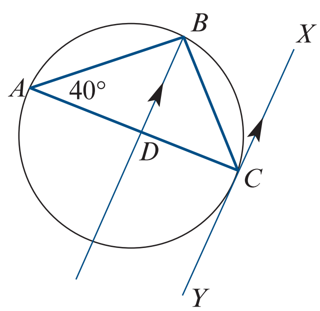
Example 5

**2** A triangle $ABC$ is inscribed in a circle, and the tangent to the circle at $C$ is parallel to the bisector of angle $ABC$.

**a** Find the magnitude of $\angle BCX$.

**b** Find the magnitude of $\angle CBD$, where $D$ is the point of intersection of the bisector of angle $ABC$ with $AC$.

**c** Find the magnitude of $\angle ABC$.




**3** Assume that $AB$ and $AC$ are two tangents to a circle, touching the circle at $B$ and $C$, and that $\angle BAC = 116°$. Find the magnitudes of the angles in the two segments into which $BC$ divides the circle.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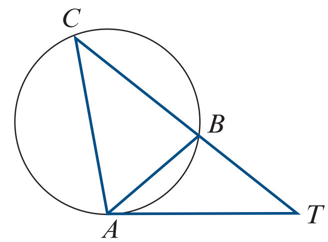
**4** $AT$ is a tangent at $A$ and $TBC$ is a secant to the circle. Given that $\angle CTA = 30°$ and $\angle CAT = 110°$, find the magnitude of angles $ACB$, $ABC$ and $BAT$.




Example 6 **5** From a point $A$ outside a circle, a secant $ABC$ is drawn cutting the circle at $B$ and $C$, and a tangent $AD$ touching it at $D$. A chord $DE$ is drawn equal in length to chord $DB$. Prove that triangles $ABD$ and $CDE$ are similar.

**6** Assume that $AB$ is a chord of a circle and that $CT$, the tangent at $C$, is parallel to $AB$. Prove that $CA = CB$.

**7** Through a point $T$, a tangent $TA$ and a secant $TPQ$ are drawn to a circle $APQ$. The chord $AB$ is drawn parallel to $PQ$. Prove that the triangles $PAT$ and $BAQ$ are similar.

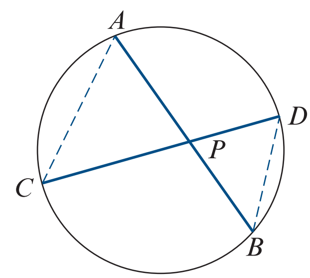
**8** $PQ$ is a diameter of a circle and $AB$ is a perpendicular chord cutting it at $N$. Prove that $PN$ is equal in length to the perpendicular from $P$ onto the tangent at $A$.

# 7C Chords in circles

**Theorem 8**

If $AB$ and $CD$ are two chords of a circle that cut at a point $P$ (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.

**Proof** Case 1: The intersection point $P$ is inside the circle.




Consider triangles $APC$ and $DPB$:

$\angle APC = \angle DPB$ (vertically opposite)

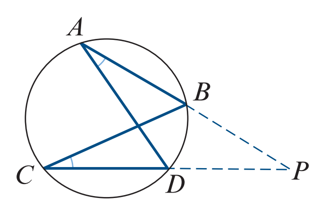
$\angle CAB = \angle BDC$ (angles in the same segment)

Thus triangle $APC$ is similar to triangle $DPB$. This gives

$$\frac{PA}{PD} = \frac{PC}{PB}$$

$$\therefore \quad PA \cdot PB = PC \cdot PD$$

Case 2: The intersection point $P$ is outside the circle.




Show that triangle $APD$ is similar to triangle $CPB$. This gives

$$\frac{PA}{PC} = \frac{PD}{PB}$$

$$\therefore \quad PA \cdot PB = PC \cdot PD$$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A converse of Theorem 8 is:

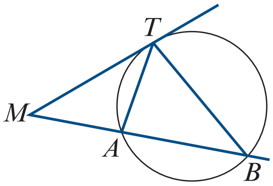
> If line segments $AB$ and $CD$ intersect at a point $M$ and $AM \cdot BM = CM \cdot DM$, then the points $A$, $B$, $C$ and $D$ are concyclic.

This is proved in Extended-response question 2.

### Theorem 9

If $M$ is a point outside a circle and $T$, $A$, $B$ are points on the circle such that $MT$ is a tangent and $MAB$ is a secant, then $MT^2 = MA \cdot MB$.

**Proof** Consider triangles $MAT$ and $MTB$:




$\angle ATM = \angle TBA$ (alternate segment theorem)

$\angle MAT = \angle MTB$ (angle sum of a triangle)

Therefore triangle $MAT$ is similar to triangle $MTB$.

This gives

$$\frac{MA}{MT} = \frac{MT}{MB}$$

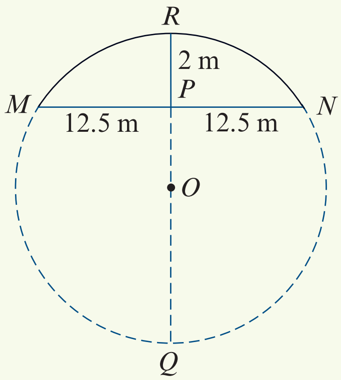
$$\therefore \quad MT^2 = MA \cdot MB$$

## Example 7

The arch of a bridge is to be in the form of an arc of a circle. The span of the bridge is to be 25 m and the height in the middle 2 m. Find the radius of the circle.

### Solution

Let $r$ be the radius of the circle. Then $PQ = 2r - 2$.




Use Theorem 8 with the chords $RQ$ and $MN$:

$$RP \cdot PQ = MP \cdot PN$$

Therefore

$$2PQ = 12.5^2$$

$$PQ = \frac{12.5^2}{2}$$

$$2r - 2 = \frac{12.5^2}{2} \qquad \text{as } PQ = 2r - 2$$

$$\therefore \quad r = \frac{1}{2}\left(\frac{12.5^2}{2} + 2\right)$$

$$= \frac{641}{16} \text{ m}$$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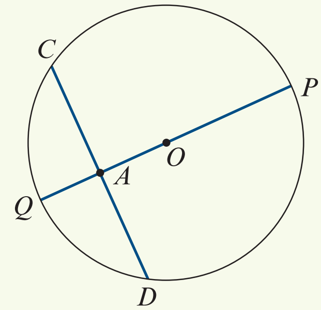
## Example 8

Let $A$ be any point inside a circle with radius $r$ and centre $O$. Show that, if $CD$ is a chord through $A$, then $CA \cdot AD = r^2 - OA^2$.

### Solution

Let $PQ$ be a diameter through $A$ as shown.




By Theorem 8:

$$CA \cdot AD = QA \cdot AP$$

Since $QA = r - OA$ and $AP = r + OA$, this gives

$$\begin{aligned} CA \cdot AD &= (r - OA)(r + OA) \\ &= r^2 - OA^2 \end{aligned}$$

## Summary 7C

- **Theorem 8** If $AB$ and $CD$ are two chords of a circle that cut at a point $P$ (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.
- **Theorem 9** If $M$ is a point outside a circle and $T$, $A$, $B$ are points on the circle such that $MT$ is a tangent and $MAB$ is a secant, then $MT^2 = MA \cdot MB$.

Skill-sheet

## Exercise 7C

Example 7

**1** Two chords $AB$ and $CD$ intersect at a point $P$ within a circle.

**a** Given that $AP = 5$ cm, $PB = 4$ cm, $CP = 2$ cm, find $PD$.

**b** Given that $AP = 4$ cm, $CP = 3$ cm, $PD = 8$ cm, find $PB$.

**2** If $AB$ is a chord and $P$ is a point on $AB$ such that $AP = 8$ cm, $PB = 5$ cm and $P$ is 3 cm from the centre of the circle, find the radius.

**3** If $AB$ is a chord of a circle with centre $O$ and $P$ is a point on $AB$ such that $BP = 4PA$, $OP = 5$ cm and the radius of the circle is 7 cm, find $AB$.

Example 8

**4** Two circles intersect at points $A$ and $B$. From any point $P$ on the line $AB$, tangents $PQ$ and $PR$ are drawn to the circles. Prove that $PQ = PR$.

**5** $PQ$ is a variable chord of the smaller of two fixed concentric circles, and $PQ$ extended meets the circumference of the larger circle at $R$. Prove that the product $RP \cdot RQ$ is constant for all positions and lengths of $PQ$.

**6** $ABC$ is an isosceles triangle with $AB = AC$. A line through $A$ meets $BC$ at $D$ and the circumcircle of the triangle at $E$. Prove that $AB^2 = AD \cdot AE$.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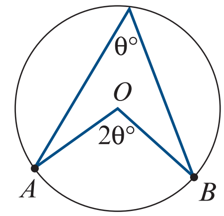
Review

## Chapter summary

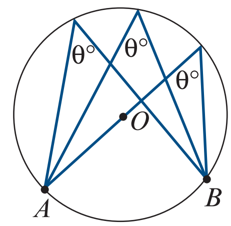
Assignment

Nrich

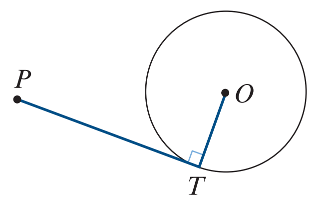
- The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



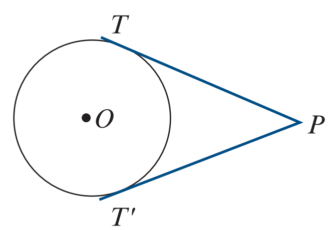

- Angles in the same segment of a circle are equal.



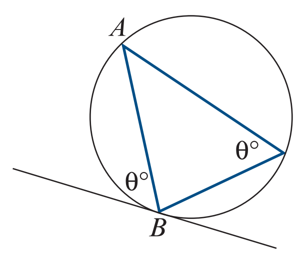

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.




- The two tangents drawn from an external point are the same length, i.e. $PT = PT'$.



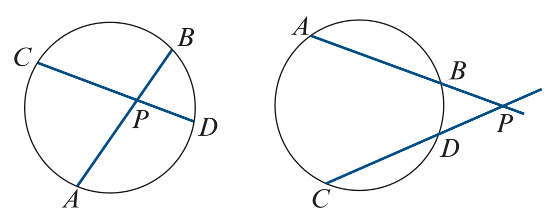

- The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.




- A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°.

- If $AB$ and $CD$ are two chords of a circle that cut at a point $P$, then

$$PA \cdot PB = PC \cdot PD$$





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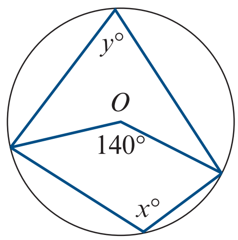
## Short-answer questions

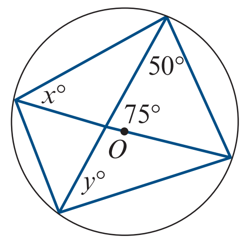
**1** $\triangle ABC$ has $\angle A = 36°$ and $\angle C = 90°$. $M$ is the midpoint of $AB$ and $CN$ is the altitude on $AB$. Find the size of $\angle MCN$.

**2** Find the values of the pronumerals in each of the following:

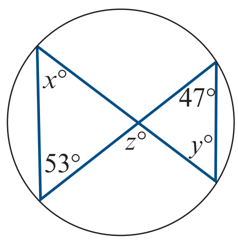
**a**



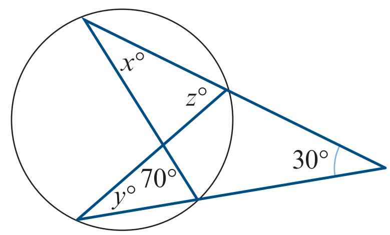

**b**




**c**




**d**




**3** Let $OP$ be a radius of a circle with centre $O$. A chord $BA$ is drawn parallel to $OP$. The lines $OA$ and $BP$ intersect at $C$. Prove that:

**a** $\angle CAB = 2\angle CBA$

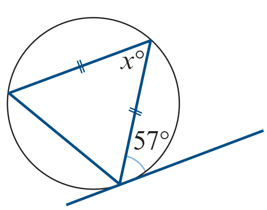
**b** $\angle PCA = 3\angle PBA$

**4** A chord $AB$ of a circle, centre $O$, is extended to $C$. The straight line bisecting $\angle OAB$ meets the circle at $E$. Prove that $EB$ bisects $\angle OBC$.

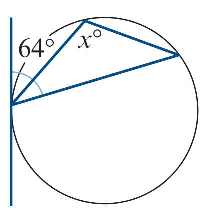
**5** Two circles intersect at $A$ and $B$. The tangent at $B$ to one circle meets the second again at $D$, and a straight line through $A$ meets the first circle at $P$ and the second at $Q$. Prove that $BP$ is parallel to $DQ$.

**6** Find the values of the pronumerals for each of the following:

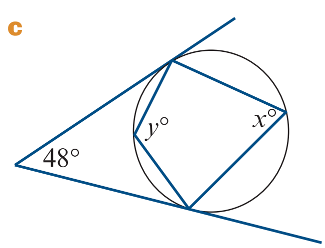
**a**




**b**




**c**





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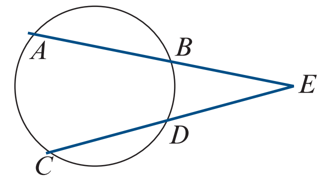
**7** Two circles intersect at $M$ and $N$. The tangent to the first circle at $M$ meets the second circle at $P$, while the tangent to the second at $N$ meets the first at $Q$. Prove that $MN^2 = NP \cdot QM$.

**8** If $AB = 10$ cm, $BE = 5$ cm and $CE = 25$ cm, find $DE$.

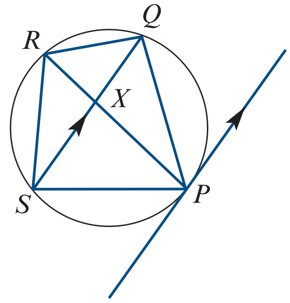



## Extended-response questions

**1** The diagonals $PR$ and $QS$ of a cyclic quadrilateral $PQRS$ intersect at $X$. The tangent at $P$ is parallel to $QS$.
Prove that:

**a** $PQ = PS$

**b** $PR$ bisects $\angle QRS$




**2** *If line segments $AB$ and $CD$ intersect at a point $M$ and $AM \cdot BM = CM \cdot DM$, then the points $A$, $B$, $C$ and $D$ are concyclic.*

To prove this claim, show that:

**a** $\dfrac{AM}{CM} = \dfrac{DM}{BM}$

**b** $\triangle AMC \sim \triangle DMB$

**c** $\angle CAM = \angle BDM$

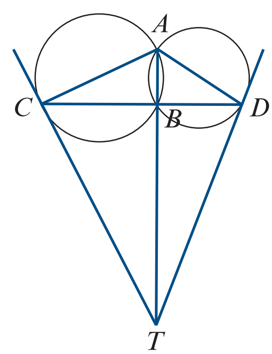
**d** $ABCD$ is cyclic

**3** Two circles intersect at $A$ and $B$. The tangents at $C$ and $D$ intersect at $T$ on the extension of $AB$. Prove that, if $CBD$ is a straight line, then:

**a** $TCAD$ is a cyclic quadrilateral

**b** $\angle TAC = \angle TAD$

**c** $TC = TD$.





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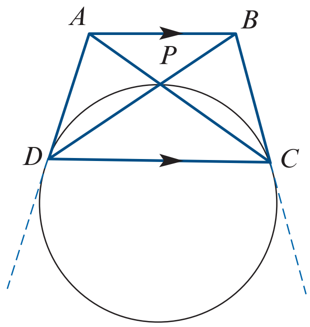
**4** $ABCD$ is a trapezium in which $AB$ is parallel to $DC$ and the diagonals meet at $P$. The circle through $D$, $P$ and $C$ touches $AD$ and $BC$ at $D$ and $C$ respectively.



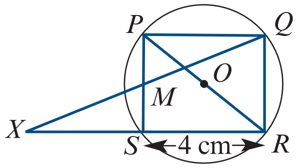

Prove that:

**a** $\angle BAC = \angle ADB$

**b** the circle through $A$, $P$ and $D$ touches $BA$ at $A$

**c** $ABCD$ is a cyclic quadrilateral.

**5** $PQRS$ is a square of side length 4 cm inscribed in a circle with centre $O$. The midpoint of the side $PS$ is $M$. The line segments $QM$ and $RS$ are extended to meet at $X$.




**a** Prove that:

**i** $\triangle XPR$ is isosceles

**ii** $PX$ is the tangent to the circle at $P$.

**b** Calculate the area of trapezium $PQRX$.

**6** **a** An isosceles triangle $ABC$, with $AB = AC$, is inscribed in a circle. A chord $AD$ intersects $BC$ at $E$. Prove that

$$AB^2 - AE^2 = BE \cdot CE$$

**b** Diameter $AB$ of a circle with centre $O$ is extended to $C$, and from $C$ a line is drawn tangent to the circle at $P$. The line $PT$ is drawn perpendicular to $AB$ at $T$. Prove that

$$CA \cdot CB - TA \cdot TB = CT^2$$

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# 8 Trigonometric functions

## In this chapter



## Syllabus references

**Topics:** The basic trigonometric functions; Trigonometric identities; Applications of trigonometric functions to model periodic phenomena

**Subtopics:** 2.1.1, 2.1.2, 2.1.5, 2.1.9

This chapter reviews and extends the study of the three trigonometric functions – sine, cosine and tangent – introduced in Mathematics Methods Units 1 & 2.

An important property of these three functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function $f$ is **periodic** if there is a positive constant $a$ such that $f(x + a) = f(x)$. The sine and cosine functions each have period $2\pi$, while the tangent function has period $\pi$.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

We will continue our study of trigonometric functions in Chapter 9, where we introduce the reciprocal trigonometric functions – secant, cosecant and cotangent – and we derive and apply several important trigonometric identities.

Note: A more detailed introduction to sine, cosine and tangent as functions is given in Mathematics Methods Units 1 & 2 and also in an online chapter for this book, available in the Interactive Textbook.



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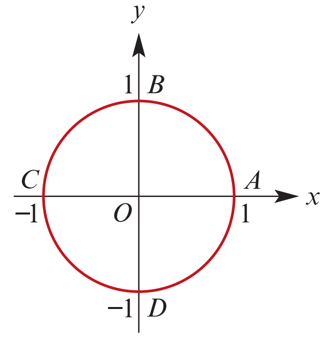
# 8A Defining the trigonometric functions

## Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.




The circumference of the unit circle $= 2\pi \times 1$

$= 2\pi$ units

Thus, the distance in an anticlockwise direction around the circle from

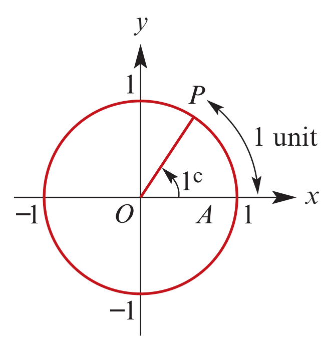
$A$ to $B = \dfrac{\pi}{2}$ units

$A$ to $C = \pi$ units

$A$ to $D = \dfrac{3\pi}{2}$ units

### Definition of a radian

In moving around the circle a distance of 1 unit from $A$ to $P$, the angle $POA$ is defined. The measure of this angle is 1 radian.




One **radian** (written $1^c$) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

### Converting between degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

**Example 1**

**a** Convert $135^\circ$ to radians.

**b** Convert $\dfrac{\pi^c}{6}$ to degrees.

**Solution**

**a** $135^\circ = \dfrac{135 \times \pi^c}{180} = \dfrac{3\pi^c}{4}$

**b** $\dfrac{\pi^c}{6} = \dfrac{\pi}{6} \times \dfrac{180^\circ}{\pi} = 30^\circ$

Note: Usually the symbol for radians, $^c$, is omitted. Any angle is assumed to be measured in radians unless indicated otherwise.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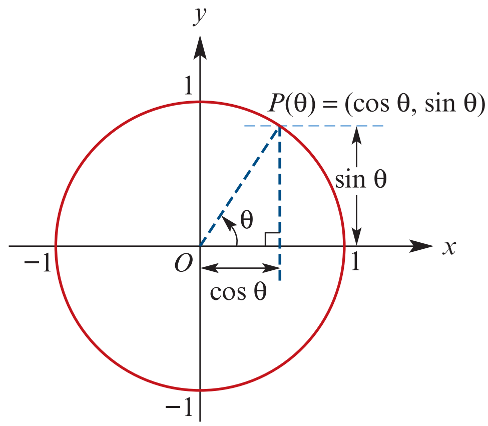
## Defining sine and cosine

Let $P(\theta)$ denote the point on the unit circle corresponding to an angle $\theta$. Then:

- $\cos\theta$ is the $x$-coordinate of $P(\theta)$
- $\sin\theta$ is the $y$-coordinate of $P(\theta)$




Hence the coordinates of the point $P(\theta)$ are $(\cos\theta, \sin\theta)$.

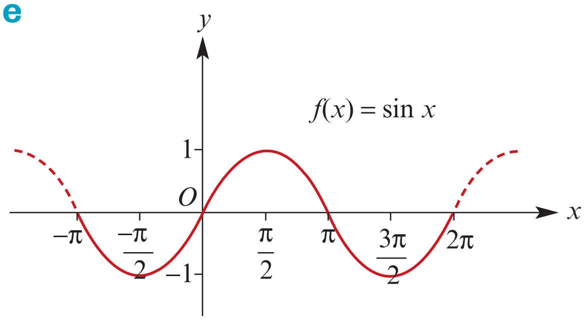
Note: Adding $2\pi$ to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

## The graphs of sine and cosine

A sketch of the graph of

$$f(x) = \sin x$$

is shown opposite.

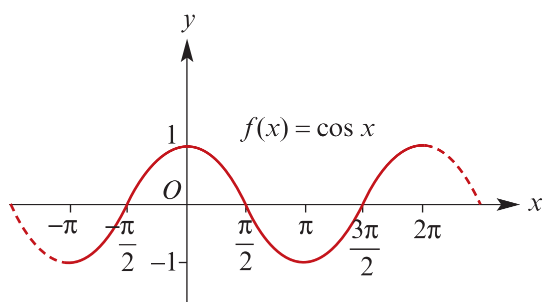



As $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$, the sine function is **periodic**. The period is $2\pi$. The amplitude is 1.

A sketch of the graph of

$$f(x) = \cos x$$

is shown opposite.




The period of the cosine function is $2\pi$. The amplitude is 1.

### Example 2

Evaluate each of the following:

**a** $\sin\left(\frac{3\pi}{2}\right)$ **b** $\cos\left(-\frac{\pi}{2}\right)$ **c** $\cos\left(\frac{21\pi}{2}\right)$

| Solution | Explanation |
|---|---|
| **a** $\sin\left(\frac{3\pi}{2}\right) = -1$ | since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$. |
| **b** $\cos\left(-\frac{\pi}{2}\right) = 0$ | since $P\left(-\frac{\pi}{2}\right)$ has coordinates $(0, -1)$. |
| **c** $\cos\left(\frac{21\pi}{2}\right) = \cos\left(10\pi + \frac{\pi}{2}\right) = 0$ | since $P\left(\frac{\pi}{2}\right)$ has coordinates $(0, 1)$. |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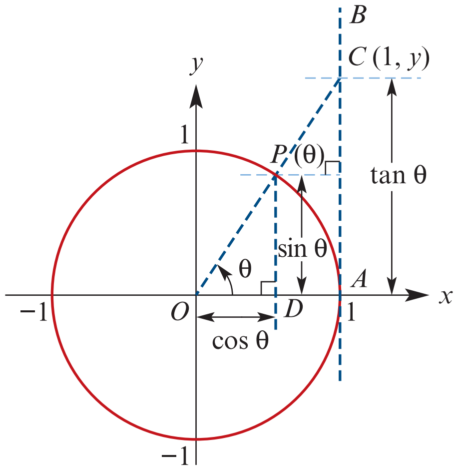
## Defining tangent

Again consider the unit circle.

If we draw a tangent to the unit circle at $A$, then the $y$-coordinate of $C$, the point of intersection of the line $OP$ and the tangent, is called **tangent** $\theta$ (abbreviated to $\tan\theta$).




By considering the similar triangles $OPD$ and $OCA$:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

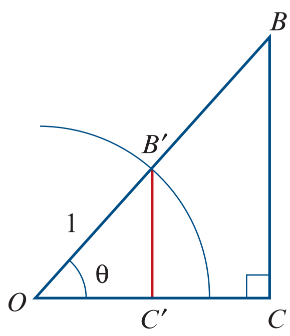
$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note that $\tan\theta$ is undefined when $\cos\theta = 0$. The domain of tan is $\mathbb{R} \setminus \{\, \theta : \cos\theta = 0 \,\}$ and so $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

Note: Adding $\pi$ to the angle does not change the line $OP$. Thus $\tan(\pi + \theta) = \tan\theta$.

## The trigonometric ratios

For a right-angled triangle $OBC$, we can construct a similar triangle $OB'C'$ that lies in the unit circle.




From the diagram:

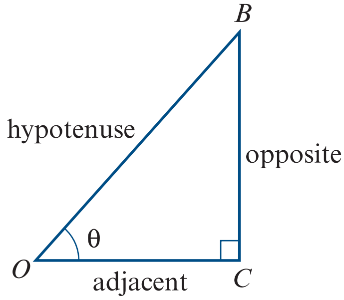
$$B'C' = \sin\theta \qquad \text{and} \qquad OC' = \cos\theta$$

The similarity factor is the length $OB$, giving

$$BC = OB\sin\theta \qquad \text{and} \qquad OC = OB\cos\theta$$

$$\therefore \quad \frac{BC}{OB} = \sin\theta \qquad \text{and} \qquad \frac{OC}{OB} = \cos\theta$$

This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle $\theta$ is as shown.




$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$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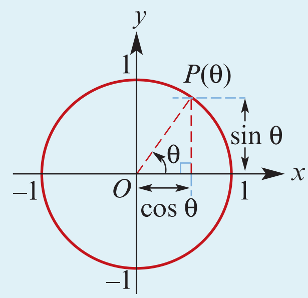
## Summary 8A

- One radian (written $1^c$) is the angle formed at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

- The point $P(\theta)$ on the unit circle corresponding to an angle $\theta$ has coordinates $(\cos\theta, \sin\theta)$.




## Exercise 8A

Example 1a **1** Convert the following angles from degrees to exact values in radians:

**a** $720^\circ$ **b** $540^\circ$ **c** $-450^\circ$ **d** $15^\circ$ **e** $-10^\circ$ **f** $-315^\circ$

Example 1b **2** Convert the following angles from radians to degrees:

**a** $\frac{5\pi}{4}$ **b** $-\frac{2\pi}{3}$ **c** $\frac{7\pi}{12}$ **d** $-\frac{11\pi}{6}$ **e** $\frac{13\pi}{9}$ **f** $-\frac{11\pi}{12}$

Example 2 **3** Find the exact value of each of the following:

**a** $\cos\left(\frac{3\pi}{2}\right)$ **b** $\sin\left(-\frac{\pi}{2}\right)$ **c** $\cos(6\pi)$ **d** $\sin\left(\frac{15\pi}{2}\right)$

**4** Find the exact value of each of the following:

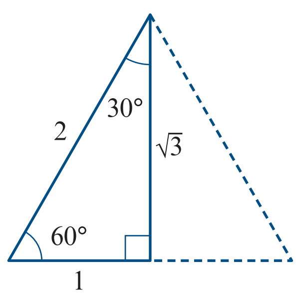
**a** $\sin(270^\circ)$ **b** $\cos(-540^\circ)$ **c** $\sin(450^\circ)$ **d** $\cos(720^\circ)$

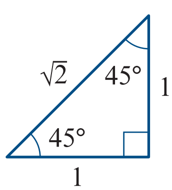
# 8B Symmetry properties and the Pythagorean identity

## Exact values of trigonometric functions

The values in this table for 30°, 45° and 60° can be determined from the two triangles shown.

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\frac{\pi}{6}$ (30°) | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ (45°) | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| $\frac{\pi}{3}$ (60°) | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ (90°) | 1 | 0 | undefined |





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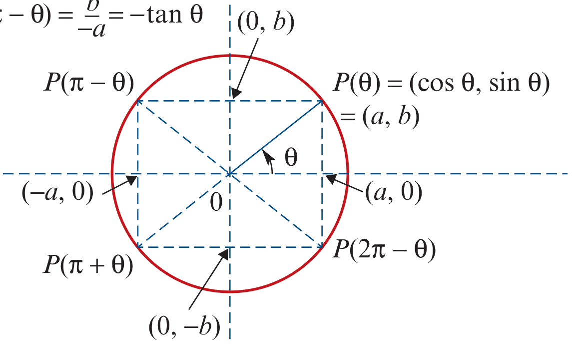
## Symmetry properties

The coordinate axes divide the unit circle into four quadrants, numbered anticlockwise from the positive direction of the $x$-axis. Using symmetry, we can determine relationships between the trigonometric functions for angles in different quadrants:

**Quadrant 2**
By symmetry
$\sin(\pi - \theta) = b = \sin\theta$
$\cos(\pi - \theta) = -a = -\cos\theta$
$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

**Quadrant 1**




**Quadrant 3**
$\sin(\pi + \theta) = -b = -\sin\theta$
$\cos(\pi + \theta) = -a = -\cos\theta$
$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

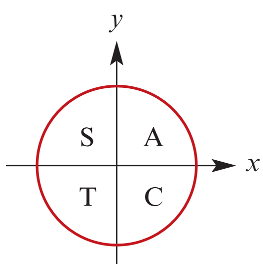
**Quadrant 4**
$\sin(2\pi - \theta) = -b = -\sin\theta$
$\cos(2\pi - \theta) = a = \cos\theta$
$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Note: $\sin(2\pi + \theta) = \sin\theta$
$\cos(2\pi + \theta) = \cos\theta$
$\tan(2\pi + \theta) = \tan\theta$

### Signs of trigonometric functions

Using these symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

| | | |
|---|---|---|
| 1st quadrant | all are positive | (A) |
| 2nd quadrant | sin is positive | (S) |
| 3rd quadrant | tan is positive | (T) |
| 4th quadrant | cos is positive | (C) |



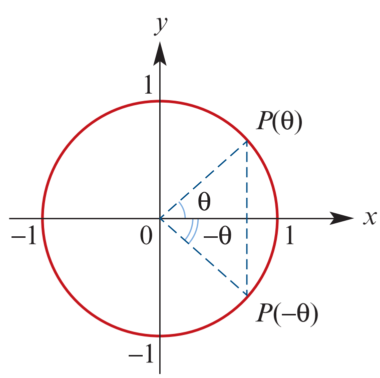

### Negative angles

By symmetry:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$





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## Example 3

Find the exact value of:

**a** $\sin\left(\frac{11\pi}{6}\right)$

**b** $\cos\left(\frac{-5\pi}{4}\right)$

### Solution

**a**
$$\begin{aligned}\sin\left(\frac{11\pi}{6}\right) &= \sin\left(2\pi - \frac{\pi}{6}\right)\\ &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

**b**
$$\begin{aligned}\cos\left(\frac{-5\pi}{4}\right) &= \cos\left(\frac{5\pi}{4}\right)\\ &= \cos\left(\pi + \frac{\pi}{4}\right)\\ &= -\cos\left(\frac{\pi}{4}\right)\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

## Example 4

Find the exact value of:

**a** $\sin 150^\circ$

**b** $\cos(-585^\circ)$

### Solution

**a**
$$\begin{aligned}\sin 150^\circ &= \sin(180^\circ - 150^\circ)\\ &= \sin 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

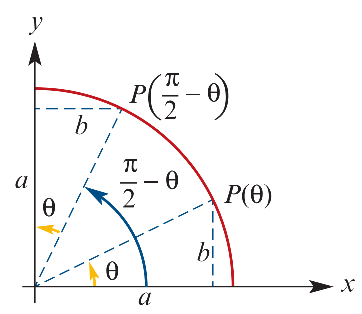
**b**
$$\begin{aligned}\cos(-585^\circ) &= \cos 585^\circ\\ &= \cos(585^\circ - 360^\circ)\\ &= \cos 225^\circ\\ &= -\cos 45^\circ\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

## Complementary relationships

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos\theta$$

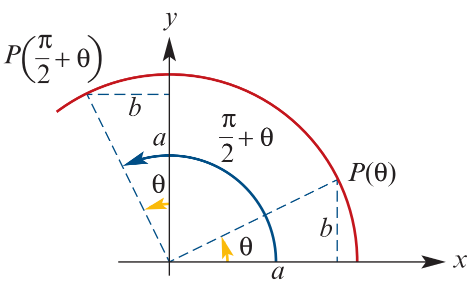
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin\theta$$




From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$





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**Example 5**

If $\sin\theta = 0.4$ and $\cos\alpha = 0.8$, find the value of:

**a** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

**Solution**

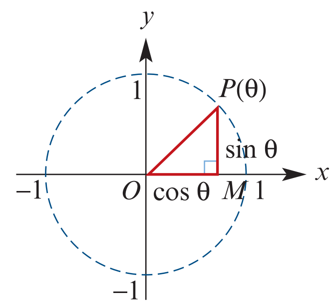
**a** $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

**b** $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.4$

**c** $\sin(-\theta) = -\sin\theta$
$= -0.4$

## The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.




By Pythagoras' theorem:

$$OP^2 = OM^2 + MP^2$$
$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Since this is true for all values of $\theta$, it is called an identity.

We can write $(\cos\theta)^2$ and $(\sin\theta)^2$ as $\cos^2\theta$ and $\sin^2\theta$, and therefore we obtain:

**Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

**Example 6**

If $\sin x = 0.3$ and $0 < x < \frac{\pi}{2}$, find:

**a** $\cos x$ **b** $\tan x$

**Solution**

**a** $\cos^2 x + \sin^2 x = 1$

$\cos^2 x + 0.09 = 1$

$\cos^2 x = 0.91$

$\therefore \quad \cos x = \pm\sqrt{0.91}$

Since the point $P(x)$ is in the 1st quadrant, this gives

$$\cos x = \sqrt{0.91} = \sqrt{\frac{91}{100}}$$
$$= \frac{\sqrt{91}}{10}$$

**b** $\tan x = \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}}$

$= \frac{3}{\sqrt{91}}$

$= \frac{3\sqrt{91}}{91}$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## Summary 8B

- Exact values

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\frac{\pi}{6}$ (30°) | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ (45°) | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| $\frac{\pi}{3}$ (60°) | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ (90°) | 1 | 0 | undefined |

- Complementary relationships

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

## Exercise 8B

Example 3

**1** Evaluate each of the following:

**a** $\cos\left(\frac{3\pi}{4}\right)$ **b** $\sin\left(\frac{5\pi}{4}\right)$ **c** $\sin\left(\frac{25\pi}{2}\right)$ **d** $\sin\left(\frac{15\pi}{6}\right)$ **e** $\cos\left(\frac{17\pi}{4}\right)$

**f** $\sin\left(-\frac{15\pi}{4}\right)$ **g** $\sin(27\pi)$ **h** $\sin\left(-\frac{17\pi}{3}\right)$ **i** $\cos\left(\frac{75\pi}{6}\right)$ **j** $\cos\left(-\frac{15\pi}{6}\right)$

**k** $\sin\left(-\frac{35\pi}{2}\right)$ **l** $\cos\left(-\frac{45\pi}{6}\right)$ **m** $\cos\left(\frac{16\pi}{3}\right)$ **n** $\sin\left(-\frac{105\pi}{2}\right)$ **o** $\cos(1035\pi)$

Example 4

**2** Find the exact value of each of the following:

**a** $\sin(135^\circ)$ **b** $\cos(-300^\circ)$ **c** $\sin(480^\circ)$

**d** $\cos(240^\circ)$ **e** $\sin(-225^\circ)$ **f** $\sin(420^\circ)$

Example 5

**3** If $\sin x = 0.3$ and $\cos\alpha = 0.6$, find the value of:

**a** $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\cos\left(\frac{\pi}{2} - x\right)$ **d** $\sin(-x)$

**e** $\cos\left(\frac{\pi}{2} + x\right)$ **f** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **g** $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **h** $\cos\left(\frac{3\pi}{2} - x\right)$

Example 6

**4** If $\sin x = 0.5$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

**5** If $\cos x = -0.7$ and $\pi < x < \frac{3\pi}{2}$, find $\sin x$ and $\tan x$.

**6** If $\sin x = -0.5$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

**7** If $\sin x = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, find $\cos x$ and $\tan x$.


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# 8C Solution of equations involving sine and cosine

If a trigonometric equation has a solution, then it will have a corresponding solution in each 'cycle' of its domain. Such an equation is solved by using the symmetry of the graph to obtain solutions within one 'cycle' of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

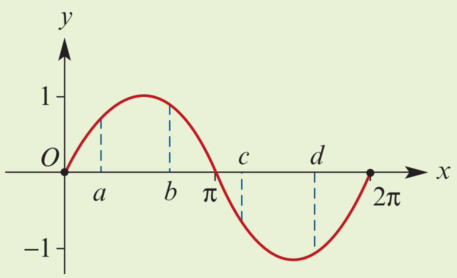
## Example 7

The graph of $y = f(x)$ for

$$f(x) = \sin x, \{ x \in \mathbb{R} : 0 \leq x \leq 2\pi \}$$

is shown.

For each pronumeral marked on the $x$-axis, find the other $x$-value which has the same $y$-value.




### Solution

For $x = a$, the other value is $\pi - a$.

For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

## Example 8

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

### Solution

Let $\theta = 2x + \frac{\pi}{3}$. Note that

$$\begin{aligned} 0 \leq x \leq 2\pi \quad &\Leftrightarrow \quad 0 \leq 2x \leq 4\pi \\ &\Leftrightarrow \quad \frac{\pi}{3} \leq 2x + \frac{\pi}{3} \leq \frac{13\pi}{3} \\ &\Leftrightarrow \quad \frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3} \end{aligned}$$

To solve $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$, we first solve $\sin\theta = \frac{1}{2}$ for $\frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3}$.

Consider $\sin\theta = \frac{1}{2}$.

$$\therefore \quad \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \ldots$$

The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required, as they lie outside the restricted domain for $\theta$.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For $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$:

$$\theta = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

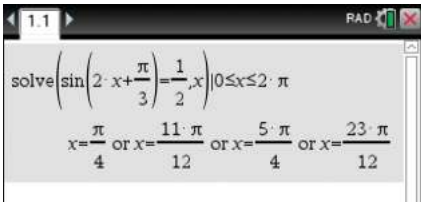
$$\therefore \quad 2x + \frac{2\pi}{6} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore \quad 2x = \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12}$$

## Using the TI-Nspire

- Ensure your calculator is in radian mode. (To change the mode, go to [on] > **Settings** > **Document Settings**.)
- Complete as shown.



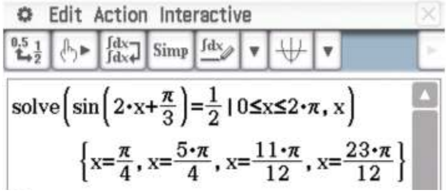

Note: The **Graph** application has its own settings, which are accessed from a **Graph** page using [menu] > **Settings**.

## Using the Casio ClassPad

- Open the Main $\sqrt{\alpha}$ application.
- Ensure your calculator is in radian mode (with **Rad** in the status bar).
- Using the [Math1] and [Math3] keyboards, enter and then highlight

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \mid 0 \le x \le 2\pi$$

- Select **Interactive** > **Equation/Inequality** > **solve**.




Skill-sheet

## Exercise 8C

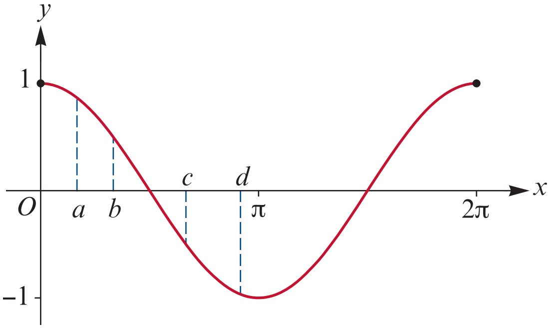
Example 7

**1** The graph of $y = f(x)$ for

$f(x) = \cos x, \{ x \in \mathbb{R} : 0 \le x \le 2\pi \}$

is shown.

For each pronumeral marked on the $x$-axis, find the other $x$-value which has the same $y$-value.





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Example 8

**2** Solve each of the following for $x \in [0, 2\pi]$:

**a** $\sin x = -\dfrac{\sqrt{3}}{2}$ **b** $\sin(2x) = -\dfrac{\sqrt{3}}{2}$ **c** $2\cos(2x) = -1$

**d** $\sin\left(x + \dfrac{\pi}{3}\right) = -\dfrac{1}{2}$ **e** $2\cos\left(2\left(x + \dfrac{\pi}{3}\right)\right) = -1$ **f** $2\sin\left(2x + \dfrac{\pi}{3}\right) = -\sqrt{3}$

**3** Solve each of the following for $x \in [-\pi, \pi]$:

**a** $\sin x + \dfrac{1}{2} = 0$ **b** $\sin(3x) = 0$ **c** $\cos\left(\dfrac{x}{2}\right) = 1$

**d** $\sin\left(x - \dfrac{\pi}{6}\right) = -\dfrac{1}{2}$ **e** $2\cos\left(2\left(x + \dfrac{\pi}{6}\right)\right) = -1$

# 8D Transformations of the graphs of sine and cosine

The graphs of functions with rules of the form

$$f(x) = a\sin(n(x + \varepsilon)) + b \qquad \text{and} \qquad f(x) = a\cos(n(x + \varepsilon)) + b$$

can be obtained from the graphs of $y = \sin x$ and $y = \cos x$ by transformations.

## Example 9

Sketch the graph of the function

$$h(x) = 3\cos\left(2x + \frac{\pi}{3}\right) + 1, \{ x \in \mathbb{R} : 0 \le x \le 2\pi \}$$

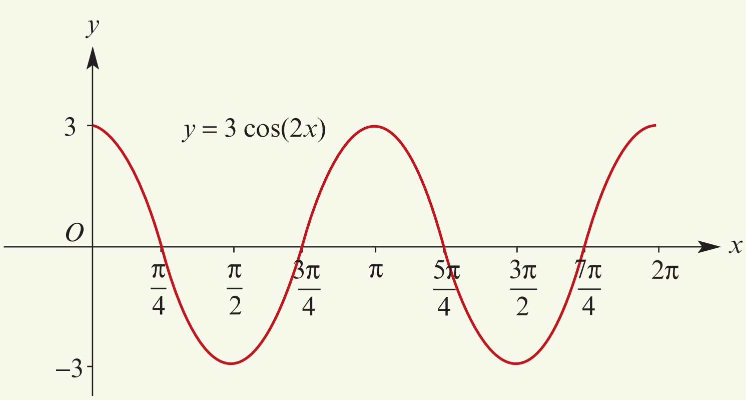
### Solution

We can write $h(x) = 3\cos\left(2\left(x + \dfrac{\pi}{6}\right)\right) + 1$.

The graph of $y = h(x)$ is obtained from the graph of $y = \cos x$ by:

- a dilation of factor $\frac{1}{2}$ parallel to the $x$-axis
- a dilation of factor 3 parallel to the $y$-axis
- a translation of $\dfrac{\pi}{6}$ units in the negative direction of the $x$-axis
- a translation of 1 unit in the positive direction of the $y$-axis.

First apply the two dilations to the graph of $y = \cos x$.





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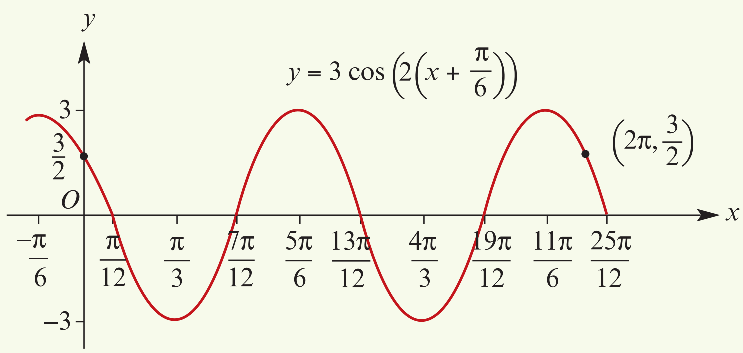
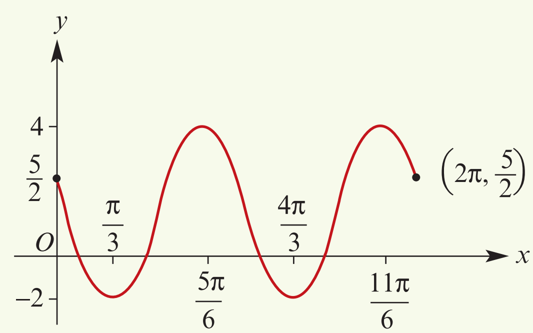
Next apply the translation $\frac{\pi}{6}$ units in the negative direction of the $x$-axis.




Apply the final translation and restrict the graph to the required domain.




## Summary 8D

For the graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$, where $a > 0$ and $n > 0$:

- Period $= \frac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a, a]$

Skill-sheet

## Exercise 8D

Example 9

**1** Sketch the graph of each of the following for the stated domain:

**a** $f(x) = \sin(2x),\ x \in [0, 2\pi]$

**b** $f(x) = \cos\left(x + \frac{\pi}{3}\right),\ x \in \left[\frac{-\pi}{3}, \pi\right]$

**c** $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right),\ x \in [0, \pi]$

**d** $f(x) = 2\sin(3x) + 1,\ x \in [0, \pi]$

**e** $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3},\ x \in [0, 2\pi]$

**f** $f(x) = \cos(2x),\ x \in [-\pi, \pi]$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**2** Sketch the graph of each of the following for the stated domain:

**a** $f(x) = \sin\left(x + \frac{\pi}{6}\right),\ x \in [-\pi, \pi]$

**b** $f(x) = \sin\left(2\left(x + \frac{\pi}{4}\right)\right),\ x \in [-\pi, \pi]$

**c** $f(x) = 2\cos\left(\frac{x}{3}\right) + 1,\ x \in [0, 6\pi]$

**d** $f(x) = 2\cos\left(x - \frac{\pi}{3}\right) + \sqrt{3},\ x \in [0, 2\pi]$

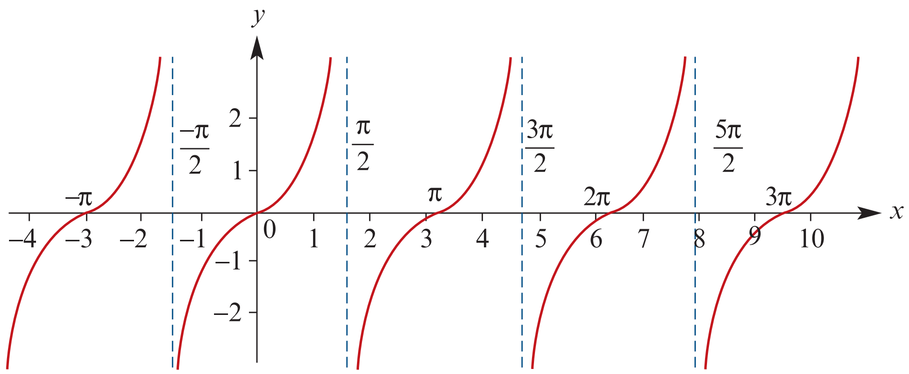
**e** $f(x) = \cos(\pi x),\ x \in [-2, 2]$

**f** $f(x) = \cos\left(\frac{\pi x}{6}\right),\ x \in [-12, 12]$

# 8E The tangent function

Recall that the tangent function is given by $\tan x = \dfrac{\sin x}{\cos x}$ for $\cos x \neq 0$.

The graph of $y = \tan x$ is shown below.




## Properties of the tangent function

- The graph repeats itself every $\pi$ units, i.e. the period of tan is $\pi$.
- The range of tan is $\mathbb{R}$.
- The vertical asymptotes have equations $x = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.

## Symmetry properties of tangent

Using symmetry properties of sine and cosine, we have

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

Similarly, we obtain:

- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$
- $\tan(-\theta) = -\tan\theta$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \dfrac{\cos\theta}{\sin\theta}$
- $\tan\left(\frac{\pi}{2} + \theta\right) = -\dfrac{\cos\theta}{\sin\theta}$

Note: We will see in Section 9A that $\dfrac{\cos\theta}{\sin\theta}$ can be written as $\cot\theta$.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**Example 10**

Find the exact value of:

**a** $\tan\left(\frac{4\pi}{3}\right)$

**b** $\tan 330°$

**Solution**

**a** $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right)$

$= \tan\left(\frac{\pi}{3}\right)$

$= \sqrt{3}$

**b** $\tan 330° = \tan(360° - 30°)$

$= -\tan 30°$

$= -\frac{1}{\sqrt{3}}$

## Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function, which can be applied to finding the $x$-axis intercepts for graphs of the tangent function.

The method is similar to that used for solving equations involving sine and cosine, except that only one solution needs to be found and then all other solutions are one period length apart.

**Example 11**

Solve the equation $3\tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

**Solution**

$3\tan(2x) = \sqrt{3}$

$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

$\therefore \quad 2x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ or $\frac{13\pi}{6}$ or $\frac{19\pi}{6}$

$x = \frac{\pi}{12}$ or $\frac{7\pi}{12}$ or $\frac{13\pi}{12}$ or $\frac{19\pi}{12}$

**Explanation**

Since we want solutions for $x$ in $(0, 2\pi)$, we find solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of $\pi$.

**Example 12**

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

**Solution**

Let $\theta = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$. Note that

$$-2\pi \le x \le 2\pi \Leftrightarrow -\frac{9\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$$

$$\Leftrightarrow -\frac{9\pi}{8} \le \frac{1}{2}\left(x - \frac{\pi}{4}\right) \le \frac{7\pi}{8}$$

$$\Leftrightarrow -\frac{9\pi}{8} \le \theta \le \frac{7\pi}{8}$$


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We first solve the equation $\tan\theta = -1$ for $-\frac{9\pi}{8} \le \theta \le \frac{7\pi}{8}$:

$$\theta = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore \quad x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

## Graphing the tangent function

When graphing a transformation of the tangent function:

- Find the period.
- Find the equations of the asymptotes.
- Find the intercepts with the axes.

### Example 13

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

**a** $y = 3\tan(2x)$

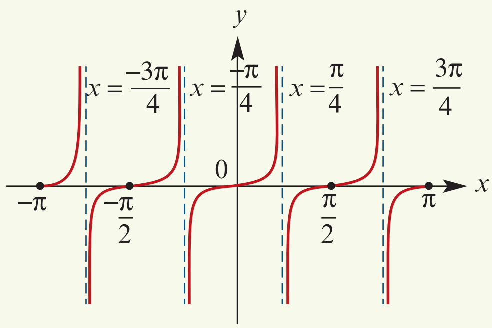
**b** $y = -2\tan(3x)$

#### Solution

**a** Period $= \frac{\pi}{n} = \frac{\pi}{2}$

Asymptotes: $x = \frac{(2k+1)\pi}{4}, \ k \in \mathbb{Z}$

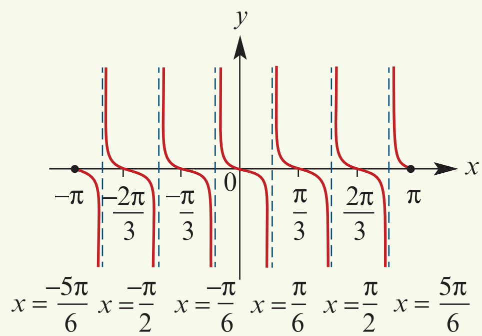
Axis intercepts: $x = \frac{k\pi}{2}, \ k \in \mathbb{Z}$




**b** Period $= \frac{\pi}{n} = \frac{\pi}{3}$

Asymptotes: $x = \frac{(2k+1)\pi}{6}, \ k \in \mathbb{Z}$

Axis intercepts: $x = \frac{k\pi}{3}, \ k \in \mathbb{Z}$





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## Summary 8E

- The tangent function is given by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$.
- The graph of $y = \tan x$:
  - The period is $\pi$.
  - The vertical asymptotes have equations $x = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
  - The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
  - $\tan(\pi + \theta) = \tan\theta$
  - $\tan(-\theta) = -\tan\theta$

## Exercise 8E

Example 10

**1** Find the exact value of each of the following:

**a** $\tan\left(\dfrac{5\pi}{4}\right)$ **b** $\tan\left(-\dfrac{2\pi}{3}\right)$ **c** $\tan\left(-\dfrac{29\pi}{6}\right)$

**2** Find the exact value of each of the following:

**a** $\tan 240^\circ$ **b** $\tan(-150^\circ)$ **c** $\tan 315^\circ$

**3** If $\tan x = \dfrac{1}{4}$ and $\pi \le x \le \dfrac{3\pi}{2}$, find the exact value of:

**a** $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(\pi - x)$

**4** If $\tan x = -\dfrac{\sqrt{3}}{2}$ and $\dfrac{\pi}{2} \le x \le \pi$, find the exact value of:

**a** $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(x - \pi)$

Example 11

**5** Solve each of the following equations for $x$ in the stated interval:

**a** $\tan x = -1,\ x \in (0, 2\pi)$ **b** $\tan x = \sqrt{3},\ x \in (0, 2\pi)$

**c** $\tan x = \dfrac{1}{\sqrt{3}},\ x \in (0, 2\pi)$ **d** $\tan(2x) = 1,\ x \in (-\pi, \pi)$

**e** $\tan(2x) = \sqrt{3},\ x \in (-\pi, \pi)$ **f** $\tan(2x) = -\dfrac{1}{\sqrt{3}},\ x \in (-\pi, \pi)$

Example 12

**6** Solve each of the following equations for $x$ in the stated interval:

**a** $\tan\left(2\left(x - \dfrac{\pi}{4}\right)\right) = 1,\ x \in (0, 2\pi)$ **b** $\tan\left(2\left(x - \dfrac{\pi}{4}\right)\right) = -1,\ x \in (-\pi, \pi)$

**c** $\tan\left(3\left(x - \dfrac{\pi}{6}\right)\right) = \sqrt{3},\ x \in (-\pi, \pi)$ **d** $\tan\left(\dfrac{1}{2}\left(x - \dfrac{\pi}{6}\right)\right) = -\dfrac{1}{\sqrt{3}},\ x \in (-\pi, \pi)$


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Example 13

**7** Sketch the graph of each of the following:

**a** $y = \tan(2x)$ **b** $y = \tan(3x)$ **c** $y = -\tan(2x)$

**d** $y = 3\tan x$ **e** $y = \tan\left(\frac{x}{2}\right)$ **f** $y = 2\tan\left(x + \frac{\pi}{4}\right)$

**g** $y = 3\tan x + 1$ **h** $y = 2\tan\left(x + \frac{\pi}{2}\right) + 1$ **i** $y = 3\tan\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$

**8** Sketch the graph of each of the following for the stated domain:

**a** $y = \tan\left(x + \frac{\pi}{3}\right) + \sqrt{3}$ for $x \in [0, 2\pi]$

**b** $y = \tan\left(\frac{x}{2}\right)$ for $x \in [0, 4\pi]$

**c** $y = \tan\left(\frac{\pi x}{2}\right)$ for $x \in [0, 4]$

## 8F General solution of trigonometric equations

We have seen how to solve equations involving trigonometric functions over a restricted domain. We now consider the general solutions of such equations over the natural domain for each function.

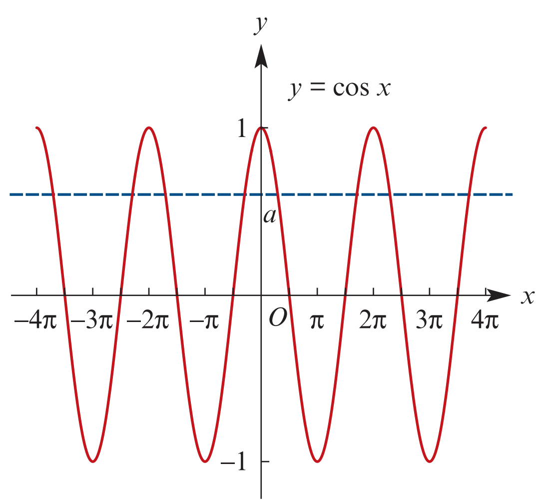
By convention:

- $\cos^{-1}$ has range $[0, \pi]$
- $\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \qquad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

If an equation involving a trigonometric function has one or more solutions in one 'cycle', then it will have corresponding solutions in each 'cycle' of its domain, i.e. there will be infinitely many solutions.




For example, consider the equation

$$\cos x = a$$

for some fixed $a \in [-1, 1]$. The solution in the interval $[0, \pi]$ is given by

$$x = \cos^{-1}(a)$$

By the symmetry properties of the cosine function, the other solutions are given by

$$-\cos^{-1}(a),\ \pm 2\pi + \cos^{-1}(a),\ \pm 2\pi - \cos^{-1}(a),\ \pm 4\pi + \cos^{-1}(a),\ \pm 4\pi - \cos^{-1}(a),\ \dots$$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In general, we have the following:

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
  $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
  $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

## Example 14

Find the general solution of each of the following equations:

**a** $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

### Solution

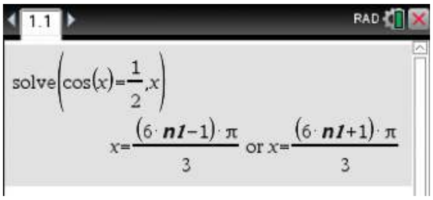
**a**
$$\cos x = 0.5$$
$$x = 2n\pi \pm \cos^{-1}(0.5)$$
$$= 2n\pi \pm \frac{\pi}{3}$$
$$= \frac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z}$$

**b**
$$\tan(3x) = \frac{1}{\sqrt{3}}$$
$$3x = n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= n\pi + \frac{\pi}{6}$$
$$= \frac{(6n+1)\pi}{6}$$
$$x = \frac{(6n+1)\pi}{18}, \quad n \in \mathbb{Z}$$

**c**
$$\sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$
$$x = 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$= 2n\pi + \frac{\pi}{4} \qquad\qquad = (2n+1)\pi - \frac{\pi}{4}$$
$$= \frac{(8n+1)\pi}{4}, \quad n \in \mathbb{Z} \qquad\qquad = \frac{(8n+3)\pi}{4}, \quad n \in \mathbb{Z}$$

### Using the TI-Nspire

- Make sure the calculator is in radian mode.
- Use **solve** from the **Algebra** menu and complete as shown. Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.





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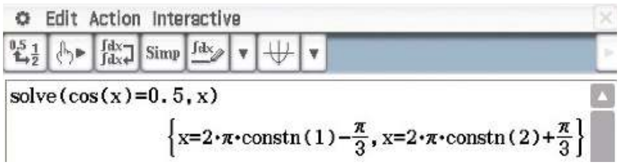
## Using the Casio ClassPad

- Check that the calculator is in radian mode.
- In Main $\sqrt{\alpha}$, enter and highlight the equation $\cos(x) = 0.5$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Then tap EXE.
- To view the entire solution, rotate the screen by selecting Rotate.

Note: Replace constn(1) and constn(2) with $n$ in the written answer.




## Example 15

Find the first three positive solutions of each of the following equations:

**a** $\cos x = 0.5$

**b** $\sqrt{3}\tan(3x) = 1$

**c** $2\sin x = \sqrt{2}$

### Solution

**a** The general solution (from Example 14a) is given by $x = \frac{(6n \pm 1)\pi}{3}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \pm\frac{\pi}{3}$, and when $n = 1$, $x = \frac{5\pi}{3}$ or $x = \frac{7\pi}{3}$.

Thus the first three positive solutions of $\cos x = 0.5$ are $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$.

**b** The general solution (from Example 14b) is given by $x = \frac{(6n + 1)\pi}{18}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{18}$, and when $n = 1$, $x = \frac{7\pi}{18}$, and when $n = 2$, $x = \frac{13\pi}{18}$.

Thus the first three positive solutions of $\sqrt{3}\tan(3x) = 1$ are $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$.

**c** The general solution (from Example 14c) is $x = \frac{(8n + 1)\pi}{4}$ or $x = \frac{(8n + 3)\pi}{4}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$, and when $n = 1$, $x = \frac{9\pi}{4}$ or $x = \frac{11\pi}{4}$.

Thus the first three positive solutions of $2\sin x = \sqrt{2}$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$.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## Example 16

Find the general solution for each of the following:

**a** $\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

**b** $\tan\left(2x - \frac{\pi}{3}\right) = 1$

### Solution

**a** $\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$x - \frac{\pi}{3} = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad x = n\pi + (-1)^n\left(\frac{\pi}{3}\right) + \frac{\pi}{3}, \qquad n \in \mathbb{Z}$$

The solutions are $x = \frac{(3n+2)\pi}{3}$ for $n$ even and $x = n\pi$ for $n$ odd.

**b** $\tan\left(2x - \frac{\pi}{3}\right) = 1$

$$2x - \frac{\pi}{3} = n\pi + \frac{\pi}{4}$$

$$2x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$$

$$\therefore \quad x = \frac{1}{2}\left(n\pi + \frac{7\pi}{12}\right)$$

$$= \frac{(12n+7)\pi}{24}, \qquad n \in \mathbb{Z}$$

## Summary 8F

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

  $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

  $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Skill-sheet

## Exercise 8F

**1** Evaluate each of the following for:

**i** $n = 1$ **ii** $n = 2$ **iii** $n = -2$

**a** $2n\pi \pm \cos^{-1}(1)$ **b** $2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

Example 14

**2** Find the general solution of each of the following equations:

**a** $\cos x = \frac{\sqrt{3}}{2}$ **b** $2\sin(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan x = 3$

Example 15

**3** Find the first two positive solutions of each of the following equations:

**a** $\sin x = 0.5$ **b** $2\cos(2x) = \sqrt{3}$ **c** $\sqrt{3}\tan(2x) = -3$

**4** Given that a trigonometric equation has general solution $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-2\pi, 2\pi]$.


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**5** Given that a trigonometric equation has general solution $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-\pi, 2\pi]$.

Example 16

**6** Find the general solution for each of the following:

**a** $\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = \frac{1}{2}$ **b** $2\tan\left(2\left(x + \frac{\pi}{4}\right)\right) = 2\sqrt{3}$ **c** $2\sin\left(x + \frac{\pi}{3}\right) = -1$

**7** Find the general solution of $2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$ and hence find all the solutions for $x$ in the interval $(-2\pi, 2\pi)$.

**8** Find the general solution of $\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$ and hence find all the solutions for $x$ in the interval $[-\pi, 0]$.

**9** Find the general solution of $2\sin(4\pi x) + \sqrt{3} = 0$ and hence find all the solutions for $x$ in the interval $[-1, 1]$.

# 8G Applications of trigonometric functions

A **sinusoidal function** has a rule of the form $y = a\sin(nt + \varepsilon) + b$ or, equivalently, of the form $y = a\cos(nt + \varepsilon) + b$. Such functions can be used to model periodic phenomena.

## Example 17

The height, $h(t)$ metres, of the tide above mean sea level at a harbour entrance over one day is given by the rule $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$, where $t$ is the number of hours after midnight.

**a** Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

**b** When was high tide?

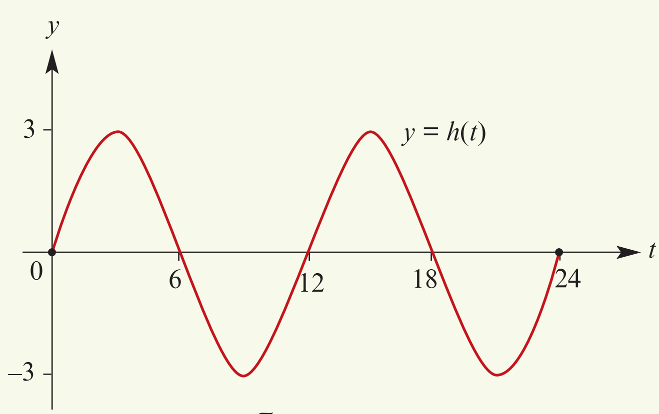
**c** What was the height of the high tide?

**d** What was the height of the tide at 8 a.m.?

**e** A boat can only enter the harbour when the tide is at least 1 metre above mean sea level. When could the boat enter the harbour during this particular day?

### Solution

**a**




Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

**b** High tide occurs when $h(t) = 3$:

$$3\sin\left(\frac{\pi t}{6}\right) = 3$$

$$\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\frac{\pi t}{6} = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).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**c** The high tide has height 3 metres above the mean height.

**d** $h(8) = 3\sin\left(\frac{8\pi}{6}\right) = 3\sin\left(\frac{4\pi}{3}\right) = 3 \times \frac{-\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$

The water is $\frac{3}{2}\sqrt{3}$ metres below the mean height at 8 a.m.

**e** First consider when $h(t) = 1$:

$$3\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\sin\left(\frac{\pi t}{6}\right) = \frac{1}{3}$$

$$\therefore \quad t = 0.649,\ 5.351,\ 12.649,\ 17.351$$

i.e. the water is at height 1 metre at 00:39, 05:21, 12:39, 17:21.
Thus the boat can enter the harbour between 00:39 and 05:21, and between 12:39 and 17:21.

## Exercise 8G

Example 17

**1** The number of hours of daylight at a point on the Arctic Circle is given approximately by $d = 12 - 12\cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$, where $t$ is the number of months which have elapsed since 1 January.

**a** **i** Find $d$ on 21 December ($t \approx 11.7$).

**ii** Find $d$ on 21 June ($t \approx 5.7$).

**b** When will there be 5 hours of daylight?

**2** The depth, $D(t)$ metres, of water at the entrance to a harbour at $t$ hours after midnight on a particular day is given by $D(t) = 12 + 4\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.

**a** Sketch the graph of $D(t)$ for $0 \le t \le 24$.

**b** Find the values of $t$ for which $D(t) \ge 12$.

**c** Boats which need a depth of $w$ metres are permitted to enter the harbour only if the depth of the water at the entrance is at least $w$ metres for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of $w$ which satisfies this condition.

**3** A particle moves on a straight line, $OX$, and its distance $x$ metres from $O$ at time $t$ seconds is given by $x = 4 + 3\sin(2\pi t)$.

**a** Find its greatest distance from $O$.

**b** Find its least distance from $O$.

**c** Find the times at which it is 7 metres from $O$ for $0 \le t \le 2$.

**d** Find the times at which it is 1 metre from $O$ for $0 \le t \le 2$.

**e** Describe the motion of the particle.


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**4** The temperature, $A$°C, inside a house at $t$ hours after 4 a.m. is given by the rule $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$. The temperature, $B$°C, outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$.

**a** Find the temperature inside the house at 8 a.m.

**b** Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.

**c** Sketch the graph of $D$ for $0 \le t \le 24$.

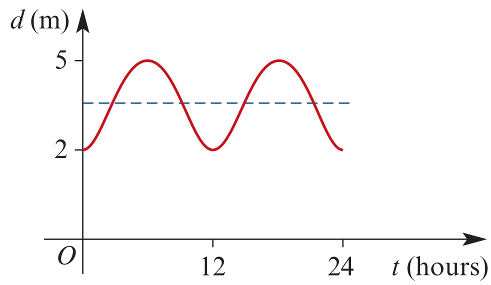
**d** Determine when the inside temperature is less than the outside temperature.

**5** The water level on a beach wall is given by $d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$, where $t$ is the number of hours after midnight and $d$ is the depth of the water in metres.

**a** What is the earliest time of day at which the water is at its highest?

**b** When is the water 2 m up the wall?

**6** The graph shows the distance, $d(t)$, of the tip of the hour hand of a large clock from the ceiling at time $t$ hours.




**a** The function $d$ is sinusoidal. Find:

**i** the amplitude

**ii** the period

**iii** the rule for $d(t)$

**iv** the length of the hour hand.

**b** At what times is the distance less than 3.5 metres from the ceiling?

**7** In a tidal river, the time between high tide and low tide is 8 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres.

**a** Sketch the graph of the depth of water at the point for the time interval from 0 to 24 hours if the relationship between time and depth is sinusoidal and there is a high tide at noon.

**b** If a boat requires a depth of 4 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

**c** If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?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**8** The population of a particular species of ant varies with time. The population, $N(t)$, at time $t$ weeks after 1 January 2016 is given by

$$N(t) = 3000\sin\left(\frac{\pi(t-10)}{26}\right) + 4000$$

**a** For the rule $N(t)$, state:

- **i** the period
- **ii** the amplitude
- **iii** the range.

**b** **i** State the values of $N(0)$ and $N(100)$.

- **ii** Sketch the graph of $y = N(t)$ for $t \in [0, 100]$.

**c** Find the values of $t \in [0, 100]$ for which the population is:

- **i** 7000
- **ii** 1000

**d** Find $\{ t \in [0, 100] : N(t) > 5500 \}$. That is, find the intervals of time during the first 100 weeks for which the population of ants is greater than 5500.

**e** A second population of ants also varies with time. The rule for the population, $M(t)$, at time $t$ weeks after 1 January 2016 is of the form

$$M(t) = a\sin\left(\frac{\pi(t-c)}{b}\right) + d$$

where $a$, $b$, $c$ and $d$ are positive constants. Find a set of possible values for the constants $a$, $b$, $c$ and $d$ given that the population has the following properties:

- the maximum population is 40 000 and occurs at $t = 10$
- the minimum population is 10 000 and occurs at $t = 20$
- the maximum and minimum values do not occur between $t = 10$ and $t = 20$.


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## Chapter summary

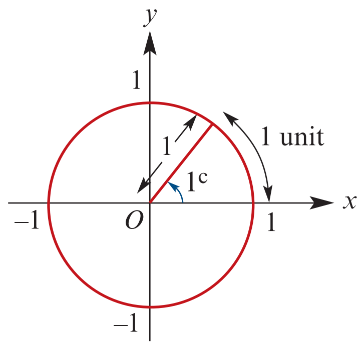
Assign-ment

Nrich

■ **Definition of a radian**

One radian (written $1^c$) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

$$1^c = \frac{180°}{\pi} \qquad 1° = \frac{\pi^c}{180}$$



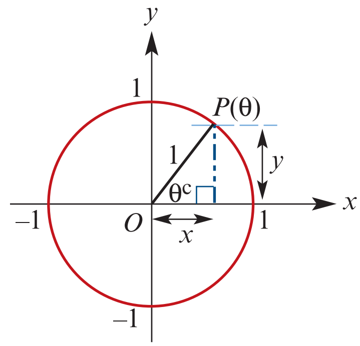

■ **Sine and cosine functions**

$x$-coordinate of $P(\theta)$ on unit circle:

$$x = \cos\theta, \qquad \theta \in \mathbb{R}$$

$y$-coordinate of $P(\theta)$ on unit circle:

$$y = \sin\theta, \qquad \theta \in \mathbb{R}$$



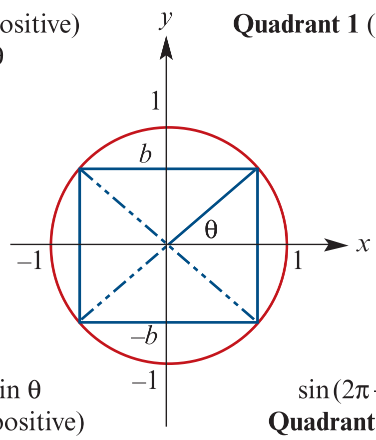

■ **Tangent function**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

■ **Symmetry properties of trigonometric functions**

**Quadrant 2** (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

**Quadrant 1** (all functions are positive)
e.g. $\sin\theta = b$




$\sin(\pi + \theta) = -b = -\sin\theta$
**Quadrant 3** (tan is positive)

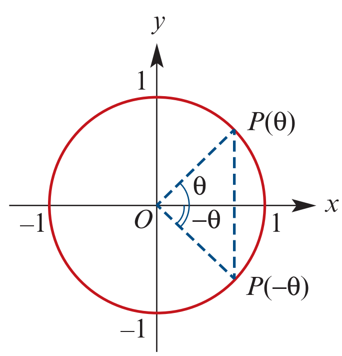
$\sin(2\pi - \theta) = -b = -\sin\theta$
**Quadrant 4** (cos is positive)

Negative angles:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$





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Review

Complementary relationships:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

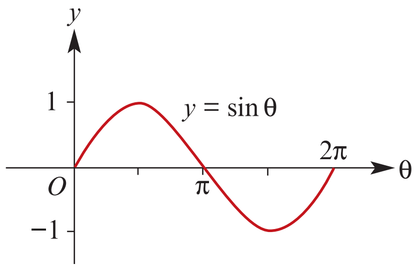
- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

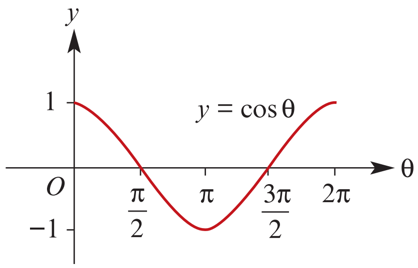
- **Exact values of trigonometric functions**

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $0$ | $0$ | $1$ | $0$ |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | $1$ | $0$ | undefined |

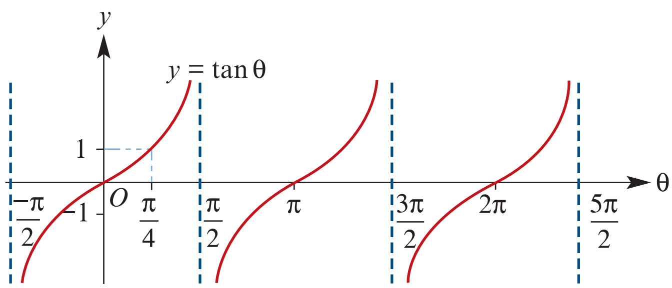
- **Graphs of trigonometric functions**




Amplitude = 1
Period = 2π




Amplitude = 1
Period = 2π




Amplitude is undefined
Period = π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- **Transformations of the graphs of trigonometric functions**

  For the graphs of $y = a\sin(n(x + \varepsilon)) + b$ and $y = a\cos(n(x + \varepsilon)) + b$, where $a, n \in \mathbb{R}^+$:

  - Period $= \dfrac{2\pi}{n}$
  - Amplitude $= a$
  - Range $= [-a + b, a + b]$

  For the graph of $y = a\tan(n(x + \varepsilon)) + b$, where $n \in \mathbb{R}^+$:

  - Period $= \dfrac{\pi}{n}$
  - Asymptotes: $x = \dfrac{(2k + 1)\pi}{2n} - \varepsilon$, where $k \in \mathbb{Z}$

- **General solution of trigonometric equations**

  - For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

    $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

  - For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

    $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

  - For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

    $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n + 1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

## Short-answer questions

**1** Convert each of the following to radian measure in terms of $\pi$:

**a** $390°$ **b** $840°$ **c** $1110°$ **d** $1065°$ **e** $165°$

**f** $450°$ **g** $420°$ **h** $390°$ **i** $40°$

**2** Convert each of the following to degree measure:

**a** $\dfrac{11\pi}{6}$ **b** $\dfrac{17\pi}{4}$ **c** $\dfrac{9\pi}{4}$ **d** $\dfrac{7\pi}{12}$ **e** $\dfrac{17\pi}{2}$

**f** $-\dfrac{11\pi}{4}$ **g** $-\dfrac{5\pi}{4}$ **h** $-\dfrac{13\pi}{4}$ **i** $\dfrac{23\pi}{4}$

**3** Give the exact value of each of the following:

**a** $\sin\left(\dfrac{9\pi}{4}\right)$ **b** $\cos\left(\dfrac{-5\pi}{4}\right)$ **c** $\sin\left(\dfrac{3\pi}{2}\right)$ **d** $\cos\left(\dfrac{-3\pi}{2}\right)$

**e** $\cos\left(\dfrac{11\pi}{6}\right)$ **f** $\sin\left(\dfrac{21\pi}{6}\right)$ **g** $\tan\left(\dfrac{-25\pi}{3}\right)$ **h** $\tan\left(\dfrac{-15\pi}{4}\right)$

**4** State the amplitude and period of each of the following:

**a** $4\sin\left(\dfrac{\theta}{2}\right)$ **b** $-5\sin(6\theta)$ **c** $\dfrac{1}{3}\sin(4\theta)$

**d** $-2\cos(5x)$ **e** $-7\sin\left(\dfrac{\pi x}{4}\right)$ **f** $\dfrac{2}{3}\sin\left(\dfrac{2\pi x}{3}\right)$

**5** Find the maximum and minimum values of the function with rule:

**a** $3 + 2\sin\theta$ **b** $4 - 5\cos\theta$


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**6** Sketch the graph of each of the following (showing one cycle):

**a** $y = 2\cos(2x)$ **b** $y = -3\sin\left(\frac{x}{3}\right)$ **c** $y = -2\cos(3x)$

**d** $y = 2\cos\left(\frac{x}{3}\right)$ **e** $y = \cos\left(x - \frac{\pi}{4}\right)$ **f** $y = \cos\left(x + \frac{2\pi}{3}\right)$

**g** $y = 2\sin\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\sin\left(x + \frac{5\pi}{6}\right)$

**7** Solve each of the following equations:

**a** $\cos\theta = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$ **b** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$

**c** $\cos\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$ **d** $\cos\left(\theta + \frac{\pi}{3}\right) = -1$ for $\theta \in [0, 2\pi]$

**e** $\cos\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$

**8** Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

**a** $f(x) = 2\cos(2x) + 1$ **b** $f(x) = 1 - 2\cos(2x)$ **c** $f(x) = 3\sin\left(x + \frac{\pi}{3}\right)$

**d** $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$ **e** $f(x) = 1 - 2\cos(3x)$

**9** Solve each of the following for $x \in [0, 2\pi]$:

**a** $\tan x = -\sqrt{3}$ **b** $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

**c** $2\tan\left(\frac{x}{2}\right) + 2 = 0$ **d** $3\tan\left(\frac{\pi}{2} + 2x\right) = -3$

**10** Sketch the graph of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:

**a** $f(x) = \tan(2x)$ **b** $f(x) = \tan\left(x - \frac{\pi}{3}\right)$

**c** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right)$ **d** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$

**11** Find the values of $\theta \in [0, 2\pi]$ for which:

**a** $\sin^2\theta = \frac{1}{4}$ **b** $\sin(2\theta) = \frac{1}{2}$ **c** $\cos(3\theta) = \frac{\sqrt{3}}{2}$ **d** $\sin^2(2\theta) = 1$

**12** Solve the equation $\tan(\theta°) = 2\sin(\theta°)$ for values of $\theta°$ from 0° to 360°.

**13** Find the general solution for each of the following:

**a** $\sin(2x) = -1$ **b** $\cos(3x) = 1$ **c** $\tan x = -1$


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## Extended-response questions

**1** The depth, $D$ metres, of sea water in a bay $t$ hours after midnight on a particular day may be represented by a function with rule

$$D(t) = a + b\cos\left(\frac{2\pi t}{k}\right)$$

where $a$, $b$ and $k$ are real numbers. The water is at a maximum depth of 15.4 metres at midnight and noon, and is at a minimum depth of 11.4 metres at 6 a.m. and 6 p.m.

**a** Find the value of:

**i** $a$ **ii** $b$ **iii** $k$

**b** Find the times when the depth of the water is 13.4 metres.

**c** Find the values of $t$ for which the depth of the water is less than 14.4 metres.

**2** The temperature, $T$°C, in a small town in the mountains over a day is modelled by the function with rule

$$T = 15 - 8\cos\left(\frac{\pi t}{12} + 6\right), \quad 0 \le t \le 24$$

where $t$ is the time in hours after midnight.

**a** What is the temperature at midnight, correct to two significant figures?

**b** What are the maximum and minimum temperatures reached?

**c** At what times of the day, to the nearest minute, are temperatures warmer than 20°C?

**d** Sketch the graph for the temperatures over a day.

**3** A particle oscillates back and forth, in a straight line, between points $A$ and $B$ about a point $O$. Its position, $x(t)$ metres, relative to $O$ at time $t$ seconds is given by the rule $x(t) = 3\sin(2\pi t - a)$. The position of the particle when $t = 1$ is $x = -1.5$.


**a** If $a \in \left[0, \frac{\pi}{2}\right]$, find the value of $a$.

**b** Sketch the graph of $x(t)$ against $t$ for $t \in [0, 2]$. Label the maximum and minimum points, the axis intercepts and the endpoints with their coordinates.

**c** How far from $O$ is point $A$?

**d** At what time does the particle first pass through $A$?

**e** How long is it before the particle returns to $A$?

**f** How long does it take for the particle to go from $A$ to $O$?

**g** How far does the particle travel in:

**i** the first 2 seconds of its motion

**ii** the first 2.5 seconds of its motion?


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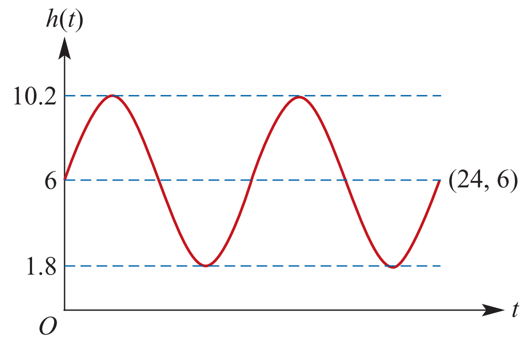
Review

**4** The depth of water, $h(t)$ m, at a particular jetty in a harbour at time $t$ hours after midnight is given by the rule

$$h(t) = p + q\sin\left(\frac{\pi t}{6}\right)$$

for constants $p$ and $q$. The graph of $h(t)$ against $t$ for $t \in [0, 24]$ is shown. The maximum depth is 10.2 m and the minimum depth is 1.8 m.




**a** Find the values of $p$ and $q$.

**b** At what times during the time interval $[0, 24]$ is the depth of water at a maximum?

**c** What is the average depth of the water over the time interval $[0, 24]$?

**d** At what times during the time interval $[0, 24]$ is the depth of the water 3.9 m?

**e** For how long during the 24-hour period from midnight is the depth of the water more than 8.1 m?

**5** Consider the function $f(x) = 2\sin(3x) + 1,\ x \in [0, 2\pi]$.

**a** Find the values of $k$ such that the equation $f(x) = k$ has:

**i** six solutions for $x \in [0, 2\pi]$

**ii** three solutions for $x \in [0, 2\pi]$

**iii** no solutions for $x \in [0, 2\pi]$.

**b** Find a sequence of transformations which takes the graph of $y = f(x)$ to the graph of $y = \sin x$.

**c** Find the values of $h \in [0, 2\pi]$ such that the graph of $y = f(x + h)$ has:

**i** a maximum at the point $\left(\frac{\pi}{3}, 3\right)$

**ii** a minimum at the point $\left(\frac{\pi}{3}, -1\right)$.

**6** **a** Find a sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = \sin x$.

**b** Find a sequence of transformations which takes the graph of $y = 2\cos x$ to the graph of $y = -\frac{1}{2}\sin(2x)$.

**c** **i** Find the rule for the image of the graph of $f(x) = \sin x$ under a dilation of factor $\frac{2}{\pi}$ from the $y$-axis, followed by a reflection in the line $y = 2$.

**ii** Find the range and period of the new function.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**7** The population, $N$, of a particular species of ant varies with the seasons. The population is modelled by the equation

$$N = 3000\sin\left(\frac{\pi(t-1)}{6}\right) + 4000$$

where $t$ is the number of months after 1 January in a given year. The population, $M$, of a second species of ant also varies with time. Its population is modelled by the equation

$$M = 3000\sin\left(\frac{\pi(t-3.5)}{5}\right) + 5500$$

where $t$ is again the number of months after 1 January in a given year.

Use your calculator to plot the graphs of both equations over a period of one year on the same axes, and hence:

**a** find the maximum and minimum populations of both species and the months in which those maximums and minimums occur

**b** find the month of the year during which the populations of both species are equal and find the population of each species at that time

**c** by formulating a third equation, find when the combined population of species $N$ and $M$ is at a maximum and find this maximum value

**d** by formulating a fourth equation, find when the difference between the two populations is at a maximum.

**8** Passengers on a ferris wheel access their seats from a platform 5 m above the ground. As each seat is filled, the ferris wheel moves around so that the next seat can be filled. Once all seats are filled, the ride begins and lasts for 6 minutes. The height, $h$ m, of Isobel's seat above the ground $t$ seconds after the ride has begun is given by $h = 15\sin(10t - 45)^\circ + 16.5$.

**a** Use a calculator to sketch the graph of $h$ against $t$ for the first 2 minutes of the ride.

**b** How far above the ground is Isobel's seat at the commencement of the ride?

**c** After how many seconds does Isobel's seat pass the access platform?

**d** How many times will her seat pass the access platform in the first 2 minutes?

**e** How many times will her seat pass the access platform during the entire ride?

Due to a malfunction, the ferris wheel stops abruptly 1 minute 40 seconds into the ride.

**f** How far above the ground is Isobel stranded?

**g** If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground is Hamish stranded?


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# 9 Trigonometric identities

## In this chapter



## Syllabus references

**Topics:** Compound angles; The reciprocal trigonometric functions, secant, cosecant and cotangent; Trigonometric identities

**Subtopics:** 2.1.3 – 2.1.8

In this chapter we build on our study of trigonometric functions from Chapter 8.

There are many interesting and useful relationships between the trigonometric functions. The most fundamental is the Pythagorean identity:

$$\sin^2 A + \cos^2 A = 1$$

Some of these identities were discovered a very long time ago. For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

They are of great importance in many areas of mathematics, including calculus.


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# 9A Reciprocal functions and the Pythagorean identity

In this section we introduce the reciprocals of the basic trigonometric functions. The graphs of these functions appear in section 9B. Here we use these functions in various forms of the Pythagorean identity.

## Reciprocal functions

The trigonometric functions sine, cosine and tangent can be used to form three other functions, called the reciprocal trigonometric functions.

### Secant, cosecant and cotangent

- $\sec\theta = \dfrac{1}{\cos\theta}$ (for $\cos\theta \neq 0$)
- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ (for $\sin\theta \neq 0$)
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (for $\sin\theta \neq 0$)

Note: For $\cos\theta \neq 0$ and $\sin\theta \neq 0$, we have

$$\cot\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}$$

### Example 1

Find the exact value of each of the following:

**a** $\sec\left(\dfrac{2\pi}{3}\right)$ **b** $\cot\left(\dfrac{5\pi}{4}\right)$ **c** $\operatorname{cosec}\left(\dfrac{7\pi}{4}\right)$

#### Solution

**a**
$$\begin{aligned}\sec\left(\frac{2\pi}{3}\right) &= \frac{1}{\cos\left(\frac{2\pi}{3}\right)} \\ &= \frac{1}{\cos\left(\pi - \frac{\pi}{3}\right)} \\ &= \frac{1}{-\cos\left(\frac{\pi}{3}\right)} \\ &= 1 \div \left(-\frac{1}{2}\right) \\ &= -2\end{aligned}$$

**b**
$$\begin{aligned}\cot\left(\frac{5\pi}{4}\right) &= \frac{\cos\left(\frac{5\pi}{4}\right)}{\sin\left(\frac{5\pi}{4}\right)} \\ &= \frac{\cos\left(\pi + \frac{\pi}{4}\right)}{\sin\left(\pi + \frac{\pi}{4}\right)} \\ &= \frac{-1}{\sqrt{2}} \div \left(\frac{-1}{\sqrt{2}}\right) \\ &= 1\end{aligned}$$

**c**
$$\begin{aligned}&\operatorname{cosec}\left(\frac{7\pi}{4}\right) \\ &= \frac{1}{\sin\left(2\pi - \frac{\pi}{4}\right)} \\ &= \frac{1}{-\sin\left(\frac{\pi}{4}\right)} \\ &= 1 \div \left(-\frac{1}{\sqrt{2}}\right) \\ &= -\sqrt{2}\end{aligned}$$

Note: In this example, we are using symmetry properties and exact values of trigonometric functions, which are revised in Section 8B.


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## Example 2

Find the values of $x$ between 0 and $2\pi$ for which:

**a** $\sec x = -2$

**b** $\cot x = -1$

### Solution

**a**
$$\sec x = -2$$
$$\frac{1}{\cos x} = -2$$
$$\cos x = \frac{-1}{2}$$
$$\therefore \quad x = \pi - \frac{\pi}{3} \quad \text{or} \quad x = \pi + \frac{\pi}{3}$$
$$\therefore \quad x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}$$

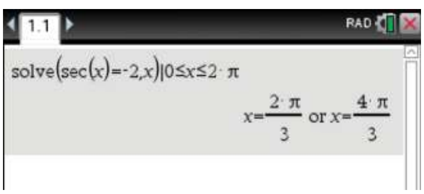
**b**
$$\cot x = -1$$
$$\tan x = -1$$
$$\therefore \quad x = \pi - \frac{\pi}{4} \quad \text{or} \quad x = 2\pi - \frac{\pi}{4}$$
$$\therefore \quad x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4}$$

### Using the TI-Nspire

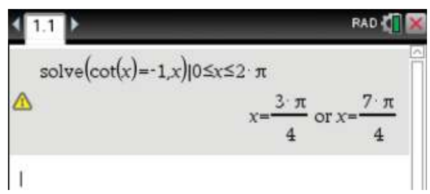
Check that your calculator is in radian mode. Use menu > **Algebra** > **Solve** as shown.

Note: Access sec and cot using trig. Access $\leq$ using ctrl =.

**a**




**b**

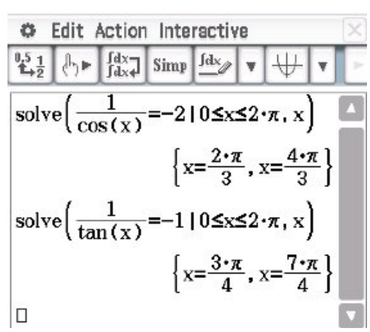



### Using the Casio ClassPad

The ClassPad does not recognise $\sec x$, $\operatorname{cosec} x$ and $\cot x$. These functions must be entered as reciprocals of $\cos x$, $\sin x$ and $\tan x$ respectively.

**a**
- Enter and highlight: $\dfrac{1}{\cos(x)} = -2 \,\Big|\, 0 \leq x \leq 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve**, ensure the variable is set to $x$ and tap OK.

**b**
- Enter and highlight: $\dfrac{1}{\tan(x)} = -1 \,\Big|\, 0 \leq x \leq 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve**, ensure the variable is set to $x$ and tap OK.




Note: The 'for' operator | is found in the Math3 keyboard and is used to specify a condition. In this case, the condition is the domain restriction.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## The Pythagorean identity

We introduced the Pythagorean identity in Section 8B.

**Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

We can now derive two other forms of this identity:

- Dividing both sides by $\cos^2\theta$ gives

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\therefore \quad 1 + \tan^2\theta = \sec^2\theta$$

- Dividing both sides by $\sin^2\theta$ gives

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\therefore \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

### Example 3

**a** If $\operatorname{cosec} x = \frac{7}{4}$, find $\cos x$.

**b** If $\sec x = -\frac{3}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find $\sin x$.

**Solution**

**a** Since $\operatorname{cosec} x = \frac{7}{4}$, we have $\sin x = \frac{4}{7}$.

Now $\cos^2 x + \sin^2 x = 1$

$$\cos^2 x + \frac{16}{49} = 1$$

$$\cos^2 x = \frac{33}{49}$$

$$\therefore \quad \cos x = \pm\frac{\sqrt{33}}{7}$$

**b** Since $\sec x = -\frac{3}{2}$, we have $\cos x = -\frac{2}{3}$.

Now $\cos^2 x + \sin^2 x = 1$

$$\frac{4}{9} + \sin^2 x = 1$$

$$\therefore \quad \sin x = \pm\frac{\sqrt{5}}{3}$$

But $\sin x$ is positive for $P(x)$ in the 2nd quadrant, and so $\sin x = \frac{\sqrt{5}}{3}$.

### Example 4

Prove the identity $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2\operatorname{cosec}^2\theta$.

**Solution**

$$\begin{aligned}
\text{LHS} &= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \\
&= \frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta} \\
&= \frac{2}{1-\cos^2\theta} \\
&= \frac{2}{\sin^2\theta} \\
&= 2\operatorname{cosec}^2\theta \\
&= \text{RHS}
\end{aligned}$$


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## Summary 9A

- **Reciprocal functions**

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{for } \cos\theta \neq 0)$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

## Exercise 9A

Example 1

**1** Find the exact value of each of the following:

**a** $\cot\left(\frac{3\pi}{4}\right)$ **b** $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$ **d** $\operatorname{cosec}\left(\frac{\pi}{2}\right)$

**e** $\sec\left(\frac{4\pi}{3}\right)$ **f** $\operatorname{cosec}\left(\frac{13\pi}{6}\right)$ **g** $\cot\left(\frac{7\pi}{3}\right)$ **h** $\sec\left(\frac{5\pi}{3}\right)$

**2** Without using a calculator, write down the exact value of each of the following:

**a** $\cot 135°$ **b** $\sec 150°$ **c** $\operatorname{cosec} 90°$ **d** $\cot 240°$ **e** $\operatorname{cosec} 225°$

**f** $\sec 330°$ **g** $\cot 315°$ **h** $\operatorname{cosec} 300°$ **i** $\cot 420°$

Example 2

**3** Find the values of $x$ between 0 and $2\pi$ for which:

**a** $\operatorname{cosec} x = 2$ **b** $\cot x = \sqrt{3}$ **c** $\sec x + \sqrt{2} = 0$ **d** $\operatorname{cosec} x = \sec x$

Example 3

**4** If $\sec\theta = \frac{-17}{8}$ and $\frac{\pi}{2} < \theta < \pi$, find:

**a** $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$

**5** If $\tan\theta = \frac{-7}{24}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\cos\theta$ and $\sin\theta$.

**6** Find the value of $\sec\theta$ if $\tan\theta = 0.4$ and $\theta$ is not in the 1st quadrant.

**7** If $\tan\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, evaluate $\frac{\sin\theta - 2\cos\theta}{\cot\theta - \sin\theta}$.

**8** If $\cos\theta = \frac{2}{3}$ and $\theta$ is in the 4th quadrant, express $\frac{\tan\theta - 3\sin\theta}{\cos\theta - 2\cot\theta}$ in simplest surd form.

Example 4

**9** Prove each of the following identities for suitable values of $\theta$ and $\varphi$:

**a** $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$ **b** $\cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta = 1$

**c** $\frac{\tan\theta}{\tan\varphi} = \frac{\tan\theta + \cot\varphi}{\cot\theta + \tan\varphi}$ **d** $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

**e** $\frac{1 + \cot^2\theta}{\cot\theta\operatorname{cosec}\theta} = \sec\theta$ **f** $\sec\theta + \tan\theta = \frac{\cos\theta}{1 - \sin\theta}$


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# 9B Graphing the reciprocal trigonometric functions

We now consider the graphs of the **reciprocal trigonometric functions**. In this section we will apply basic transformations to the graphs of these functions.

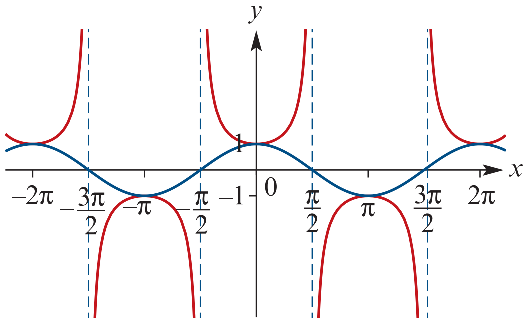
## The secant function

The secant function is defined by

$$\sec x = \frac{1}{\cos x}$$

provided $\cos x \neq 0$.

The graphs of $y = \cos x$ and $y = \sec x$ are shown here on the same axes.




The significant features of the two graphs are listed in the following table.

| Function $y = \cos x$ | Reciprocal function $y = \sec x$ |
|---|---|
| $x$-axis intercepts at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$ | vertical asymptotes at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$ |
| domain $= \mathbb{R}$ | domain $= \{ x \in \mathbb{R} : x \neq (2n+1)\frac{\pi}{2} \text{ for } n \in \mathbb{Z} \}$ |
| local maximums at $(2n\pi, 1),\ n \in \mathbb{Z}$ | local minimums at $(2n\pi, 1),\ n \in \mathbb{Z}$ |
| local minimums at $((2n+1)\pi, -1),\ n \in \mathbb{Z}$ | local maximums at $((2n+1)\pi, -1),\ n \in \mathbb{Z}$ |
| range $= [-1, 1]$ | range $= (-\infty, -1] \cup [1, \infty)$ |

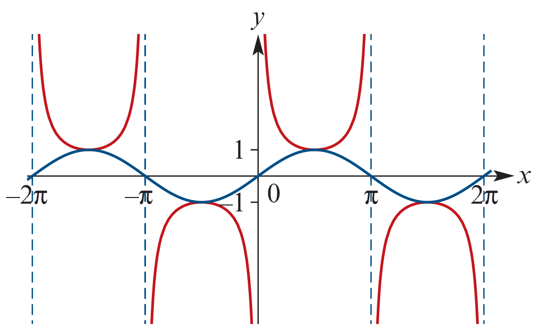
## The cosecant function

The cosecant function is defined by

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

provided $\sin x \neq 0$.

The graphs of $y = \sin x$ and $y = \operatorname{cosec} x$ are shown here on the same axes.




| Function $y = \sin x$ | Reciprocal function $y = \operatorname{cosec} x$ |
|---|---|
| $x$-axis intercepts at $x = n\pi,\ n \in \mathbb{Z}$ | vertical asymptotes at $x = n\pi,\ n \in \mathbb{Z}$ |
| domain $= \mathbb{R}$ | domain $= \{ x \in \mathbb{R} : x \neq n\pi \text{ for } n \in \mathbb{Z} \}$ |
| local maximums at $\left(2n\pi + \frac{\pi}{2}, 1\right),\ n \in \mathbb{Z}$ | local minimums at $\left(2n\pi + \frac{\pi}{2}, 1\right),\ n \in \mathbb{Z}$ |
| local minimums at $\left(2n\pi - \frac{\pi}{2}, -1\right),\ n \in \mathbb{Z}$ | local maximums at $\left(2n\pi - \frac{\pi}{2}, -1\right),\ n \in \mathbb{Z}$ |
| range $= [-1, 1]$ | range $= (-\infty, -1] \cup [1, \infty)$ |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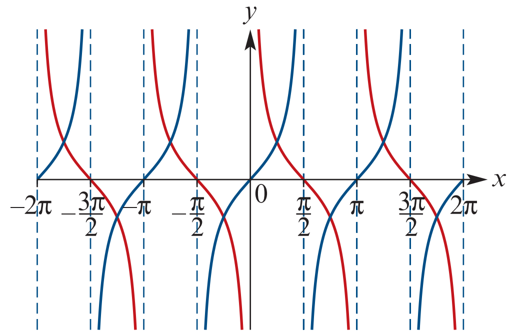
## The cotangent function

The cotangent function is defined by

$$\cot x = \frac{\cos x}{\sin x}$$

provided $\sin x \neq 0$.

This diagram shows the graph of $y = \tan x$ in blue and the graph of $y = \cot x$ in red.




| Function $y = \tan x$ | Function $y = \cot x$ |
|---|---|
| $x$-axis intercepts at $x = n\pi,\ n \in \mathbb{Z}$ | vertical asymptotes at $x = n\pi,\ n \in \mathbb{Z}$ |
| vertical asymptotes at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$ | $x$-axis intercepts at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$ |
| domain $= \{ x \in \mathbb{R} : x \neq (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z} \}$ | domain $= \{ x \in \mathbb{R} : x \neq n\pi$ for $n \in \mathbb{Z} \}$ |
| range $= \mathbb{R}$ | range $= \mathbb{R}$ |

Note the similarity between the graphs of $y = \cot x$ and $y = \tan x$. Using the complementary relationship between sine and cosine, we have

$$\cot x = \frac{\cos x}{\sin x} = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \tan\left(\frac{\pi}{2} - x\right) = \tan\left(-\left(x - \frac{\pi}{2}\right)\right) = -\tan\left(x - \frac{\pi}{2}\right)$$

Therefore the graph of $y = \cot x$ can be obtained from the graph of $y = \tan x$ by a reflection in the $x$-axis followed by a translation of $\frac{\pi}{2}$ units in the positive direction of the $x$-axis.

## Transformations of the reciprocal trigonometric functions

We now look at the effect of the basic transformations (dilations, reflections and translations) on the reciprocal trigonometric functions.

### Example 5

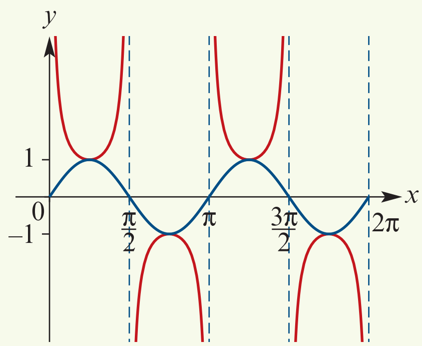
Sketch the graph of each of the following over the interval $[0, 2\pi]$:

**a** $y = \operatorname{cosec}(2x)$ **b** $y = 2\sec x$ **c** $y = -\cot x$

#### Solution

**a** The graph of $y = \operatorname{cosec}(2x)$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the $y$-axis.

It is helpful to draw the graph of $y = \sin(2x)$ on the same axes.





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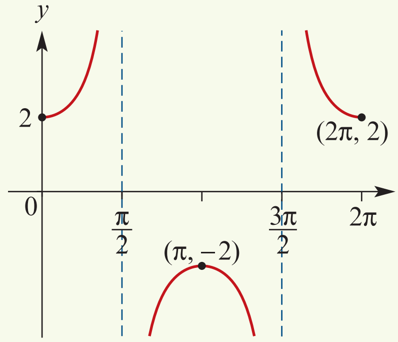
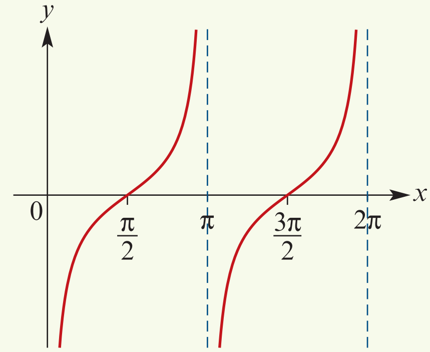
**b** The graph of $y = 2\sec x$ can be obtained from the graph of $y = \sec x$ by a dilation of factor 2 from the $x$-axis.




**c** The graph of $y = -\cot x$ can be obtained from the graph of $y = \cot x$ by a reflection in the $x$-axis.




## Example 6

Sketch the graph of each of the following over the interval $[0, 2\pi]$:

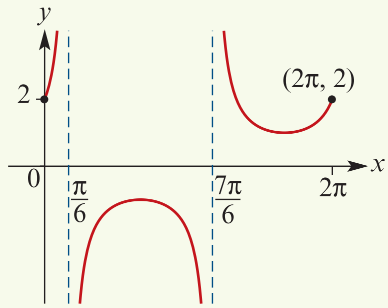
**a** $y = \sec\left(x + \frac{\pi}{3}\right)$ **b** $y = \operatorname{cosec}(x) - 2$ **c** $y = \cot\left(x - \frac{\pi}{4}\right)$

### Solution

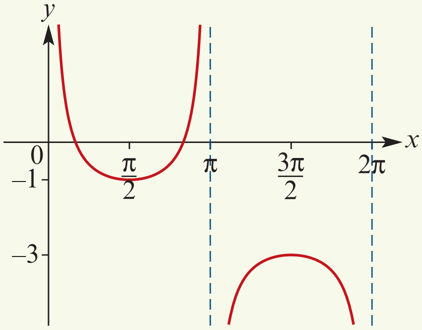
**a** The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ can be obtained from the graph of $y = \sec x$ by a translation of $\frac{\pi}{3}$ units in the negative direction of the $x$-axis.

The $y$-axis intercept is $\sec\left(\frac{\pi}{3}\right) = 2$.

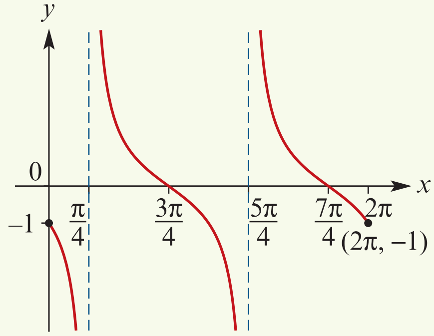
The asymptotes are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.




**b** The graph of $y = \operatorname{cosec}(x) - 2$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a translation of 2 units in the negative direction of the $y$-axis.




**c** The graph of $y = \cot\left(x - \frac{\pi}{4}\right)$ can be obtained from the graph of $y = \cot x$ by a translation of $\frac{\pi}{4}$ units in the positive direction of the $x$-axis.





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## Example 7

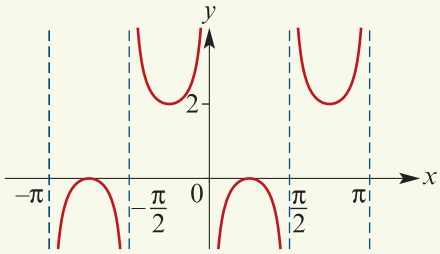
Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -\sec\left(2x - \frac{\pi}{2}\right) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

### Solution

It helps to write the equation of the transformed graph as $y = -\sec\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

An appropriate sequence is:

1. reflection in the $x$-axis
2. dilation of factor $\frac{1}{2}$ parallel to the $x$-axis
3. translation of $\frac{\pi}{4}$ units to the right and 1 unit up.




## Summary 9B

### Reciprocal trigonometric functions

- $\sec x = \dfrac{1}{\cos x}$ (provided $\cos x \neq 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (provided $\sin x \neq 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$ (provided $\sin x \neq 0$)

## Exercise 9B

Example 5

**1** Sketch the graph of each of the following over the interval $[0, 2\pi]$:

**a** $y = \sec(2x)$ **b** $y = \cot(2x)$ **c** $y = 3\sec x$
**d** $y = 2\operatorname{cosec} x$ **e** $y = -\operatorname{cosec} x$ **f** $y = -2\sec x$

Example 6

**2** Sketch the graph of each of the following over the interval $[0, 2\pi]$:

**a** $y = \sec\left(x - \frac{\pi}{2}\right)$ **b** $y = \cot\left(x + \frac{\pi}{4}\right)$ **c** $y = -\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$
**d** $y = 1 + \sec x$ **e** $y = 2 - \operatorname{cosec} x$ **f** $y = 1 + \cot\left(x + \frac{\pi}{4}\right)$

Example 7

**3** Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -2\sec\left(x - \frac{\pi}{2}\right)$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

**4** Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = \operatorname{cosec}(-2x) + 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

**5** Describe a sequence of transformations that will take the graph of $y = \cot x$ to the graph of $y = -\cot\left(2x - \frac{\pi}{2}\right) - 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

**6** On the one set of axes, sketch the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ over the interval $[0, 2\pi]$. Find and label the points of intersection.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# 9C Addition formulas and double angle formulas

## Addition formulas

### Addition formulas for cosine

**1** $\cos(u + v) = \cos u \cos v - \sin u \sin v$

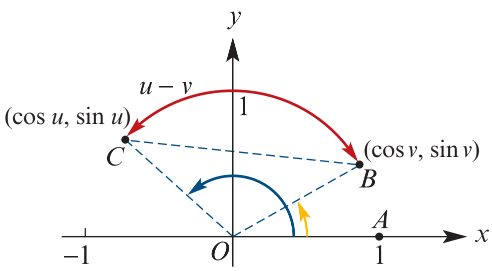
**2** $\cos(u - v) = \cos u \cos v + \sin u \sin v$

**Proof** Consider a unit circle as shown:

arc length $AB = v$ units

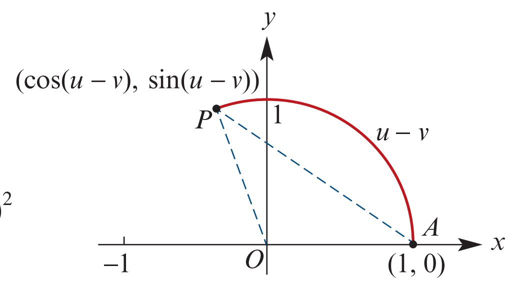
arc length $AC = u$ units

arc length $BC = u - v$ units




Rotate $\Delta OCB$ so that $B$ is coincident with $A$. Then $C$ is moved to

$P(\cos(u - v), \sin(u - v))$




Since the triangles $CBO$ and $PAO$ are congruent, we have $CB = PA$.

Using the coordinate distance formula:

$$\begin{aligned} CB^2 &= (\cos u - \cos v)^2 + (\sin u - \sin v)^2 \\ &= 2 - 2(\cos u \cos v + \sin u \sin v) \end{aligned}$$

$$\begin{aligned} PA^2 &= (\cos(u - v) - 1)^2 + (\sin(u - v) - 0)^2 \\ &= 2 - 2\cos(u - v) \end{aligned}$$

Since $CB = PA$, this gives

$$2 - 2\cos(u - v) = 2 - 2(\cos u \cos v + \sin u \sin v)$$

$$\therefore \quad \cos(u - v) = \cos u \cos v + \sin u \sin v$$

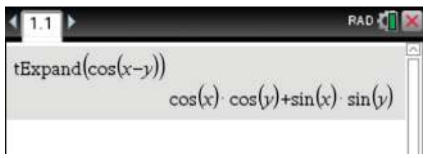
We can now obtain the first formula from the second by replacing $v$ with $-v$:

$$\begin{aligned} \cos(u + v) &= \cos(u - (-v)) \\ &= \cos u \cos(-v) + \sin u \sin(-v) \\ &= \cos u \cos v - \sin u \sin v \end{aligned}$$

Note: Here we used $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

### Using the TI-Nspire

Access the **tExpand( )** command from menu > **Algebra** > **Trigonometry** > **Expand** and complete as shown.





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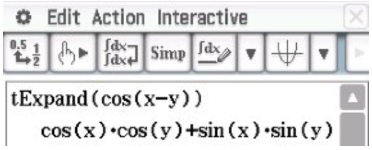
## Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight $\cos(x - y)$.
- Go to **Interactive** > **Transformation** > **tExpand** and tap OK.




## Example 8

Evaluate $\cos 75°$.

### Solution

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

### Addition formulas for sine

**1** $\sin(u + v) = \sin u \cos v + \cos u \sin v$

**2** $\sin(u - v) = \sin u \cos v - \cos u \sin v$

**Proof** We use the symmetry properties $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$:

$$\begin{aligned}\sin(u + v) &= \cos\left(\frac{\pi}{2} - (u + v)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - u\right) - v\right)\\ &= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v\\ &= \sin u \cos v + \cos u \sin v\end{aligned}$$

We can now obtain the second formula from the first by replacing $v$ with $-v$:

$$\begin{aligned}\sin(u - v) &= \sin u \cos(-v) + \cos u \sin(-v)\\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$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## Example 9

Evaluate:

**a** $\sin 75^\circ$

**b** $\sin 15^\circ$

### Solution

**a**
$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ)\\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

**b**
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

### Addition formulas for tangent

**1** $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

**2** $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

**Proof** To obtain the first formula, we write

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v}$$

Now divide the numerator and denominator by $\cos u \cos v$. The second formula can be obtained from the first by using $\tan(-\theta) = -\tan\theta$.

## Example 10

If $u$ and $v$ are acute angles such that $\tan u = 4$ and $\tan v = \frac{3}{5}$, show that $u - v = \frac{\pi}{4}$.

### Solution

$$\begin{aligned}\tan(u-v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v}\\ &= \frac{4 - \frac{3}{5}}{1 + 4\times\frac{3}{5}}\\ &= \frac{20-3}{5+4\times 3}\\ &= 1\end{aligned}$$

$$\therefore \quad u - v = \frac{\pi}{4}$$

Note: Since $u$ and $v$ are acute angles with $u > v$, the angle $u - v$ is in the interval $\left(0, \frac{\pi}{2}\right)$.


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## Double angle formulas

Using the addition formulas, we can easily derive useful expressions for $\sin(2u)$, $\cos(2u)$ and $\tan(2u)$.

### Double angle formulas for cosine

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \qquad (\text{since } \sin^2 u = 1 - \cos^2 u) \\ &= 1 - 2\sin^2 u \qquad (\text{since } \cos^2 u = 1 - \sin^2 u)\end{aligned}$$

**Proof** $\cos(u + u) = \cos u \cos u - \sin u \sin u$
$= \cos^2 u - \sin^2 u$

### Double angle formula for sine

$$\sin(2u) = 2\sin u \cos u$$

**Proof** $\sin(u + u) = \sin u \cos u + \cos u \sin u$
$= 2\sin u \cos u$

### Double angle formula for tangent

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

**Proof** $\tan(u + u) = \dfrac{\tan u + \tan u}{1 - \tan u \tan u}$
$= \dfrac{2\tan u}{1 - \tan^2 u}$

### Example 11

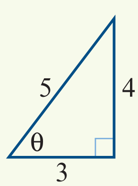
If $\tan\theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate:

**a** $\sin(2\theta)$ **b** $\tan(2\theta)$

**Solution**

**a** $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

$$\begin{aligned}\therefore\ \sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2 \times \frac{4}{5} \times \frac{3}{5} \\ &= \frac{24}{25}\end{aligned}$$




**b**
$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2 \times \frac{4}{3}}{1 - \frac{16}{9}} \\ &= \frac{2 \times 4 \times 3}{9 - 16} \\ &= -\frac{24}{7}\end{aligned}$$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## Example 12

Prove each of the following identities:

**a** $\dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tan(2\theta)$

**b** $\dfrac{\sin\theta}{\sin\varphi} + \dfrac{\cos\theta}{\cos\varphi} = \dfrac{2\sin(\theta+\varphi)}{\sin(2\varphi)}$

**c** $\dfrac{1}{\cos\theta + \sin\theta} + \dfrac{1}{\cos\theta - \sin\theta} = \tan(2\theta)\operatorname{cosec}\theta$

### Solution

**a**
$$\begin{aligned}\text{LHS} &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)}\\ &= \tan(2\theta)\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$.

**b**
$$\begin{aligned}\text{LHS} &= \frac{\sin\theta}{\sin\varphi} + \frac{\cos\theta}{\cos\varphi}\\ &= \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\sin\varphi\cos\varphi}\\ &= \frac{\sin(\theta+\varphi)}{\frac{1}{2}\sin(2\varphi)}\\ &= \frac{2\sin(\theta+\varphi)}{\sin(2\varphi)}\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\sin(2\varphi) \neq 0$.

**c**
$$\begin{aligned}\text{LHS} &= \frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta}\\ &= \frac{\cos\theta - \sin\theta + \cos\theta + \sin\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{2\cos\theta}{\cos(2\theta)}\\ &= \frac{2\cos\theta}{\cos(2\theta)}\cdot\frac{\sin\theta}{\sin\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)\sin\theta}\\ &= \frac{\tan(2\theta)}{\sin\theta}\\ &= \tan(2\theta)\operatorname{cosec}\theta\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$ and $\sin\theta \neq 0$.


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Sometimes the easiest way to prove that two expressions are equal is to simplify each of them separately. This is demonstrated in the following example.

### Example 13

Prove that $(\sec A - \cos A)(\operatorname{cosec} A - \sin A) = \dfrac{1}{\tan A + \cot A}$.

**Solution**

$$\begin{aligned}\text{LHS} &= (\sec A - \cos A)(\operatorname{cosec} A - \sin A)\\ &= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{1}{\sin A} - \sin A\right)\\ &= \frac{1-\cos^2 A}{\cos A} \times \frac{1-\sin^2 A}{\sin A}\\ &= \frac{\sin^2 A\cos^2 A}{\cos A \sin A}\\ &= \cos A \sin A\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{\tan A + \cot A}\\ &= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}\\ &= \frac{\cos A \sin A}{\sin^2 A + \cos^2 A}\\ &= \cos A \sin A\end{aligned}$$

We have shown that LHS = RHS.

## Summary 9C

- **Addition formulas**

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$
$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$
$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

- **Double angle formulas**

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$
$$\sin(2u) = 2\sin u \cos u$$
$$\tan(2u) = \frac{2\tan u}{1-\tan^2 u}$$

Skill-sheet

## Exercise 9C

Example 8

**1** By using the appropriate addition formulas, find exact values for the following:

**a** $\cos 15°$ **b** $\cos 105°$

Example 9

**2** By using the appropriate addition formulas, find exact values for the following:

**a** $\sin 165°$ **b** $\tan 75°$

**3** Find the exact value of:

**a** $\cos\left(\frac{5\pi}{12}\right)$ **b** $\sin\left(\frac{11\pi}{12}\right)$ **c** $\tan\left(\frac{-\pi}{12}\right)$


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Example 10

**4** If $\sin u = \frac{12}{13}$ and $\sin v = \frac{3}{5}$, evaluate $\sin(u + v)$. (Note: There is more than one answer.)

**5** Simplify the following:

**a** $\sin\left(\theta + \frac{\pi}{6}\right)$ **b** $\cos\left(\varphi - \frac{\pi}{4}\right)$ **c** $\tan\left(\theta + \frac{\pi}{3}\right)$ **d** $\sin\left(\theta - \frac{\pi}{4}\right)$

**6** Simplify:

**a** $\cos(u - v)\sin v + \sin(u - v)\cos v$ **b** $\sin(u + v)\sin v + \cos(u + v)\cos v$

Example 11

**7** If $\sin\theta = \frac{-3}{5}$, with θ in the 3rd quadrant, and $\cos\varphi = \frac{-5}{13}$, with φ in the 2nd quadrant, evaluate each of the following without using a calculator:

**a** $\cos(2\varphi)$ **b** $\sin(2\theta)$ **c** $\tan(2\theta)$ **d** $\sec(2\varphi)$
**e** $\sin(\theta + \varphi)$ **f** $\cos(\theta - \varphi)$ **g** $\operatorname{cosec}(\theta + \varphi)$ **h** $\cot(2\theta)$

**8** For acute angles $u$ and $v$ such that $\tan u = \frac{4}{3}$ and $\tan v = \frac{5}{12}$, evaluate:

**a** $\tan(u + v)$ **b** $\tan(2u)$ **c** $\cos(u - v)$ **d** $\sin(2u)$

**9** If $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{24}{25}$, with $\frac{\pi}{2} < \beta < \alpha < \pi$, evaluate:

**a** $\cos(2\alpha)$ **b** $\sin(\alpha - \beta)$ **c** $\tan(\alpha + \beta)$ **d** $\sin(2\beta)$

**10** If $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$, evaluate:

**a** $\sin(2\theta)$ **b** $\cos(2\theta)$

**11** Simplify each of the following expressions:

**a** $(\sin\theta - \cos\theta)^2$ **b** $\cos^4\theta - \sin^4\theta$

Example 12
Example 13

**12** Prove the following identities:

**a** $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta - \cos\theta$

**b** $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

**c** $\tan\left(\theta + \frac{\pi}{4}\right)\tan\left(\theta - \frac{\pi}{4}\right) = -1$

**d** $\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$

**e** $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$

**f** $\frac{\sin(u + v)}{\cos u \cos v} = \tan v + \tan u$

**g** $\frac{\tan u + \tan v}{\tan u - \tan v} = \frac{\sin(u + v)}{\sin(u - v)}$

**h** $\cos(2\theta) + 2\sin^2\theta = 1$

**i** $\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

**j** $\frac{1 - \sin(2\theta)}{\sin\theta - \cos\theta} = \sin\theta - \cos\theta$


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# 9D Simplifying $a\cos x + b\sin x$

In this section, we see how to rewrite the rule of a function $f(x) = a\cos x + b\sin x$ in terms of a single trigonometric function.

$$a\cos x + b\sin x = r\cos(x - \alpha) \quad \text{where } r = \sqrt{a^2 + b^2},\ \cos\alpha = \frac{a}{r} \text{ and } \sin\alpha = \frac{b}{r}$$

**Proof** Let $r = \sqrt{a^2 + b^2}$. Consider the point $P\left(\frac{a}{r}, \frac{b}{r}\right)$ and its distance from the origin $O$:

$$OP^2 = \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

The point $P$ is on the unit circle, and so $\frac{a}{r} = \cos\alpha$ and $\frac{b}{r} = \sin\alpha$, for some angle $\alpha$.

We can now write

$$\begin{aligned} a\cos x + b\sin x &= r\left(\frac{a}{r}\cos x + \frac{b}{r}\sin x\right) \\ &= r(\cos\alpha\,\cos x + \sin\alpha\,\sin x) \\ &= r\cos(x - \alpha) \end{aligned}$$

Similarly, it may be shown that

$$a\cos x + b\sin x = r\sin(x + \beta) \quad \text{where } r = \sqrt{a^2 + b^2},\ \sin\beta = \frac{a}{r},\ \cos\beta = \frac{b}{r}$$

## Example 14

Express $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x - \alpha)$. Hence find the range of the function $f$ with rule $f(x) = \cos x - \sqrt{3}\sin x$ and find the maximum and minimum values of $f$.

### Solution

Here $a = 1$ and $b = -\sqrt{3}$. Therefore

$$r = \sqrt{1 + 3} = 2, \quad \cos\alpha = \frac{a}{r} = \frac{1}{2} \quad \text{and} \quad \sin\alpha = \frac{b}{r} = \frac{-\sqrt{3}}{2}$$

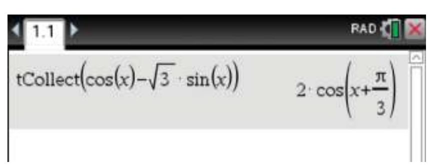
We see that $\alpha = -\frac{\pi}{3}$ and so

$$f(x) = \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

Thus the range of $f$ is $[-2, 2]$, the maximum value is 2 and the minimum value is $-2$.

### Using the TI-Nspire

Access the **tCollect( )** command from menu > **Algebra** > **Trigonometry** > **Collect** and complete as shown.





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## Example 15

Solve $\cos x - \sqrt{3}\sin x = 1$ for $x \in [0, 2\pi]$.

### Solution

From Example 14, we have

$$\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

$$\therefore \quad 2\cos\left(x + \frac{\pi}{3}\right) = 1$$

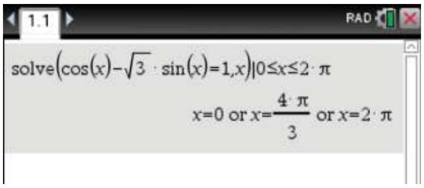
$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3} \text{ or } \frac{7\pi}{3}$$

$$x = 0, \frac{4\pi}{3} \text{ or } 2\pi$$

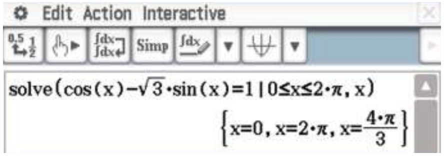
### Using the TI-Nspire

Use **solve( )** from the **Algebra** menu as shown.




### Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, enter and highlight the equation $\cos(x) - \sqrt{3}\sin(x) = 1 \mid 0 \le x \le 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve** and tap OK.




## Example 16

Express $\sqrt{3}\sin(2x) - \cos(2x)$ in the form $r\sin(2x + \alpha)$.

### Solution

A slightly different technique is used. Assume that

$$\sqrt{3}\sin(2x) - \cos(2x) = r\sin(2x + \alpha)$$

$$= r\left(\sin(2x)\cos\alpha + \cos(2x)\sin\alpha\right)$$

This is to hold for all $x$.

For $x = \frac{\pi}{4}$: $\quad \sqrt{3} = r\cos\alpha \qquad (1)$

For $x = 0$: $\quad -1 = r\sin\alpha \qquad (2)$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Squaring and adding (1) and (2) gives

$$r^2\cos^2\alpha + r^2\sin^2\alpha = 4$$
$$r^2 = 4$$
$$\therefore \quad r = \pm 2$$

We take the positive solution. Substituting in (1) and (2) gives

$$\frac{\sqrt{3}}{2} = \cos\alpha \quad \text{and} \quad -\frac{1}{2} = \sin\alpha$$

Thus $\alpha = -\frac{\pi}{6}$ and hence

$$\sqrt{3}\sin(2x) - \cos(2x) = 2\sin\left(2x - \frac{\pi}{6}\right)$$

Check: Expand the right-hand side of the equation using an addition formula.

## Summary 9D

- $a\cos x + b\sin x = r\cos(x - \alpha)$ where $r = \sqrt{a^2 + b^2}$, $\cos\alpha = \frac{a}{r}$, $\sin\alpha = \frac{b}{r}$
- $a\cos x + b\sin x = r\sin(x + \beta)$ where $r = \sqrt{a^2 + b^2}$, $\sin\beta = \frac{a}{r}$, $\cos\beta = \frac{b}{r}$

Skill-sheet

## Exercise 9D

Example 14

**1** Find the maximum and minimum values of the following:

**a** $4\cos x + 3\sin x$
**b** $\sqrt{3}\cos x + \sin x$
**c** $\cos x - \sin x$
**d** $\cos x + \sin x$
**e** $3\cos x + \sqrt{3}\sin x$
**f** $\sin x - \sqrt{3}\cos x$
**g** $\cos x - \sqrt{3}\sin x + 2$
**h** $5 + 3\sin x - 2\cos x$

Example 15

**2** Solve each of the following for $x \in [0, 2\pi]$ or for $\theta \in [0, 360]$:

**a** $\sin x - \cos x = 1$
**b** $\sqrt{3}\sin x + \cos x = 1$
**c** $\sin x - \sqrt{3}\cos x = -1$
**d** $3\cos x - \sqrt{3}\sin x = 3$
**e** $4\sin\theta° + 3\cos\theta° = 5$
**f** $2\sqrt{2}\sin\theta° - 2\cos\theta° = 3$

**3** Write $\sqrt{3}\cos(2x) - \sin(2x)$ in the form $r\cos(2x + \alpha)$.

Example 16

**4** Write $\cos(3x) - \sin(3x)$ in the form $r\sin(3x - \alpha)$.

**5** Sketch the graph of each of the following, showing one cycle:

**a** $f(x) = \sin x - \cos x$
**b** $f(x) = \sqrt{3}\sin x + \cos x$
**c** $f(x) = \sin x + \cos x$
**d** $f(x) = \sin x - \sqrt{3}\cos x$


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# 9E Sums and products of sines and cosines

In Section 9C, we derived the addition formulas for sine and cosine. We use them in this section to obtain new identities which allow us to rewrite products of sines and cosines as sums or differences, and vice versa.

## Expressing products as sums or differences

### Product-to-sum identities

$$2\cos A\cos B = \cos(A - B) + \cos(A + B)$$
$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$
$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$

**Proof** We use the addition formulas for sine and cosine:

$$\cos(A + B) = \cos A\cos B - \sin A\sin B \qquad (1)$$
$$\cos(A - B) = \cos A\cos B + \sin A\sin B \qquad (2)$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \qquad (3)$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B \qquad (4)$$

The first product-to-sum identity is obtained by adding (2) and (1), the second identity is obtained by subtracting (1) from (2), and the third by adding (3) and (4).

### Example 17

Express each of the following products as sums or differences:

**a** $2\sin(3\theta)\cos(\theta)$

**b** $2\sin 50^\circ \cos 60^\circ$

**c** $2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right)$

#### Solution

**a** Use the third product-to-sum identity:

$$\begin{aligned} 2\sin(3\theta)\cos(\theta) &= \sin(3\theta + \theta) + \sin(3\theta - \theta) \\ &= \sin(4\theta) + \sin(2\theta) \end{aligned}$$

**b** Use the third product-to-sum identity:

$$\begin{aligned} 2\sin 50^\circ \cos 60^\circ &= \sin 110^\circ + \sin(-10)^\circ \\ &= \sin 110^\circ - \sin 10^\circ \end{aligned}$$

**c** Use the first product-to-sum identity:

$$\begin{aligned} 2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2}\right) + \cos(2\theta) \\ &= \cos(2\theta) \end{aligned}$$


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## Expressing sums and differences as products

**Sum-to-product identities**

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

**Proof** Using the first product-to-sum identity, we have

$$\begin{aligned} 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) &= \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) \\ &= \cos B + \cos A \\ &= \cos A + \cos B \end{aligned}$$

The other three sum-to-product identities can be obtained similarly.

### Example 18

Express each of the following as products:

**a** $\sin 36° + \sin 10°$

**b** $\cos 36° + \cos 10°$

**c** $\sin 36° - \sin 10°$

**d** $\cos 36° - \cos 10°$

**Solution**

**a** $\sin 36° + \sin 10° = 2\sin 23° \cos 13°$

**b** $\cos 36° + \cos 10° = 2\cos 23° \cos 13°$

**c** $\sin 36° - \sin 10° = 2\cos 23° \sin 13°$

**d** $\cos 36° - \cos 10° = -2\sin 23° \sin 13°$

### Example 19

Prove that $\dfrac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} = \tan(2\theta)$.

**Solution**

$$\begin{aligned} \text{LHS} &= \frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} \\ &= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \frac{2\sin(2\theta)\sin(\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \tan(2\theta) \\ &= \text{RHS} \end{aligned}$$


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## Example 20

Solve the equation $\sin(3x) + \sin(11x) = 0$ for $x \in [0, \pi]$.

### Solution

$$\begin{aligned}
& \sin(3x) + \sin(11x) = 0 \\
\Leftrightarrow \quad & 2\sin(7x)\cos(4x) = 0 \\
\Leftrightarrow \quad & \sin(7x) = 0 \quad \text{or} \quad \cos(4x) = 0 \\
\Leftrightarrow \quad & 7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi \quad \text{or} \quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\
\Leftrightarrow \quad & x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8} \text{ or } \frac{7\pi}{8}
\end{aligned}$$

## Summary 9E

- Product-to-sum identities

$$\begin{aligned}
2\cos A\cos B &= \cos(A - B) + \cos(A + B) \\
2\sin A\sin B &= \cos(A - B) - \cos(A + B) \\
2\sin A\cos B &= \sin(A + B) + \sin(A - B)
\end{aligned}$$

- Sum-to-product identities

$$\begin{aligned}
\cos A + \cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\cos A - \cos B &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \\
\sin A + \sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\sin A - \sin B &= 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)
\end{aligned}$$

## Exercise 9E

Example 17

**1** Express each of the following products as sums or differences:

**a** $2\sin(3\pi t)\cos(2\pi t)$

**b** $\sin 20° \cos 30°$

**c** $2\cos\left(\frac{\pi x}{4}\right)\sin\left(\frac{3\pi x}{4}\right)$

**d** $2\sin\left(\frac{A+B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)$

**2** Express $2\sin(3\theta)\sin(2\theta)$ as a difference of cosines.

**3** Use a product-to-sum identity to derive the expression for $2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$ as a difference of sines.


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**4** Show that $\sin 75^\circ \sin 15^\circ = \frac{1}{4}$.

Example 18

**5** Express each of the following as products:

**a** $\sin 56^\circ + \sin 22^\circ$ **b** $\cos 56^\circ + \cos 22^\circ$

**c** $\sin 56^\circ - \sin 22^\circ$ **d** $\cos 56^\circ - \cos 22^\circ$

**6** Express each of the following as products:

**a** $\sin(6A) + \sin(2A)$ **b** $\cos(x) + \cos(2x)$

**c** $\sin(4x) - \sin(3x)$ **d** $\cos(3A) - \cos(A)$

Example 19

**7** Show that $\sin(A) + 2\sin(3A) + \sin(5A) = 4\cos^2(A)\sin(3A)$.

**8** For any three angles α, β and γ, show that

$$\sin(\alpha + \beta)\sin(\alpha - \beta) + \sin(\beta + \gamma)\sin(\beta - \gamma) + \sin(\gamma + \alpha)\sin(\gamma - \alpha) = 0$$

**9** Show that $\cos 70^\circ + \sin 40^\circ = \cos 10^\circ$.

**10** Show that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

Example 20

**11** Solve each of the following equations for $x \in [-\pi, \pi]$:

**a** $\cos(5x) + \cos(x) = 0$ **b** $\cos(5x) - \cos(x) = 0$

**c** $\sin(5x) + \sin(x) = 0$ **d** $\sin(5x) - \sin(x) = 0$

**12** Solve each of the following equations for $\theta \in [0, \pi]$:

**a** $\cos(2\theta) - \sin(\theta) = 0$ **b** $\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$

**c** $\sin(7\theta) - \sin(\theta) = \sin(3\theta)$ **d** $\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$

**13** Prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A + B}{2}\right)$.

**14** Prove the identity:

$$4\sin(A + B)\sin(B + C)\sin(C + A) = \sin(2A) + \sin(2B) + \sin(2C) - \sin(2A + 2B + 2C)$$

**15** Prove that $\dfrac{\cos(2A) - \cos(2B)}{\sin(2A - 2B)} = -\dfrac{\sin(A + B)}{\cos(A - B)}$.

**16** Prove each of the following identities:

**a** $\dfrac{\sin(A) + \sin(3A) + \sin(5A)}{\cos(A) + \cos(3A) + \cos(5A)} = \tan(3A)$

**b** $\cos^2(A) + \cos^2(B) - 1 = \cos(A + B)\cos(A - B)$

**c** $\cos^2(A - B) - \cos^2(A + B) = \sin(2A)\sin(2B)$

**d** $\cos^2(A - B) - \sin^2(A + B) = \cos(2A)\cos(2B)$

**17** Find the sum $\sin(x) + \sin(3x) + \sin(5x) + \cdots + \sin(99x)$.
Hint: First multiply this sum by $2\sin(x)$.


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## Chapter summary

Assign-ment

Nrich

- **Reciprocal trigonometric functions**

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \quad (\text{if } \cos\theta \neq 0)$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

- **Addition formulas**

$$\cos(u+v) = \cos u\cos v - \sin u\sin v$$
$$\cos(u-v) = \cos u\cos v + \sin u\sin v$$
$$\sin(u+v) = \sin u\cos v + \cos u\sin v$$
$$\sin(u-v) = \sin u\cos v - \cos u\sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$$
$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$$

- **Double angle formulas**

$$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$
$$\sin(2u) = 2\sin u\cos u$$
$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

- **Linear combinations**

$$a\cos x + b\sin x = r\cos(x - \alpha) \quad \text{where } r = \sqrt{a^2+b^2},\quad \cos\alpha = \frac{a}{r},\quad \sin\alpha = \frac{b}{r}$$
$$a\cos x + b\sin x = r\sin(x + \beta) \quad \text{where } r = \sqrt{a^2+b^2},\quad \sin\beta = \frac{a}{r},\quad \cos\beta = \frac{b}{r}$$

- **Product-to-sum identities**

$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

- **Sum-to-product identities**

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$


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Review

## Short-answer questions

**1** Find the maximum and minimum values of each of the following:

**a** $3 + 2\sin\theta$ **b** $1 - 3\cos\theta$ **c** $4\sin\left(\frac{3\theta}{2}\right)$ **d** $2\sin^2\left(\frac{\theta}{2}\right)$ **e** $\frac{1}{2 + \cos\theta}$

**2** Find the values of $\theta \in [0, 2\pi]$ for which:

**a** $\tan^2\theta = \frac{1}{3}$ **b** $\tan(2\theta) = -1$ **c** $\sin(3\theta) = -1$ **d** $\sec(2\theta) = \sqrt{2}$

**3** Prove each of the following identities:

**a** $\sec\theta + \operatorname{cosec}\theta\cot\theta = \sec\theta\operatorname{cosec}^2\theta$ **b** $\sec\theta - \sin\theta = \frac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

**4** If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where $A$ and $B$ are acute, find:

**a** $\cos(A + B)$ **b** $\sin(A - B)$ **c** $\tan(A + B)$

**5** Find:

**a** $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$ **b** $\frac{\tan 15^\circ + \tan 30^\circ}{1 - \tan 15^\circ \tan 30^\circ}$

**6** If $A + B = \frac{\pi}{2}$, find the value of:

**a** $\sin A\cos B + \cos A\sin B$ **b** $\cos A\cos B - \sin A\sin B$

**7** Prove each of the following:

**a** $\sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A - \sin^2 B$

**b** $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = \frac{2}{\sin\theta}$ **c** $\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

**8** Given that $\sin A = \frac{\sqrt{5}}{3}$ and that $A$ is obtuse, find the value of:

**a** $\cos(2A)$ **b** $\sin(2A)$ **c** $\sin(4A)$

**9** Prove:

**a** $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos(2A)$ **b** $\sqrt{2r^2(1 - \cos\theta)} = 2r\sin\left(\frac{\theta}{2}\right)$ for $r > 0$ and $\theta$ acute

**10** Find $\tan 15^\circ$ in simplest surd form.

**11** Solve each of the following equations for $x \in [0, 2\pi]$:

**a** $\sin x + \cos x = 1$ **b** $\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right) = -\frac{1}{4}$

**c** $3\tan(2x) = 2\tan x$ **d** $\sin^2 x = \cos^2 x + 1$

**e** $\sin(3x)\cos x - \cos(3x)\sin x = \frac{\sqrt{3}}{2}$ **f** $2\cos\left(2x - \frac{\pi}{3}\right) = -\sqrt{3}$


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Review

**12** Sketch the graph of:

**a** $y = 2\cos^2 x$ **b** $y = 1 - 2\sin\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right)$ **c** $f(x) = \tan(2x)$

**13** If $\tan A = 2$ and $\tan(\theta + A) = 4$, find the exact value of $\tan\theta$.

**14** **a** Express $2\cos\theta + 9\sin\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

**b** **i** Give the maximum value of $2\cos\theta + 9\sin\theta$.

**ii** Give the cosine of $\theta$ for which this maximum occurs.

**iii** Find the smallest positive solution of the equation $2\cos\theta + 9\sin\theta = 1$.

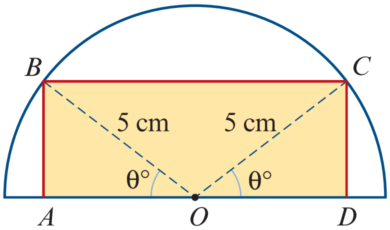
**15** Solve each of the following equations for $\theta \in [0, \pi]$:

**a** $\sin(4\theta) + \sin(2\theta) = 0$ **b** $\sin(2\theta) - \sin(\theta) = 0$

**16** Prove that $\dfrac{\cos A - \cos B}{\sin A + \sin B} = \tan\left(\dfrac{B - A}{2}\right)$.

## Extended-response questions

**1** The diagram shows a rectangle $ABCD$ inside a semicircle, centre $O$ and radius 5 cm, with $\angle BOA = \angle COD = \theta°$.



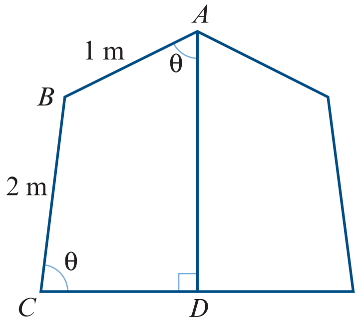

**a** Show that the perimeter, $P$ cm, of the rectangle is given by $P = 20\cos\theta + 10\sin\theta$.

**b** Express $P$ in the form $r\cos(\theta - \alpha)$ and hence find the value of $\theta$ for which $P = 16$.

**c** Find the value of $k$ for which the area of the rectangle is $k\sin(2\theta)$ cm$^2$.

**d** Find the value of $\theta$ for which the area is a maximum.

**2** The diagram shows a vertical section through a tent in which $AB = 1$ m, $BC = 2$ m and $\angle BAD = \angle BCD = \theta$. The line $CD$ is horizontal, and the diagram is symmetrical about the vertical $AD$.




**a** Obtain an expression for $AD$ in terms of $\theta$.

**b** Express $AD$ in the form $r\cos(\theta - \alpha)$, where $r$ is positive.

**c** State the maximum length of $AD$ and the corresponding value of $\theta$.

**d** Given that $AD = 2.15$ m, find the value of $\theta$ for which $\theta > \alpha$.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**3** **a** Prove the identity

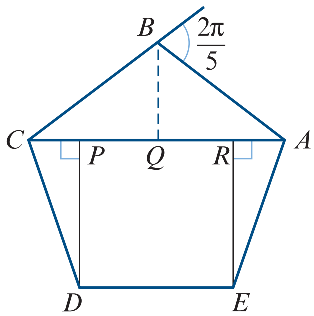
$$\cos(2\theta) = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

**b** **i** Use the result of part **a** to show that $1 + x^2 = \sqrt{2}x^2 - \sqrt{2}$, where $x = \tan\left(67\frac{1}{2}\right)^\circ$.

**ii** Hence find the values of integers $a$ and $b$ such that $\tan\left(67\frac{1}{2}\right)^\circ = a + b\sqrt{2}$.

**c** Find the value of $\tan\left(7\frac{1}{2}\right)^\circ$.

**4** $ABCDE$ is a regular pentagon with side length one unit. The exterior angles of a regular pentagon each have magnitude $\frac{2\pi}{5}$.




**a** **i** Show that the magnitude of $\angle BCA$ is $\frac{\pi}{5}$.

**ii** Find the length of $CA$.

**b** **i** Show the magnitude of $\angle DCP$ is $\frac{2\pi}{5}$.

**ii** Use the fact that $AC = 2CQ = 2CP + PR$ to show that $2\cos\left(\frac{\pi}{5}\right) = 2\cos\left(\frac{2\pi}{5}\right) + 1$.

**iii** Use the identity $\cos(2\theta) = 2\cos^2\theta - 1$ to form a quadratic equation in terms of $\cos\left(\frac{\pi}{5}\right)$.

**iv** Find the exact value of $\cos\left(\frac{\pi}{5}\right)$.

**5** **a** Prove each of the identities:

**i** $\cos\theta = \dfrac{1 - \tan^2(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$

**ii** $\sin\theta = \dfrac{2\tan(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$

**b** Use the results of part **a** to find the value of $\tan\left(\frac{1}{2}\theta\right)$, given that $8\cos\theta - \sin\theta = 4$.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# 10 Revision of Chapters 6–9



## 10A Short-answer questions

**1** Suppose that $n$ is odd. Prove that $n^2 + n$ is even.

**2** Prove that if $m$ and $n$ are consecutive integers, then $n^2 - m^2 = n + m$.

**3** Let $n \in \mathbb{Z}$. Consider the statement: If $5n + 3$ is even, then $n$ is odd.

- **a** Write down the converse statement.
- **b** Prove the converse.
- **c** Write down the contrapositive statement.
- **d** Prove the contrapositive.

**4** Let $n \in \mathbb{Z}$. Prove that $3n + 1$ is even if and only if $n$ is odd.

**5** Prove that each of the following statements is false by finding a counterexample:

- **a** The sum of two prime numbers cannot be a prime number.
- **b** For all $x \in \mathbb{R}$, we have $x^3 > x^2$.

**6** Show that this statement is false: There exists $n \in \mathbb{N}$ such that $25n^2 - 9$ is a prime number.


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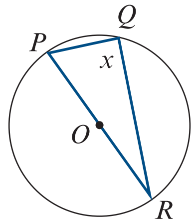
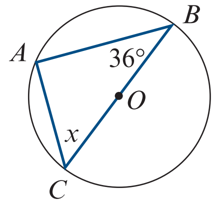
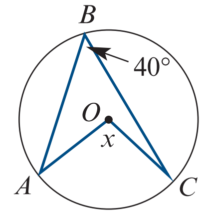
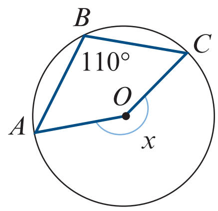
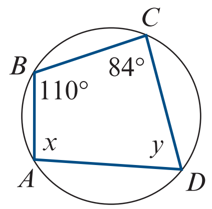
**7** Find the values of $x$ and $y$ in each of the following diagrams, giving reasons.

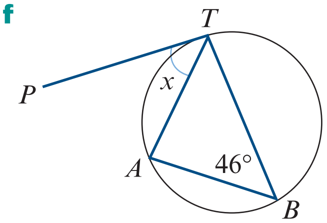
**a**



**b**


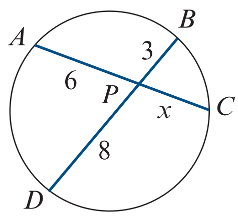

**c**


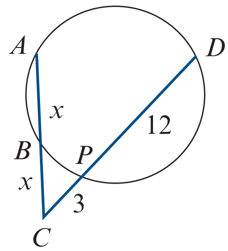

**d**


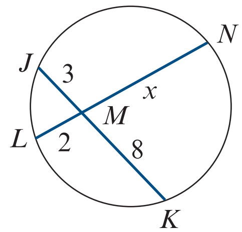

**e**



**f**



**8** In each of the following, find the value of $x$:

**a**



**b**



**c**



**9** Solve each of the following equations:

**a** $\sin\left(\dfrac{x}{3}\right) = \dfrac{1}{2}$ for $x \in [0, 12\pi]$

**b** $\sqrt{2}\cos\left(2x + \dfrac{\pi}{6}\right) + 1 = 0$ for $x \in [-\pi, \pi]$

**10** Solve the equation $\cos x = \dfrac{\sqrt{3}}{2}$, giving the general solution.

**11** Given that $\sin A = \dfrac{3}{5}$, where $A$ is acute, and that $\cos B = -\dfrac{1}{2}$, where $B$ is obtuse, find the exact values of:

**a** $\sec A$

**b** $\cot A$

**c** $\cot B$

**d** $\operatorname{cosec} B$

**12** Given that $\cos A = \dfrac{1}{3}$, find the possible values of $\cos\left(\dfrac{A}{2}\right)$.

**13** Prove the identity $\dfrac{1}{1 + \sin A} + \dfrac{1}{1 - \sin A} = 2\sec^2 A$.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**14** **a** Write the product $\cos(3x)\sin(x)$ as a sum or a difference.

**b** Find the general solution of the equation $\cos(3\theta) + \cos(\theta) = 0$.

**15** Prove each of the following:

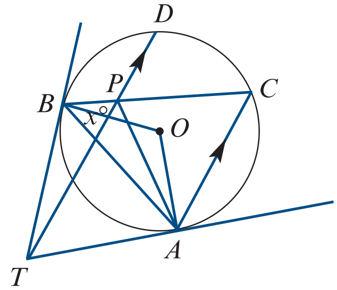
**a** $\dfrac{\sin(3x) + \sin(x)}{\cos(3x) + \cos(x)} = \tan(2x)$

**b** $\dfrac{\sin(x) + \sin(2x)}{1 + \cos(x) + \cos(2x)} = \tan(x)$

## 10B Extended-response questions

**1** Let $a$, $b$ and $c$ be positive real numbers. Prove that if $b > a$, then $\dfrac{a+c}{b+c} > \dfrac{a}{b}$.

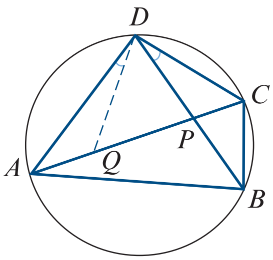
**2** In the figure, $O$ is the centre of a circle. $TD$ and $AC$ are parallel. $TA$ and $TB$ are tangents to the circle. Let $\angle BPT = x°$.




**a** Prove that $TBOA$ is a cyclic quadrilateral.

**b** Find $\angle BCA$, $\angle BOA$, $\angle TAB$ and $\angle TBA$ in terms of $x$.

**3** Ptolemy's theorem: *In a cyclic quadrilateral, the sum of the products of the opposite sides is equal to the product of the diagonals.*




**a** To prove Ptolemy's theorem, we need to show that $BC \cdot AD + AB \cdot CD = AC \cdot BD$.
Line $DQ$ is drawn such that $\angle BDC = \angle ADQ$.

**i** Prove that $\triangle ADQ \sim \triangle BDC$.

**ii** Prove that $\triangle ADB \sim \triangle QDC$.

**iii** Show that $AQ = \dfrac{BC \cdot AD}{BD}$ and $QC = \dfrac{AB \cdot CD}{BD}$.

**iv** Complete the proof of Ptolemy's theorem.
(Hint: $AC = AQ + QC$.)

**b** Prove Pythagoras' theorem by applying Ptolemy's theorem to a rectangle.

**c** Use Ptolemy's theorem to prove that, in a regular pentagon with side length 1, the length of each diagonal is the golden ratio $\varphi$.

Hint: Draw the diagonals and take three sides of the pentagon and one diagonal to form a quadrilateral.

**d** Given an equilateral triangle inscribed in a circle and a point on the circle, use Ptolemy's theorem to prove that the distance from the point to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.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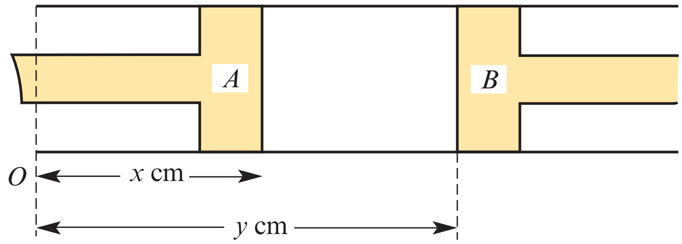
**4** Two pistons $A$ and $B$ move backwards and forwards in a cylinder as shown.




The distance, $x$ cm, of the right-hand end of piston $A$ from the point $O$ at time $t$ seconds is modelled by the rule

$$x = 4\sin(3t) + 4$$

The distance, $y$ cm, of the left-hand end of piston $B$ from the point $O$ at time $t$ seconds is modelled by the rule

$$y = 2\sin\left(2t - \frac{\pi}{6}\right) + 10$$

The two pistons are set in motion at time $t = 0$.

**a** State the value of $x$ and the value of $y$ when $t = 0$.

**b** **i** State the amplitude of the motion of piston $A$.

**ii** State the amplitude of the motion of piston $B$.

**c** **i** State the maximum and minimum values of $x$.

**ii** State the maximum and minimum values of $y$.

**d** **i** State the period of the motion of piston $A$.

**ii** State the period of the motion of piston $B$.

**e** Find the time(s) in the first cycle of $A$ that its distance from $O$ is a maximum.

**f** Find the next four values of $t$ for which $x$ takes its maximum value.

**g** Find the values of $t \in [0, 4\pi]$ for which $y$ attains its minimum value.

**h** On one set of axes, sketch the graphs of $x = 4\sin(3t) + 4$ and $y = 2\sin\left(2t - \frac{\pi}{6}\right) + 10$ over the interval $[0, \pi]$.

**i** State the time when the pistons first touch each other.

**j** How many seconds are there between the first and second times the pistons touch?

**5** The pistons $A$ and $B$ (from Question 4) are adjusted so that the distance, $x$ cm, of the right-hand end of piston $A$ from the point $O$ at time $t$ seconds is modelled by the rule

$$x = a\sin(nt) + b$$

and the distance, $y$ cm, of the left-hand end of piston $B$ from the point $O$ at time $t$ seconds is modelled by the rule

$$y = c\sin(mt) + d$$

The pistons meet every second at a point 8 cm from $O$. The right-hand end of piston $A$ cannot go to the left of the point $O$.

**a** Find one possible set of values of $a, b, n$ and $c, d, m$ and explain your solution.

**b** Using the set of values found in part **a**, sketch the graphs of $x$ against $t$ and $y$ against $t$ on the one set of axes.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**6** Suppose that $k$ is a real number and consider the function $f(x) = 2\sin(x) + k$, $0 \le x \le 2\pi$.

**a** Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when:

**i** $k = 1$

**ii** $k = 3$

**b** For what value of $k$ does the graph of $y = \dfrac{1}{f(x)}$ have only one vertical asymptote?

**c** For this value of $k$, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

## Triple angle formulas

**7** **a** Using the addition formulas and the double angle formulas, prove each of the following identities:

**i** $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$

**ii** $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

Now let $\alpha = \cos\left(\dfrac{\pi}{9}\right)$.

**b** Use an identity from part **a** to show that $x = \alpha$ is a solution of the equation $f(x) = 0$, where $f(x) = x^3 - \dfrac{3}{4}x - \dfrac{1}{8}$.

**c** Use your calculator to plot the graph of $y = f(x)$. Hence determine that $x = \alpha$ is the unique positive real solution of the cubic equation $f(x) = 0$.


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# 11 Matrices

## In this chapter



## Syllabus references

**Topics:** Matrix arithmetic; Systems of linear equations

**Subtopics:** 2.2.1 – 2.2.3, 2.2.11

A **matrix** is a rectangular array of numbers. An example of a matrix is

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

Matrix algebra was first studied in England in the middle of the nineteenth century. Matrices are now used in many areas of science: for example, in physics, medical research, encryption and internet search engines.

In this chapter we will show how addition and multiplication of matrices can be defined and how matrices can be used to solve simultaneous linear equations. In Chapter 12 we will see how they can be used to study transformations of the plane.



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# 11A Matrix notation

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The following are examples of matrices:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

## The size of a matrix

Matrices vary in size. The **size** of the matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines) that occur in the matrix.

The sizes of the above matrices are, in order:

$$3 \times 2, \quad 1 \times 4, \quad 3 \times 3, \quad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

An $m \times n$ matrix has $m$ rows and $n$ columns.

### Example 1

Write down the sizes of the following matrices:

**a** $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$

**Solution**

**a** $2 \times 3$ **b** $4 \times 1$ **c** $1 \times 3$

## Storing information in matrices

The use of matrices to store information is demonstrated by the following example.

Four exporters $A$, $B$, $C$ and $D$ sell refrigerators ($r$), CD players ($c$), washing machines ($w$) and televisions ($t$). The sales in a particular month can be represented by a $4 \times 4$ array of numbers. This array of numbers is called a matrix.

| | $r$ | $c$ | $w$ | $t$ | |
|---|---|---|---|---|---|
| $A$ | 120 | 95 | 370 | 250 | row 1 |
| $B$ | 430 | 380 | 950 | 900 | row 2 |
| $C$ | 60 | 50 | 150 | 100 | row 3 |
| $D$ | 200 | 100 | 470 | 50 | row 4 |
| | column 1 | column 2 | column 3 | column 4 | |


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From this matrix it can be seen that:

- Exporter $A$ sold 120 refrigerators, 95 CD players, 370 washing machines, 250 televisions.
- Exporter $B$ sold 430 refrigerators, 380 CD players, 950 washing machines, 900 televisions.

The entries for the sales of refrigerators are in column 1.

The entries for the sales of exporter $A$ are in row 1.

**Example 2**

A minibus has four rows of seats, with three seats in each row. If 0 indicates that a seat is vacant and 1 indicates that a seat is occupied, write down a matrix to represent:

**a** the 1st and 3rd rows are occupied, but the 2nd and 4th rows are vacant

**b** only the seat at the front-left corner of the minibus is occupied.

**Solution**

**a** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

**b** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Example 3**

There are four clubs in a local football league:

- Club A has 2 senior teams and 3 junior teams.
- Club B has 2 senior teams and 4 junior teams.
- Club C has 1 senior team and 2 junior teams.
- Club D has 3 senior teams and 3 junior teams.

Represent this information in a matrix.

**Solution**

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$$

**Explanation**

The rows represent clubs A, B, C, D and the columns represent the number of senior and junior teams.

## Entries and equality

We will use uppercase letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ to denote matrices.

If $\mathbf{A}$ is a matrix, then $a_{ij}$ will be used to denote the entry that occurs in row $i$ and column $j$ of $\mathbf{A}$. Thus a $3 \times 4$ matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$


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Two matrices **A** and **B** are **equal**, and we can write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

For example:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1 & -1 \\ 1-1 & 1 & \frac{6}{2} \end{bmatrix}$$

**Example 4**

If matrices **A** and **B** are equal, find the values of $x$ and $y$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ x & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -3 & y \end{bmatrix}$$

**Solution**

$x = -3$ and $y = 4$

Although a matrix is made from a set of numbers, it is important to think of a matrix as a single entity, somewhat like a 'super number'.

### Summary 11A

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has $m$ rows and $n$ columns.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same entry at corresponding positions.

## Exercise 11A

Example 1

**1** Write down the sizes of the following matrices:

**a** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} a & b & c & d \end{bmatrix}$ **d** $\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$

Example 2

**2** There are 25 seats arranged in five rows and five columns. Using 0 to indicate that a seat is vacant and 1 to indicate that a seat is occupied, write down a matrix to represent the situation when:

**a** only the seats on the two diagonals are occupied

**b** all seats are occupied.


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**3** Seating arrangements are again represented by matrices, as in Question 2. Describe the seating arrangement represented by each of the following matrices:

**a** the entry $a_{ij}$ is 1 if $i = j$, but 0 if $i \neq j$

**b** the entry $a_{ij}$ is 1 if $i > j$, but 0 if $i \leq j$

**c** the entry $a_{ij}$ is 1 if $i = j + 1$, but 0 otherwise.

Example 3

**4** At a certain school there are 200 girls and 110 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 117 in Year 8, 135 and 98 in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11, and 28 and 33 in Year 12. Summarise this information in a matrix.

Example 4

**5** From the following, select those pairs of matrices which could be equal, and write down the values of $x$ and $y$ which would make them equal:

**a** $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ x \end{bmatrix}$, $\begin{bmatrix} 0 & x \end{bmatrix}$, $\begin{bmatrix} 0 & 4 \end{bmatrix}$

**b** $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$, $\begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 4 & x & 1 & -2 \end{bmatrix}$

**c** $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$

**6** Find the values of the pronumerals so that matrices **A** and **B** are equal:

**a** $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} x & 1 & -1 \\ 0 & 1 & y \end{bmatrix}$

**b** $\mathbf{A} = \begin{bmatrix} x \\ 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 \\ y \end{bmatrix}$

**c** $\mathbf{A} = \begin{bmatrix} -3 & x \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} y & 4 \end{bmatrix}$

**d** $\mathbf{A} = \begin{bmatrix} 1 & y \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$

**7** The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5
Player B points 8, rebounds 2, assists 3
Player C points 4, rebounds 1, assists 1
Player D points 14, rebounds 8, assists 60
Player E points 0, rebounds 1, assists 2

Express this information in a $5 \times 3$ matrix.

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# 11B Addition, subtraction and multiplication by a real number

### Addition of matrices

If **A** and **B** are two matrices of the same size, then the sum **A** + **B** is the matrix obtained by adding together the corresponding entries of the two matrices.

For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

### Multiplication of a matrix by a real number

If **A** is any matrix and $k$ is a real number, then the product $k$**A** is the matrix obtained by multiplying each entry of **A** by $k$.

For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Note: If a matrix is added to itself, then the result is twice the matrix, i.e. **A** + **A** = 2**A**. Similarly, for any natural number $n$, the sum of $n$ matrices each equal to **A** is $n$**A**.

If **B** is any matrix, then −**B** denotes the product (−1)**B**.

### Subtraction of matrices

If **A** and **B** are matrices of the same size, then **A** − **B** is defined to be the sum

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} + (-1)\mathbf{B}$$

For two matrices **A** and **B** of the same size, the difference **A** − **B** can be found by subtracting corresponding entries.

### Example 5

Find:

**a** $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

**b** $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

#### Solution

**a** $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & -1 \end{bmatrix}$

**b** $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$


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### Zero matrix

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**, and will be denoted by **O**.

For any $m \times n$ matrix **A** and the $m \times n$ zero matrix **O**, we have

$$\mathbf{A} + \mathbf{O} = \mathbf{A} \qquad \text{and} \qquad \mathbf{A} + (-\mathbf{A}) = \mathbf{O}$$

## Example 6

Let $\mathbf{X} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $3\mathbf{A} + \mathbf{B}$.

### Solution

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$2\mathbf{X} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4\mathbf{Y} + \mathbf{X} = 4\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$-3\mathbf{A} = -3\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix}$$

$$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

## Example 7

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find the matrix **X** such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

### Solution

If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

$$\mathbf{X} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix}$$


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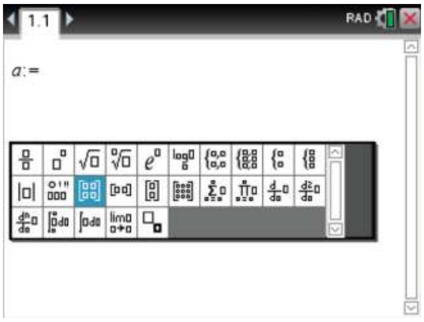
## Using the TI-Nspire

### The matrix template

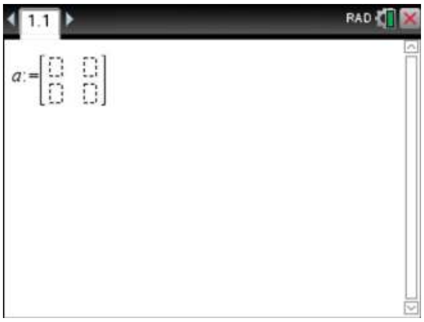
Matrices can be assigned (or stored) as variables for further computations.

Assign matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as follows:

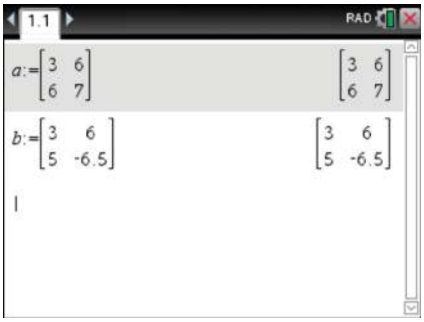
- In a **Calculator** page, type $a :=$ and then enter the matrix. (Access the assignment symbol := using (ctrl)(⊞).)




- The simplest way to enter a 2 × 2 matrix is using the 2 × 2 matrix template as shown. (Access the templates using either (⊞) or (ctrl)(menu) > **Math Templates**.)
- Notice that there is also a template for entering $m \times n$ matrices.
- Use the touchpad arrows (or (tab)) to move between the entries of the 2 × 2 matrix template.




Assign the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$ similarly.




### Operations on matrices

Once **A** and **B** are defined as above, the matrices **A** + **B**, **A** − **B** and $k$**A** can easily be determined.



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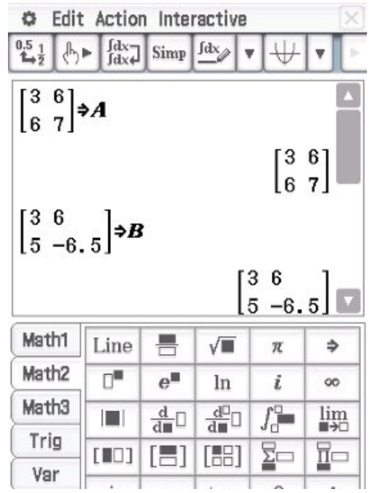
## Using the Casio ClassPad

### Entering a matrix

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, select the Math2 keyboard.
- To enter a $2 \times 2$ matrix, tap [⊞].
- Type the values into the matrix template.

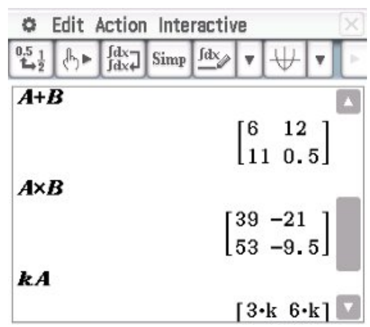
Note: Tap at each new position to enter the value, or use the black cursor key on the hard keyboard to navigate to each new position.




### Assigning a matrix

- Move the cursor to the right-hand side of the matrix. Then tap the variable assignment key ⇒ followed by Var CAPS $A$.
- Tap EXE to confirm your choice.
- Enter the second matrix and assign it the variable name $B$ as shown.

Note: Until it is reassigned, the variable $A$ will represent the matrix as defined above.




### Operations on matrices

- Calculate $\mathbf{A} + \mathbf{B}$, $\mathbf{AB}$ and $k\mathbf{A}$ as shown. (Use the Var keyboard to enter the variable names.)

## Summary 11B

- If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size, then:
  - the matrix $\mathbf{A} + \mathbf{B}$ is obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$
  - the matrix $\mathbf{A} - \mathbf{B}$ is obtained by subtracting the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$.
- If $\mathbf{A}$ is any matrix and $k$ is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of $\mathbf{A}$ by $k$.

## Exercise 11B

Example 6

**1** Let $\mathbf{X} = \begin{bmatrix}1\\-2\end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix}3\\0\end{bmatrix}$, $\mathbf{A} = \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}4 & 0\\-1 & 2\end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

**2** Let $\mathbf{A} = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$. Find $2\mathbf{A}$, $-3\mathbf{A}$ and $-6\mathbf{A}$.

**3** For $m \times n$ matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, is it always true that:

**a** $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

**b** $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$?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**4** Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix}$. Calculate:

**a** $2\mathbf{A}$ **b** $3\mathbf{B}$ **c** $2\mathbf{A} + 3\mathbf{B}$ **d** $3\mathbf{B} - 2\mathbf{A}$

**5** Let $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$. Calculate:

**a** $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} + 3\mathbf{Q}$ **c** $2\mathbf{P} - \mathbf{Q} + \mathbf{R}$

Example 7

**6** If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -10 \\ -2 & 17 \end{bmatrix}$, find matrices $\mathbf{X}$ and $\mathbf{Y}$ such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

**7** Matrices $\mathbf{X}$ and $\mathbf{Y}$ show the production of four models of cars $a$, $b$, $c$, $d$ at two factories $P$, $Q$ in successive weeks. Find $\mathbf{X} + \mathbf{Y}$ and describe what this sum represents.

Week 1: $\mathbf{X} = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 150 & 90 & 100 & 50 \\ 100 & 0 & 75 & 0 \end{bmatrix} \end{array}$

Week 2: $\mathbf{Y} = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 160 & 90 & 120 & 40 \\ 100 & 0 & 50 & 0 \end{bmatrix} \end{array}$

## 11C Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous section. The definition for multiplication of matrices is less straightforward. The procedure for multiplying two $2 \times 2$ matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

Then
$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

and
$$\begin{aligned} \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.


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If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row $i$ and column $j$ of $\mathbf{AB}$, single out row $i$ in matrix $\mathbf{A}$ and column $j$ in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

## Example 8

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, find $\mathbf{AB}$.

### Solution

$\mathbf{A}$ is a $2 \times 2$ matrix and $\mathbf{B}$ is a $2 \times 1$ matrix. Therefore the product $\mathbf{AB}$ is defined and will be a $2 \times 1$ matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

## Example 9

Matrix $\mathbf{X}$ shows the number of cars of models $a$ and $b$ bought by four dealers $A$, $B$, $C$, $D$. Matrix $\mathbf{Y}$ shows the cost in dollars of cars $a$ and $b$. Find $\mathbf{XY}$ and explain what it represents.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \mathbf{Y} = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix} \begin{array}{c} a \\ b \end{array}$$

### Solution

$\mathbf{X}$ is a $4 \times 2$ matrix and $\mathbf{Y}$ is a $2 \times 1$ matrix. Therefore $\mathbf{XY}$ is a $4 \times 1$ matrix.

$$\mathbf{XY} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix} \begin{array}{c} a \\ b \end{array}$$

$$= \begin{bmatrix} 3 \times 26\,000 + 1 \times 32\,000 \\ 2 \times 26\,000 + 2 \times 32\,000 \\ 1 \times 26\,000 + 4 \times 32\,000 \\ 1 \times 26\,000 + 1 \times 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$$

The matrix $\mathbf{XY}$ shows that dealer $A$ spent \$110 000, dealer $B$ spent \$116 000, dealer $C$ spent \$154 000 and dealer $D$ spent \$58 000.


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## Example 10

For $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, find **AB**.

### Solution

**A** is a $2 \times 3$ matrix and **B** is a $3 \times 2$ matrix. Therefore **AB** is a $2 \times 2$ matrix.

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 4 + 3 \times 1 + 4 \times 0 & 2 \times 0 + 3 \times 2 + 4 \times 3 \\ 5 \times 4 + 6 \times 1 + 7 \times 0 & 5 \times 0 + 6 \times 2 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix} \end{aligned}$$

## Summary 11C

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row $i$ and column $j$ of **AB**, single out row $i$ in matrix **A** and column $j$ in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

- The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.

Skill-sheet

## Exercise 11C

Example 8
Example 10

**1** Let $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Find the products **AX**, **BX**, **AY**, **IX**, **AC**, **CA**, **(AC)X**, **C(BX)**, **AI**, **IB**, **AB**, **BA**, $\mathbf{A}^2$, $\mathbf{B}^2$, **A(CA)** and $\mathbf{A}^2\mathbf{C}$.

**2** Which of the following products of matrices from Question 1 are defined?
**AY**, **YA**, **XY**, $\mathbf{X}^2$, **CI**, **XI**

**3** If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$, find **AB**.

**4** Let **A** and **B** be $2 \times 2$ matrices and let **O** be the $2 \times 2$ zero matrix. Is the following argument correct?
'If $\mathbf{AB} = \mathbf{O}$ and $\mathbf{A} \neq \mathbf{O}$, then $\mathbf{B} = \mathbf{O}$.'

**5** Find a matrix **A** such that $\mathbf{A} \neq \mathbf{O}$ but $\mathbf{A}^2 = \mathbf{O}$.

**6** If $\mathbf{L} = \begin{bmatrix} 2 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find **LX** and **XL**.


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**7** Assume that both **A** and **B** are $m \times n$ matrices. Are **AB** and **BA** defined and, if so, how many rows and columns do they have?

**8** Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**a** Show that $ad - bc = 1$.

**b** What is the product matrix if the order of multiplication on the left-hand side is reversed?

**9** Using the result of Question 8, write down a pair of matrices **A** and **B** such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**10** Choose any three $2 \times 2$ matrices **A**, **B** and **C**. Find $\mathbf{A}(\mathbf{B} + \mathbf{C})$, $\mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A}$.

**11** Find matrices **A** and **B** such that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Example 9

**12** It takes John 5 minutes to drink a milk shake which costs $2.50, and 12 minutes to eat a banana split which costs $3.00.

**a** Find the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in milk-bar economics.

**b** Two friends join John. Find $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

**13** Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$ and use your answer to find $\mathbf{A}^4$ and $\mathbf{A}^8$.

**14** Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Write down a formula for $\mathbf{A}^n$.

# 11D Identities, inverses and determinants for $2 \times 2$ matrices

## Identities

A matrix with the same number of rows and columns is called a **square matrix**. For square matrices of a given size (e.g. $2 \times 2$), a multiplicative identity **I** exists.

For $2 \times 2$ matrices, the **identity matrix** is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$, and this result holds for any square matrix multiplied by the appropriate multiplicative identity.

For $3 \times 3$ matrices, the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.


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## Inverses

Given a 2 × 2 matrix **A**, is there a matrix **B** such that **AB** = **BA** = **I**?

For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then **AB** = **I** implies

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2x + 3u & 2y + 3v \\ x + 4u & y + 4v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $2x + 3u = 1$ and $2y + 3v = 0$

$x + 4u = 0$ $\quad$ $y + 4v = 1$

These simultaneous equations can be solved to find $x, y, u, v$ and hence **B**.

$$\mathbf{B} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

In general:

> If **A** is a square matrix and if a matrix **B** can be found such that
>
> **AB** = **BA** = **I**
>
> then **A** is said to be **invertible** and **B** is called the **inverse** of **A**.

We leave it as an exercise to show that the inverse of an invertible matrix is unique.

We will denote the inverse of **A** by $\mathbf{A}^{-1}$.

> For an invertible matrix **A**, we have
>
> $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$

### The inverse of a general 2 × 2 matrix

Now consider $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then **AB** = **I** implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $ax + bu = 1$ and $ay + bv = 0$

$cx + du = 0$ $\quad$ $cy + dv = 1$

These form two pairs of simultaneous equations, the first for $x, u$ and the second for $y, v$.


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The first pair of equations gives

$$(ad - bc)x = d \qquad \text{(eliminating } u\text{)}$$

$$(bc - ad)u = c \qquad \text{(eliminating } x\text{)}$$

These two equations can be solved for $x$ and $u$ provided $ad - bc \neq 0$:

$$x = \frac{d}{ad - bc} \qquad \text{and} \qquad u = \frac{c}{bc - ad} = \frac{-c}{ad - bc}$$

In a similar way, we obtain

$$y = \frac{-b}{ad - bc} \qquad \text{and} \qquad v = \frac{-a}{bc - ad} = \frac{a}{ad - bc}$$

We have established the following result.

### Inverse of a 2 × 2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{(provided } ad - bc \neq 0\text{)}$$

## The determinant

The quantity $ad - bc$ that appears in the formula for $\mathbf{A}^{-1}$ has a name: the **determinant** of $\mathbf{A}$. This is denoted $\det(\mathbf{A})$.

### Determinant of a 2 × 2 matrix

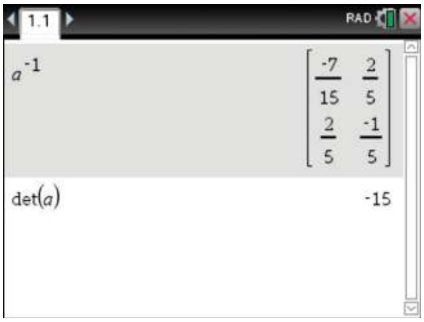
If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

A $2 \times 2$ matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

### Using the TI-Nspire

- The inverse of a matrix is obtained by raising the matrix to the power of $-1$.
- The determinant command (menu > **Matrix and Vector** > **Determinant**) is used as shown.

Hint: You can also type in det($a$).




(Here $a$ is the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ defined on page 215.)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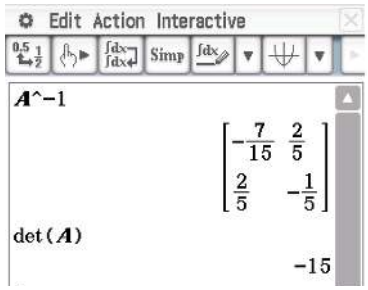
## Using the Casio ClassPad

- To find the inverse matrix, type $A$^−1 and tap EXE.

Note: If the matrix has no inverse, then the calculator will give the message **Undefined**.

- To find the determinant, enter and highlight $A$. Select **Interactive** > **Matrix** > **Calculation** > **det**.




## Example 11

For the matrix $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, find:

**a** det(**A**)

**b** $\mathbf{A}^{-1}$

### Solution

**a** $\det(\mathbf{A}) = 5 \times 1 - 2 \times 3 = -1$

**b** $\mathbf{A}^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

## Example 12

For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, find:

**a** det(**A**)

**b** $\mathbf{A}^{-1}$

**c** $\mathbf{X}$, if $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

**d** $\mathbf{Y}$, if $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

### Solution

**a** $\det(\mathbf{A}) = 3 \times 6 - 2 = 16$

**b** $\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$

**c** $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the left) by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{IX} = \mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

**d** $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the right) by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y} = \frac{1}{16}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$$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## Summary 11D

- For a $2 \times 2$ matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of $\mathbf{A}$ is given by
  $$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad - bc \neq 0)$$
  - the determinant of $\mathbf{A}$ is given by
  $$\det(\mathbf{A}) = ad - bc$$
- A $2 \times 2$ matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

Skill-sheet

## Exercise 11D

Example 11

**1** For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, find:

**a** $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\det(\mathbf{B})$ **d** $\mathbf{B}^{-1}$

**2** Find the inverse of each of the following invertible matrices (where $k$ is any non-zero real number):

**a** $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

**3** If the matrix $\mathbf{A}$ is invertible, show that the inverse is unique.

**4** Let $\mathbf{A}$ and $\mathbf{B}$ be the invertible matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

**a** Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

**b** Find $\mathbf{AB}$ and hence find, if possible, $(\mathbf{AB})^{-1}$.

**c** From $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, find the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

Example 12

**5** Let $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

**a** Find $\mathbf{A}^{-1}$. **b** If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{X}$. **c** If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{Y}$.

**6** Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$.

**a** Find $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$. **b** Find $\mathbf{Y}$ such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$.

**7** Assume that $\mathbf{A}$ is a $2 \times 2$ matrix such that $a_{12} = a_{21} = 0$, $a_{11} \neq 0$ and $a_{22} \neq 0$. Show that $\mathbf{A}$ is invertible and find $\mathbf{A}^{-1}$.

**8** Let $\mathbf{A}$ be an invertible $2 \times 2$ matrix, let $\mathbf{B}$ be a $2 \times 2$ matrix and assume that $\mathbf{AB} = \mathbf{O}$. Show that $\mathbf{B} = \mathbf{O}$.


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**9** Find all $2 \times 2$ matrices such that $\mathbf{A}^{-1} = \mathbf{A}$.

**10** For what values of $a$ does the matrix $\mathbf{A} = \begin{bmatrix} a & 1 \\ 2 & a \end{bmatrix}$ not have an inverse?

**11** Let $n$ be a natural number and let

$$\mathbf{A} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+2} \end{bmatrix}$$

Show that all the entries of the inverse matrix $\mathbf{A}^{-1}$ are integers.

# 11E Solution of simultaneous equations using matrices

Inverse matrices can be used to solve some systems of simultaneous linear equations.

## Simultaneous equations with a unique solution

For example, consider the pair of simultaneous equations

$$3x - 2y = 5$$
$$5x - 3y = 9$$

This can be written as a matrix equation:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$. The determinant of $\mathbf{A}$ is $3(-3) - (-2)5 = 1$.

Since the determinant is non-zero, the inverse matrix exists:

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$$

Now multiply both sides of the original matrix equation on the left by $\mathbf{A}^{-1}$:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{I} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} \qquad \text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is the solution to the simultaneous equations. Check by substituting $x = 3$ and $y = 2$ into the two equations.


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## Simultaneous equations without a unique solution

If a pair of simultaneous linear equations in two variables corresponds to two parallel lines, then a non-invertible matrix results.

For example, the following pair of simultaneous equations has no solution:

$$x + 2y = 3$$
$$-2x - 4y = 6$$

The associated matrix equation is

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The determinant of the matrix $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is $1(-4) - 2(-2) = 0$, so the matrix has no inverse.

### Example 13

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

**Solution**

If $\mathbf{AX} = \mathbf{K}$, then

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$
$$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

### Example 14

Solve the following simultaneous equations:

$$3x - 2y = 6$$
$$7x + 4y = 7$$

**Solution**

The matrix equation is

$$\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$. Then $\mathbf{A}^{-1} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$.

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 38 \\ -21 \end{bmatrix}$$


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Skill-sheet

## Exercise 11E

Example 13

**1** Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where:

**a** $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

**b** $\mathbf{K} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Example 14

**2** Use matrices to solve each of the following pairs of simultaneous equations:

**a** $-2x + 4y = 6$
$3x + y = 1$

**b** $-x + 2y = -1$
$-x + 4y = 2$

**3** Use matrices to find the point of intersection of the lines given by the equations $2x - 3y = 7$ and $3x + y = 5$.

**4** Two children spend their pocket money buying some books and some CDs. One child spends \$120 and buys four books and four CDs. The other child spends \$114 and buys three CDs and five books. Set up a system of simultaneous equations and use matrices to find the cost of a single book and a single CD.

**5** Consider the system

$$2x - 3y = 3$$
$$4x - 6y = 6$$

**a** Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

**b** Is $\mathbf{A}$ an invertible matrix?

**c** Can any solutions be found for this system of equations?

**d** How many pairs does the solution set contain?

**6** Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$ are $2 \times 2$ matrices and that both $\mathbf{A}$ and $\mathbf{B}$ are invertible. Solve the following for $\mathbf{X}$:

**a** $\mathbf{AX} = \mathbf{C}$

**b** $\mathbf{ABX} = \mathbf{C}$

**c** $\mathbf{AXB} = \mathbf{C}$

**d** $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{C}$

**e** $\mathbf{AX} + \mathbf{B} = \mathbf{C}$

**f** $\mathbf{XA} + \mathbf{B} = \mathbf{A}$


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Review

## Chapter summary

Assign-ment

Nrich

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same entry at corresponding positions.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has $m$ rows and $n$ columns.
- Addition is defined for two matrices only when they have the same size. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

  Subtraction is performed in a similar way.
- If **A** is any matrix and $k$ is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of **A** by $k$.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row $i$ and column $j$ of **AB**, single out row $i$ in matrix **A** and column $j$ in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

  Note that the product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.
- If **A** is a square matrix and if a matrix **B** can be found such that **AB** = **BA** = **I**, then **A** is said to be **invertible** and **B** is called the **inverse** of **A**.
- For a $2 \times 2$ matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad-bc \neq 0)$$

  - the determinant of **A** is given by

$$\det(\mathbf{A}) = ad - bc$$

- A $2 \times 2$ matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.
- Simultaneous equations can be solved using inverse matrices. For example, the system of equations

$$ax + by = c$$
$$dx + ey = f$$

  can be written as $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ and solved using $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} c \\ f \end{bmatrix}$.


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## Short-answer questions

**1** If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, find:

**a** $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$

**b** $\mathbf{A}^2 - \mathbf{B}^2$

**2** Find all possible matrices $\mathbf{A}$ which satisfy the equation $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$.

**3** Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

**a** State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**.

**b** Find **DA** and $\mathbf{A}^{-1}$.

**4** If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find **AB** and $\mathbf{C}^{-1}$.

**5** Find the $2 \times 2$ matrix $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.

**6** If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, find $\mathbf{A}^2$ and hence find $\mathbf{A}^{-1}$.

**7** If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ does not have an inverse, find the value of $x$.

**8** **a** If $\mathbf{M} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, find:

**i** $\mathbf{MM} = \mathbf{M}^2$

**ii** $\mathbf{MMM} = \mathbf{M}^3$

**iii** $\mathbf{M}^{-1}$

**b** Find $x$ and $y$, given that $\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.


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## Extended-response questions

**1** **a** Consider the system of equations

$$2x - 3y = 3$$
$$4x + y = 5$$

**i** Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

**ii** Find $\det(\mathbf{A})$ and $\mathbf{A}^{-1}$.

**iii** Solve the system of equations.

**iv** Interpret your solution geometrically.

**b** Consider the system of equations

$$2x + y = 3$$
$$4x + 2y = 8$$

**i** Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

**ii** Find $\det(\mathbf{A})$ and explain why $\mathbf{A}^{-1}$ does not exist.

**c** Interpret your findings in part **b** geometrically.

**2** The final grades for Physics and Chemistry are made up of three components: tests, practical work and exams. Each semester, a mark out of 100 is awarded for each component. Wendy scored the following marks in the three components for Physics:

Semester 1 tests 79, practical work 78, exam 80
Semester 2 tests 80, practical work 78, exam 82

**a** Represent this information in a $2 \times 3$ matrix.

To calculate the final grade for each semester, the three components are weighted: tests are worth 20%, practical work is worth 30% and the exam is worth 50%.

**b** Represent this information in a $3 \times 1$ matrix.

**c** Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in the three components for Chemistry:

Semester 1 tests 86, practical work 82, exam 84
Semester 2 tests 81, practical work 80, exam 70

**d** Calculate Wendy's final grade for Chemistry in each semester.

Students who gain a total score of 320 or more for Physics and Chemistry over the two semesters are awarded a Certificate of Merit in Science.

**e** Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to re-mark her Semester 2 Chemistry exam, hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

**f** How many extra marks on the exam does she need?

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**3** A company runs computing classes and employs full-time and part-time teaching staff, as well as technical staff, catering staff and cleaners. The number of staff employed depends on demand from term to term.

In one year the company employed the following teaching staff:

Term 1 full-time 10, part-time 2
Term 2 full-time 8, part-time 4
Term 3 full-time 8, part-time 8
Term 4 full-time 6, part-time 10

**a** Represent this information in a $4 \times 2$ matrix.

Full-time teachers are paid \$70 per hour and part-time teachers are paid \$60 per hour.

**b** Represent this information in a $2 \times 1$ matrix.

**c** Calculate the cost per hour to the company for teaching staff for each term.

In the same year the company also employed the following support staff:

Term 1 technical 2, catering 2, cleaning 1
Term 2 technical 2, catering 2, cleaning 1
Term 3 technical 3, catering 4, cleaning 2
Term 4 technical 3, catering 4, cleaning 2

**d** Represent this information in a $4 \times 3$ matrix.

Technical staff are paid \$60 per hour, catering staff are paid \$55 per hour and cleaners are paid \$40 per hour.

**e** Represent this information in a $3 \times 1$ matrix.

**f** Calculate the cost per hour to the company for support staff for each term.

**g** Calculate the total cost per hour to the company for teaching and support staff for each term.

**4** Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $2 \times 2$ matrices.

**a** Prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

**b** Hence prove that if both $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $\mathbf{AB}$ is invertible.


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# 12 Transformations of the plane

## In this chapter



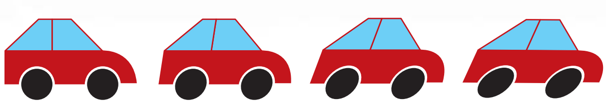
## Syllabus references

**Topic:** Transformations in the plane

**Subtopics:** 2.2.4 – 2.2.10

Modern animations are largely created with the use of computers. Many basic visual effects can be understood in terms of simple transformations of the plane.

For example, suppose that an animator wants to give the car below a sense of movement. This can be achieved by gradually tilting the car so that it leans forwards. We will see later how this can easily be done using a transformation called a **shear**.



Aside from computer graphics, linear transformations play an important role in many diverse fields such as mathematics, physics, engineering and economics.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# 12A Linear transformations

Each point in the plane can be denoted by an ordered pair $(x, y)$. The set of all ordered pairs is often called the **Cartesian plane**: $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

A **transformation** of the plane maps each point $(x, y)$ in the plane to a new point $(x', y')$. We say that $(x', y')$ is the **image** of $(x, y)$.

We will mainly be concerned with **linear transformations**, which have rules of the form

$$(x, y) \to (ax + by, cx + dy)$$

**Example 1**

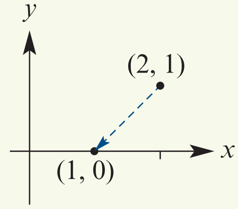
Find the image of the point (2, 1) under the transformation with rule

$$(x, y) \to (3x - 5y, 2x - 4y)$$

**Solution**

We let $x = 2$ and $y = 1$, giving

$$(2, 1) \to (3 \times 2 - 5 \times 1, 2 \times 2 - 4 \times 1) = (1, 0)$$




## Matrices and linear transformations

Each ordered pair can also be written as a $2 \times 1$ matrix, which (recalling from Chapter 3) we call a **column vector**:

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

This is a very useful observation, since we can now easily perform the linear transformation $(x, y) \to (ax + by, cx + dy)$ by using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

**Example 2**

**a** Find the matrix of the linear transformation with rule $(x, y) \to (x - 2y, 3x + y)$.

**b** Use the matrix to find the image of the point (2, 3) under the transformation.

**Solution**

**a** $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

**b** $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 - 2 \times 3 \\ 3 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$

Therefore the image of (2, 3) is $(-4, 9)$.

**Explanation**

The rows of the matrix are given by the coefficients of $x$ and $y$.

We write the point (2, 3) as a column vector and multiply by the transformation matrix.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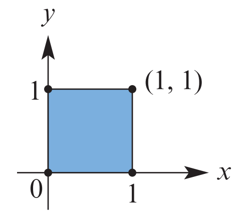
## Transforming the unit square

The **unit square** is the square with vertices (0, 0), (1, 0), (0, 1) and (1, 1). The effect of a linear transformation can often be demonstrated by studying its effect on the unit square.




### Example 3

A linear transformation is represented by the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

**a** Find the image of the unit square under this transformation.

**b** Sketch the unit square and its image.

**Solution**

**a** We could find the images of the four vertices of the square one at a time:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

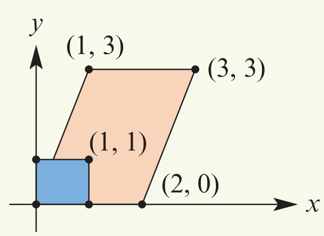
However, this can be done in a single step by multiplying the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex of the square:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

The columns of the result give the images of the vertices:

$$(0, 0), \quad (2, 0), \quad (1, 3), \quad (3, 3)$$

**b** The unit square is shown in blue and its image in red.




## Mapping the standard unit vectors

Let's express the points (1, 0) and (0, 1) as column vectors:

$$(1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

As we saw in Chapter 3, these are the **standard unit vectors** in $\mathbb{R}^2$.

We now consider the images of these points under the transformation with matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$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We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \text{first column of the matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \text{second column of the matrix}$$

To find the matrix of a linear transformation:

- The first column is the image of $(1, 0)$, written as a column vector.
- The second column is the image of $(0, 1)$, written as a column vector.

This observation allows us to write down the matrix of a linear transformation given just two pieces of information.

### Example 4

A linear transformation maps the points $(1, 0)$ and $(0, 1)$ to the points $(1, 1)$ and $(-2, 3)$ respectively.

**a** Find the matrix of the transformation.

**b** Find the image of the point $(-3, 4)$.

**Solution**

**a** $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

*Explanation:* The image of $(1, 0)$ is $(1, 1)$, and the image of $(0, 1)$ is $(-2, 3)$. We write these images as the columns of a matrix.

**b** $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \end{bmatrix}$

Therefore $(-3, 4) \to (-11, 9)$.

*Explanation:* Write the point $(-3, 4)$ as a column vector and multiply by the transformation matrix.

### Summary 12A

- A **transformation** maps each point $(x, y)$ in the plane to a new point $(x', y')$.
- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- The **unit square** has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can be seen by looking at the image of the unit square.
- In the matrix of a linear transformation:
  - the first column is the image of $(1, 0)$, written as a column vector
  - the second column is the image of $(0, 1)$, written as a column vector.


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Skill-sheet

## Exercise 12A

Example 1

**1** Find the image of the point (2, −4) under the transformation with rule:

**a** $(x, y) \to (x + y, x - y)$ **b** $(x, y) \to (2x + 3y, 3x - 4y)$

**c** $(x, y) \to (3x - 5y, x)$ **d** $(x, y) \to (y, -x)$

Example 2

**2** Find the image of the point (2, 3) under the linear transformation with matrix:

**a** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

**3** Find the matrix of the linear transformation defined by the rule:

**a** $(x, y) \to (2x + 3y, 4x + 5y)$ **b** $(x, y) \to (11x - 3y, 3x - 8y)$

**c** $(x, y) \to (2x, x - 3y)$ **d** $(x, y) \to (y, -x)$

Example 3

**4** Find and sketch the image of the unit square under the linear transformation represented by the matrix:

**a** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$

**5** Find the image of the triangle with vertices (1, 1), (1, 2) and (2, 1) under the linear transformation represented by the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

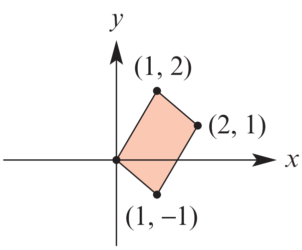
Example 4

**6** Find the matrix of the linear transformation that maps the points (1, 0) and (0, 1) to the points (3, 4) and (5, 6) respectively. Hence find the image of the point (−2, 4).

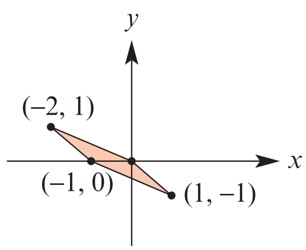
**7** Find the matrix of the linear transformation that maps the points (1, 0) and (0, 1) to the points (−3, 2) and (1, −1) respectively. Hence find the image of the point (2, 3).

**8** Find a matrix that transforms the unit square to each of the following parallelograms.
Note: There are two possible answers for each part.

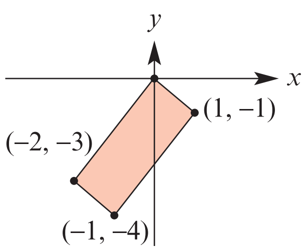
**a**




**b**




**c**





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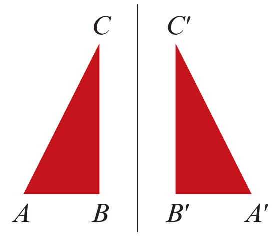
# 12B Geometric transformations

We now look at various important transformations that are geometric in nature.

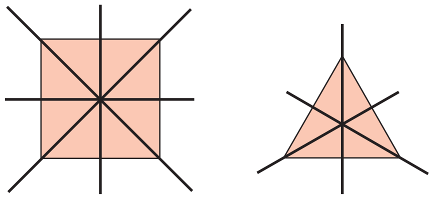
## Reflections




A **reflection** in a line $\ell$ maps each point in the plane to its mirror image on the other side of the line. The point $A$ and its image $A'$ are the same distance from $\ell$ and the line $AA'$ is perpendicular to $\ell$.

These transformations are important for studying figures with **reflective symmetry**, that is, figures that look the same when reflected in a **line of symmetry**.

A square has four lines of symmetry, while an equilateral triangle has just three.



Note: A reflection is an example of a transformation that does not change lengths. Such a transformation is called an **isometry**.

### Reflection in the $x$-axis

A reflection in the $x$-axis is defined by

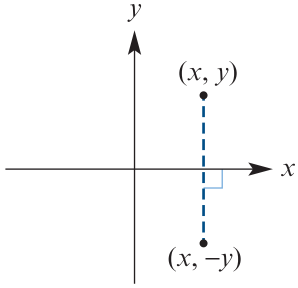
$$(x, y) \to (x, -y)$$

So if $(x', y')$ is the image of the point $(x, y)$, then

$$x' = x \quad \text{and} \quad y' = -y$$

This transformation can also be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$




### Reflection in the $y$-axis

A reflection in the $y$-axis is defined by

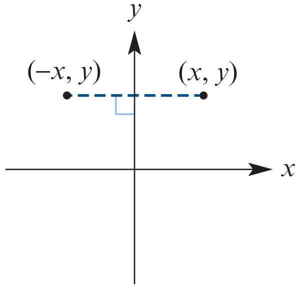
$$(x, y) \to (-x, y)$$

So if $(x', y')$ is the image of the point $(x, y)$, then

$$x' = -x \quad \text{and} \quad y' = y$$

Once again, this transformation can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$





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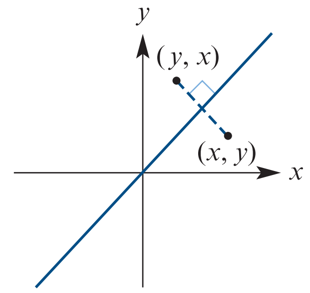
### Reflection in the line $y = x$

If the point $(x, y)$ is reflected in the line $y = x$, then it is mapped to the point $(y, x)$. So if $(x', y')$ is the image of $(x, y)$, then

$$x' = y \quad \text{and} \quad y' = x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



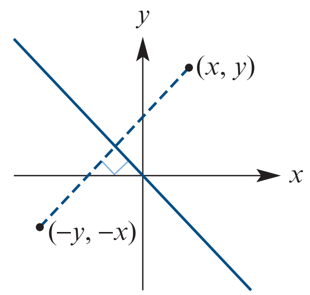

### Reflection in the line $y = -x$

If the point $(x, y)$ is reflected in the line $y = -x$, it is mapped to $(-y, -x)$. So if $(x', y')$ is the image of $(x, y)$, then

$$x' = -y \quad \text{and} \quad y' = -x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$




| Transformation | Rule | Matrix |
|---|---|---|
| Reflection in the $x$-axis | $x' = 1x + 0y$<br>$y' = 0x - 1y$ | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| Reflection in the $y$-axis | $x' = -1x + 0y$<br>$y' = 0x + 1y$ | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Reflection in the line $y = x$ | $x' = 0x + 1y$<br>$y' = 1x + 0y$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| Reflection in the line $y = -x$ | $x' = 0x - 1y$<br>$y' = -1x + 0y$ | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ |

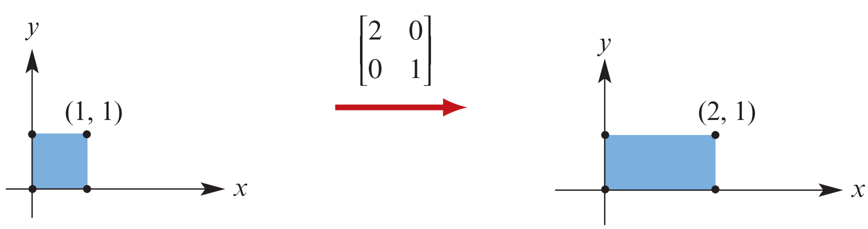
## Dilations

### Dilation parallel to the $x$-axis

A dilation parallel to the $x$-axis is a transformation of the form

$$(x, y) \to (cx, y)$$

where $c > 0$. The $x$-coordinate is scaled by a factor of $c$, but the $y$-coordinate is unchanged.





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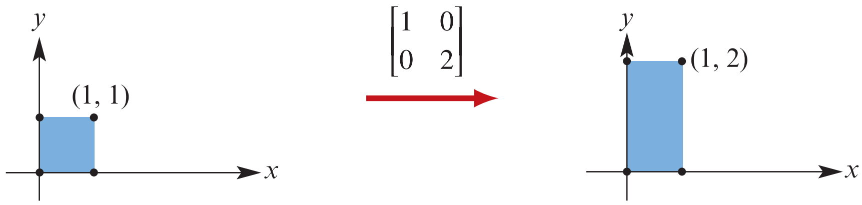
### Dilation parallel to the $y$-axis

Likewise, a dilation parallel to the $y$-axis is a transformation of the form

$$(x, y) \to (x, cy)$$

where $c > 0$. The $y$-coordinate is scaled by a factor of $c$, but the $x$-coordinate is unchanged.




### Dilation from the $x$- and $y$-axes

We can also simultaneously scale along the $x$- and $y$-axes using the transformation

$$(x, y) \to (cx, dy)$$

with scale factors $c > 0$ and $d > 0$.

| Transformation | Rule | Matrix |
|---|---|---|
| Dilation parallel to the $x$-axis | $x' = cx + 0y$<br>$y' = 0x + 1y$ | $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ |
| Dilation parallel to the $y$-axis | $x' = 1x + 0y$<br>$y' = 0x + cy$ | $\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$ |
| Dilation from the $x$- and $y$-axes | $x' = cx + 0y$<br>$y' = 0x + dy$ | $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ |

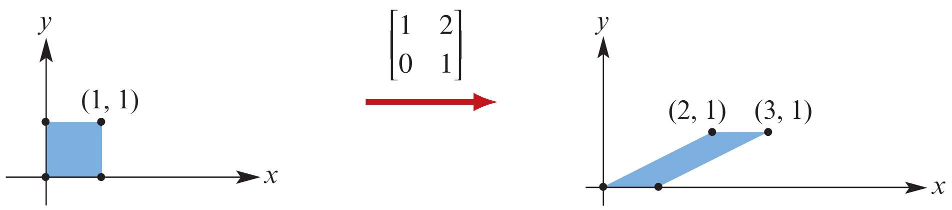
## Shears

### Shear parallel to the $x$-axis

A shear parallel to the $x$-axis is a transformation of the form

$$(x, y) \to (x + cy, y)$$

Notice that each point is moved in the $x$-direction by an amount proportional to the distance from the $x$-axis. This means that the unit square is tilted in the $x$-direction.





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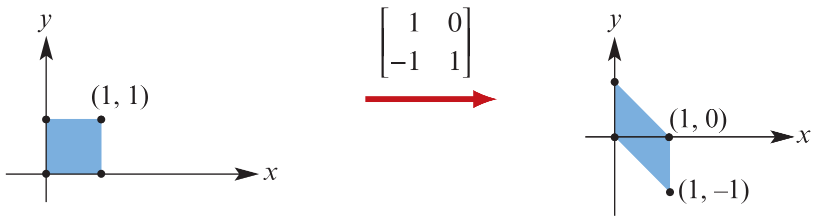
### Shear parallel to the $y$-axis

A shear parallel to the $y$-axis is a transformation of the form

$$(x, y) \to (x, cx + y)$$

Here, each point is moved in the $y$-direction by an amount proportional to the distance from the $y$-axis. Now the unit square is tilted in the $y$-direction.

Note that if $c < 0$, then we obtain a shear in the negative direction.




| Transformation | Rule | Matrix |
|---|---|---|
| Shear parallel to the $x$-axis | $x' = 1x + cy$<br>$y' = 0x + 1y$ | $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ |
| Shear parallel to the $y$-axis | $x' = 1x + 0y$<br>$y' = cx + 1y$ | $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ |

## Projections

The transformation defined by

$$(x, y) \to (x, 0)$$

will project the point $(x, y)$ onto the $x$-axis.

Likewise, the transformation defined by

$$(x, y) \to (0, y)$$

will project the point $(x, y)$ onto the $y$-axis.

| Transformation | Rule | Matrix |
|---|---|---|
| Projection onto the $x$-axis | $x' = 1x + 0y$<br>$y' = 0x + 0y$ | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ |
| Projection onto the $y$-axis | $x' = 0x + 0y$<br>$y' = 0x + 1y$ | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ |

Projections are an important class of transformations. For example, the image on a television screen is the projection of a three-dimensional scene onto a two-dimensional surface.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## Example 5

Find the image of the point (3, 4) under each of the following transformations:

**a** reflection in the $y$-axis

**b** dilation of factor 2 parallel to the $x$-axis

**c** shear of factor 4 parallel to the $x$-axis

**d** projection onto the $y$-axis

### Solution

**a** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$(3, 4) \to (-3, 4)$

**b** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$(3, 4) \to (6, 4)$

**c** $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix}$

$(3, 4) \to (19, 4)$

**d** $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$(3, 4) \to (0, 4)$

## Translations

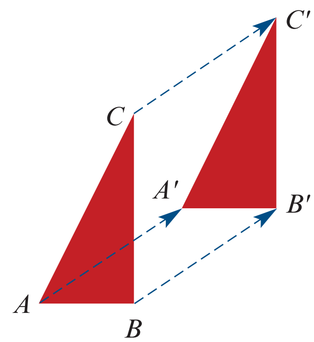
A **translation** moves a figure so that every point in the figure moves in the same direction and over the same distance.

A translation of $a$ units in the $x$-direction and $b$ units in the $y$-direction is defined by the rule

$$(x, y) \to (x + a, y + b)$$

This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$




Note: Translations cannot be represented using matrix multiplication. To see this, note that matrix multiplication will always map the point (0, 0) to itself. Therefore, there is no matrix that will translate the point (0, 0) to $(a, b)$, unless $a = b = 0$.

## Example 6

Find the rule for a translation of 2 units in the $x$-direction and −1 units in the $y$-direction, and sketch the image of the unit square under this translation.

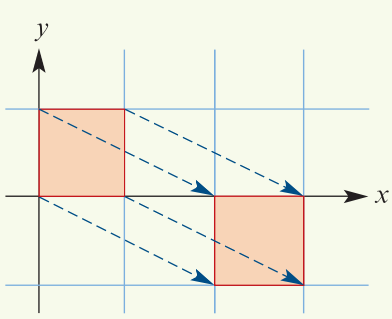
### Solution

Using vector addition, this translation can be defined by the rule

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x + 2 \\ y - 1 \end{bmatrix}$$

or equivalently

$$x' = x + 2 \quad \text{and} \quad y' = y - 1$$





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## Summary 12B

- Important geometric transformation matrices are summarised in the table below.

| Transformation | Matrix | Transformation | Matrix |
|---|---|---|---|
| Reflection in the $x$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | Reflection in the $y$-axis | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Reflection in the line $y = x$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | Reflection in the line $y = -x$ | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ |
| Dilation parallel to the $x$-axis | $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ | Dilation parallel to the $y$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$ |
| Shear parallel to the $x$-axis | $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ | Shear parallel to the $y$-axis | $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ |
| Projection onto the $x$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ | Projection onto the $y$-axis | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ |

- A translation of $a$ units in the $x$-direction and $b$ units in the $y$-direction is defined by the rule $(x, y) \to (x + a, y + b)$. This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Skill-sheet

## Exercise 12B

Example 5

**1** For each of the transformations described below:

**i** find the matrix of the transformation

**ii** sketch the image of the unit square under this transformation.

**a** dilation of factor 2 parallel to the $y$-axis

**b** dilation of factor 3 parallel to the $x$-axis

**c** shear of factor 3 parallel to the $x$-axis

**d** shear of factor $-1$ parallel to the $y$-axis

**e** reflection in the $x$-axis

**f** reflection in the line $y = -x$

Example 6

**2** For each of the translations described below:

**i** find the rule for the translation using column vectors

**ii** sketch the image of the unit square under this translation.

**a** translation of 2 units in the $x$-direction

**b** translation of $-3$ units in the $y$-direction

**c** translation of $-2$ units in the $x$-direction and $-4$ units in the $y$-direction

**d** translation by the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

**e** translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$


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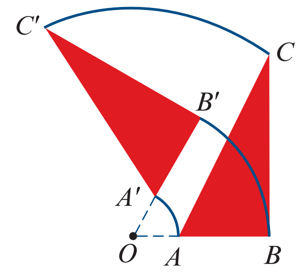
# 12C Rotations and general reflections

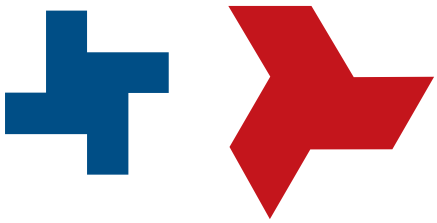
## Rotations




A **rotation** turns an object about a point, but keeps its distance to the point fixed. A rotation does not change lengths, and so is another example of an isometry.

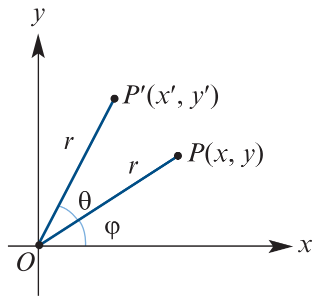
Rotations are important for studying figures with **rotational symmetry**, that is, figures that look the same when rotated through a certain angle.

These two figures have rotational symmetry, but no reflective symmetry.



### Finding the rotation matrix

Consider the transformation that rotates each point in the plane about the origin by θ degrees anticlockwise. We will show that this is a linear transformation and find its matrix.




Let $O$ be the origin and let $P(x, y)$ be a point in the plane.

Then we can write

$$x = r\cos\varphi \quad \text{and} \quad y = r\sin\varphi$$

where $r$ is the distance $OP$ and $\varphi$ is the angle between $OP$ and the positive direction of the $x$-axis.

Now let $P'(x', y')$ be the image of $P(x, y)$ under a rotation about $O$ by angle $\theta$ anticlockwise.

As $OP' = r$, we can use the addition formulas to show that

$$\begin{aligned} x' &= r\cos(\varphi + \theta) \\ &= r\cos\varphi\cos\theta - r\sin\varphi\sin\theta \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

and

$$\begin{aligned} y' &= r\sin(\varphi + \theta) \\ &= r\sin\varphi\cos\theta + r\cos\varphi\sin\theta \\ &= y\cos\theta + x\sin\theta \end{aligned}$$

Writing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$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## Example 7

Find the matrix that represents a rotation of the plane about the origin by:

**a** 90° anticlockwise

**b** 45° clockwise.

**Solution**

**a** $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

**b** $\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

**Explanation**

An anticlockwise rotation means that we let $\theta = 90°$ in the formula for the rotation matrix.

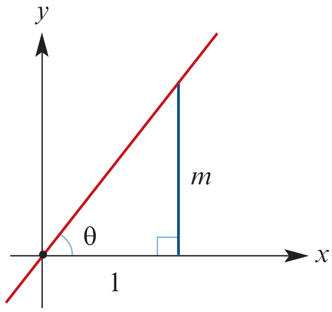
A clockwise rotation means that we let $\theta = -45°$ in the formula for the rotation matrix.

## Reflection in the line $y = mx$

Reflection in a line that passes through the origin is also a linear transformation. We will find the matrix that will reflect the point $(x, y)$ in the line $y = mx$.

Let's suppose that the angle between the positive direction of the $x$-axis and the line $y = mx$ is $\theta$. Then $\tan\theta = m$ and so

$$y = mx = x\tan\theta$$




### Finding the reflection matrix

We will use the fact that the first column of the required matrix will be the image $A$ of $C(1, 0)$, written as a column vector, and the second column will be the image $B$ of $D(0, 1)$, written as a column vector.

Since $\angle AOC = 2\theta$, we have

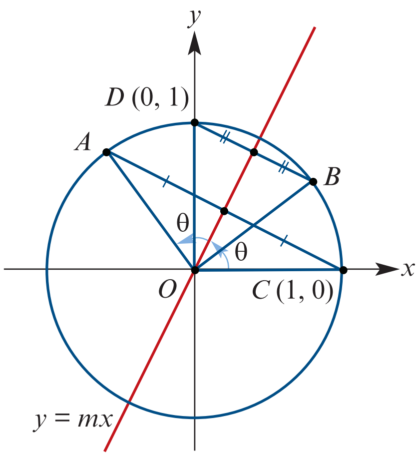
$$(1, 0) \to (\cos(2\theta), \sin(2\theta))$$

Moreover, since $\angle BOC = 2\theta - 90°$, we have

$$(0, 1) \to (\cos(2\theta - 90°), \sin(2\theta - 90°))$$
$$= (\sin(2\theta), -\cos(2\theta))$$

Writing these images as column vectors gives the reflection matrix:

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$





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## Example 8

**a** Find the matrix that will reflect the point $(x, y)$ in the line through the origin at an angle of 30° to the positive direction of the $x$-axis.

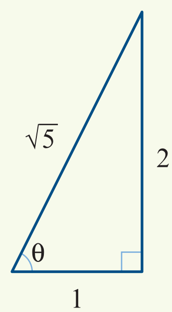
**b** Find the matrix that will reflect the point $(x, y)$ in the line $y = 2x$.

### Solution

**a** We simply let $\theta = 30°$, and so the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$$

**b** Since $\tan\theta = 2 = \frac{2}{1}$, we draw a right-angled triangle with opposite and adjacent lengths 2 and 1 respectively.




Pythagoras' theorem gives the hypotenuse as $\sqrt{5}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \sin\theta = \frac{2}{\sqrt{5}}$$

We then use the double angle formulas to show that

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = \frac{4}{5}$$

Therefore the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

## Summary 12C

### Rotation matrix

The matrix that will rotate the plane about the origin by angle $\theta$ anticlockwise is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

### Reflection matrix

The matrix that will reflect the plane in the line $y = mx = x\tan\theta$ is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

## Exercise 12C

Example 7

**1** Find the matrix for each of the following rotations about the origin:

**a** 270° anticlockwise
**b** 30° anticlockwise
**c** 60° clockwise
**d** 135° clockwise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**2** Find the image of the point $(2, 3)$ under a rotation about the origin by:

**a** $90°$ anticlockwise **b** $45°$ clockwise.

Example 8a

**3** Find the matrix that will reflect the point $(x, y)$ in the line through the origin that makes an angle with the positive direction of the $x$-axis of:

**a** $45°$ **b** $60°$ **c** $-30°$ **d** $15°$

Example 8b

**4** Find the matrix that will reflect the point $(x, y)$ in the line:

**a** $y = 3x$ **b** $y = 5x$ **c** $y = \frac{2x}{3}$ **d** $y = -3x$

**5** **a** Find a formula (in terms of $m$) for the matrix that will reflect the point $(x, y)$ in the line $y = mx$.

**b** Hence find the image of $(1, 1)$ when reflected in the line $y = 6x$.

**6** The unit square is rotated about the origin by $45°$ anticlockwise.

**a** Find the matrix of this transformation.

**b** Draw the unit square and its image on the same set of axes.

**c** Find the area of the overlapping region.

**7** The point $A(1, 0)$ is rotated to point $B$ through angle $120°$ anticlockwise. The point $A(1, 0)$ is also rotated to point $C$ through angle $240°$ anticlockwise.

**a** Find the coordinates of $B$ and $C$.

**b** What sort of triangle is $ABC$?

**c** Triangle $ABC$ has three lines of symmetry. Find the equations of all three lines.

# 12D Composition of transformations

## Composition as matrix multiplication

The matrix representation of linear transformations makes it easy to find the effect of one transformation followed by another. Suppose that we would like to:

**1** reflect the point $(x, y)$ in the $y$-axis

**2** then dilate the result by a factor of 2 parallel to the $y$-axis.

To reflect the point $(x, y)$, we would first evaluate

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

To dilate the result by a factor of 2 parallel to the $y$-axis, we would then multiply

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore the matrix that will achieve both transformations is the product

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$


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If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

**Example 9**

Find the matrix that corresponds to:

**a** a reflection in the $x$-axis and then a rotation about the origin by 90° anticlockwise

**b** a rotation about the origin by 90° anticlockwise and then a reflection in the $x$-axis.

**Solution**

**a** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

**Explanation**

A reflection in the $x$-axis has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

An anticlockwise rotation by 90° has matrix

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We then multiply these two matrices together in the correct order.

Note: In this example, we get a different matrix when the same two transformations take place in reverse order. This should not be a surprise, as matrix multiplication is not commutative in general.

## Compositions involving translations

**Example 10**

**a** Find the rule for the transformation that will reflect $(x, y)$ in the $x$-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

**b** Find the rule for the transformation if the translation takes place before the reflection.

**Solution**

**a**
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} x - 3 \\ -y + 4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x - 3, -y + 4)$.

**b**
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x - 3 \\ y + 4 \end{bmatrix}$$
$$= \begin{bmatrix} x - 3 \\ -y - 4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x - 3, -y - 4)$.


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## Summary 12D

If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

The order is important, as matrix multiplication is not commutative in general.

Skill-sheet

## Exercise 12D

Example 9

**1** Find the matrix that represents a reflection in the $y$-axis followed by a dilation of factor 3 parallel to the $y$-axis.

**2** Find the matrix that represents a rotation about the origin by 90° anticlockwise followed by a reflection in the $x$-axis.

**3** **a** Find the matrix that represents a reflection in the $x$-axis followed by a reflection in the $y$-axis.

**b** Show that this matrix corresponds to a rotation about the origin by 180°.

**4** Consider these two transformations:

- $T_1$: A reflection in the $x$-axis.
- $T_2$: A dilation of factor 2 parallel to the $x$-axis.

**a** Find the matrix of $T_1$ followed by $T_2$. **b** Find the matrix of $T_2$ followed by $T_1$.

**c** Does the order of transformation matter in this instance?

**5** Consider these two transformations:

- $T_1$: A rotation about the origin by 90° clockwise.
- $T_2$: A reflection in the line $y = x$.

**a** Find the matrix of $T_1$ followed by $T_2$. **b** Find the matrix of $T_2$ followed by $T_1$.

**c** Does the order of transformation matter in this instance?

Example 10

**6** Consider these two transformations:

- $T_1$: A reflection in the $y$-axis.
- $T_2$: A translation of $-3$ units in the $x$-direction and 5 units in the $y$-direction.

**a** Find the rule for $T_1$ followed by $T_2$. **b** Find the rule for $T_2$ followed by $T_1$.

**c** Does the order of transformation matter in this instance?

**7** Express each of the following transformation matrices as the product of a dilation matrix and a reflection matrix:

**a** $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$


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**8** **a** Find the matrix for the transformation that is a reflection in the $x$-axis followed by a reflection in the line $y = x$.

**b** Show that these two reflections can be achieved with one rotation.

**9** Suppose that matrix **A** gives a rotation about the origin by $\theta$ degrees anticlockwise and that matrix **B** gives a reflection in the line $y = x$. If **AB** = **BA**, find the angle $\theta$.

**10** Suppose that matrix **A** rotates the plane about the origin by angle $\theta$ anticlockwise.

**a** Through what angle will the matrix $\mathbf{A}^2$ rotate the plane?

**b** Evaluate $\mathbf{A}^2$.

**c** Hence find formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

**11** A transformation $T$ consists of a reflection in the line $y = x$ followed by a translation by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

**a** Find the rule for the transformation $T$.

**b** Show that the transformation $T$ can also be obtained by a translation and then a reflection in the line $y = x$. Find the translation vector.

**12** **a** Find the rotation matrix for an angle of 60° anticlockwise.

**b** Find the rotation matrix for an angle of 45° clockwise.

**c** By multiplying these two matrices, find the rotation matrix for an angle of 15° anticlockwise.

**d** Hence write down the exact values of sin 15° and cos 15°.

**13** A transformation consists of a reflection in the line $y = x\tan\varphi$ and then in the line $y = x\tan\theta$. Show that this is equivalent to a single rotation.

# 12E Inverse transformations

If transformation $T$ maps the point $(x, y)$ to the point $(x', y')$, then the **inverse transformation** $T^{-1}$ maps the point $(x', y')$ to the point $(x, y)$.

For a linear transformation $T$, we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{X}' = \mathbf{AX}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, then we have

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{X}' &= \mathbf{A}^{-1}\mathbf{AX} \\ \mathbf{A}^{-1}\mathbf{X}' &= \mathbf{IX} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{X}' \end{aligned}$$

Therefore $\mathbf{A}^{-1}$ is the matrix of the inverse transformation $T^{-1}$.


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If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse exists if and only if $\det(\mathbf{A}) = ad - bc \neq 0$.

## Example 11

Find the inverse of the transformation with rule $(x, y) \to (3x + 2y, 5x + 4y)$.

### Solution

Since the matrix of this linear transformation is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

the inverse transformation will have matrix

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3 \times 4 - 2 \times 5}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \end{aligned}$$

Therefore the rule of the inverse transformation is $(x, y) \to \left(2x - y, -\frac{5}{2}x + \frac{3}{2}y\right)$.

## Example 12

Find the matrix of the linear transformation such that $(4, 3) \to (9, 10)$ and $(2, 1) \to (5, 6)$.

### Solution

We need to find a matrix $\mathbf{A}$ such that

$$\mathbf{A}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This can be written as a single equation:

$$\mathbf{A}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}$$

Therefore

$$\mathbf{A} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$


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## Inverses of important transformations

For important geometric transformations, it is often obvious what the inverse transformation should be.

### Example 13

Let **R** be the matrix corresponding to a rotation of the plane by angle θ anticlockwise. Show that $\mathbf{R}^{-1}$ corresponds to a rotation by angle θ clockwise.

**Solution**

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

This matrix corresponds to a rotation of the plane by angle θ clockwise.

**Explanation**

We find the inverse matrix using the formula

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We also use the symmetry properties:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

The following table summarises the important geometric transformations along with their inverses. You will demonstrate some of these results in the exercises.

| Transformation | Matrix **A** | Inverse matrix $\mathbf{A}^{-1}$ | Inverse transformation |
|---|---|---|---|
| Dilation parallel to the $x$-axis | $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$ | Dilation parallel to the $x$-axis |
| Dilation parallel to the $y$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$ | Dilation parallel to the $y$-axis |
| Shear parallel to the $x$-axis | $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ | Shear parallel to the $x$-axis |
| Rotation by θ anticlockwise | $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ | $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$ | Rotation by θ clockwise |
| Reflection in the $x$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | Reflection in the $x$-axis |
| Reflection in the $y$-axis | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ | Reflection in the $y$-axis |
| Reflection in the line $y = mx$ | $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ | $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ | Reflection in the line $y = mx$ |


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## Summary 12E

If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## Exercise 12E

**1** Find the inverse matrix of each of the following transformation matrices:

**a** $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$

**c** $\begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} -1 & 3 \\ -4 & 5 \end{bmatrix}$

Example 11

**2** For each of the following transformations, find the rule for their inverse:

**a** $(x, y) \to (5x - 2y, 2x - y)$ **b** $(x, y) \to (x - y, x)$

**3** Find the point $(x, y)$ that is mapped to $(1, 1)$ by the transformation with matrix:

**a** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

Example 12

**4** Find the matrix of the linear transformation such that $(1, 2) \to (2, 1)$ and $(2, 3) \to (1, 1)$.

**5** Find the vertices of the rectangle that is mapped to the unit square by the transformation with matrix $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

Example 13

**6** Consider a dilation of factor $k$ parallel to the $x$-axis, where $k > 0$.

**a** Write down the matrix of this transformation.

**b** Show that the inverse matrix corresponds to a dilation of factor $\frac{1}{k}$ parallel to the $x$-axis.

**7** Consider a shear of factor $k$ parallel to the $x$-axis.

**a** Write down the matrix of this transformation.

**b** Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the $x$-axis.

**8** Consider the transformation that reflects each point in the $x$-axis.

**a** Write down the matrix $\mathbf{A}$ of this transformation.

**b** Show that $\mathbf{A}^{-1} = \mathbf{A}$, and explain why you should expect this result.


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**9** Consider the transformation that reflects each point in the line $y = mx = x \tan\theta$.

**a** Write down the matrix **B** of this transformation.

**b** Show that $\mathbf{B}^{-1} = \mathbf{B}$, and explain why you should expect this result.

# 12F Transformations of straight lines and other graphs

We have considered the effect of various transformations on points and figures in the plane. We will now turn our attention to graphs.

Here, we will aim to find the equations of transformed graphs. We will also investigate the effects of linear transformations on straight lines. You will study this application in much greater detail in Mathematics Methods.

## Linear transformations of straight lines

We will first investigate the effect of linear transformations on straight lines.

### Example 14

Find the equation of the image of the line $y = 2x + 3$ under a reflection in the $x$-axis followed by a dilation of factor 2 parallel to the $x$-axis.

**Solution**

The matrix of the combined transformation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

If $(x', y')$ are the coordinates of the image of $(x, y)$, then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$$

Therefore

$$x' = 2x \quad \text{and} \quad y' = -y$$

Rearranging gives

$$x = \frac{x'}{2} \quad \text{and} \quad y = -y'$$

Therefore the equation $y = 2x + 3$ becomes

$$-y' = 2\left(\frac{x'}{2}\right) + 3$$

$$-y' = x' + 3$$

$$y' = -x' - 3$$

We now ignore the dashes, and so the equation of the image is simply

$$y = -x - 3$$


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## Example 15

Consider the graph of $y = x + 1$. Find the equation of its image under the linear transformation $(x, y) \to (x + 2y, y)$.

### Solution

Let $(x', y')$ be the coordinates of the image of $(x, y)$. Then this transformation can be written in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x' - 2y' \\ y' \end{bmatrix}$$

and so $x = x' - 2y'$ and $y = y'$.

The equation $y = x + 1$ becomes

$$\begin{aligned} y' &= x' - 2y' + 1 \\ 3y' &= x' + 1 \\ y' &= \frac{x'}{3} + \frac{1}{3} \end{aligned}$$

The equation of the image is $y = \dfrac{x}{3} + \dfrac{1}{3}$.

In the previous two examples, you will have noticed that the image of each straight line was another straight line. In fact, linear transformations get their name in part from the following fact, which is proved in the exercises.

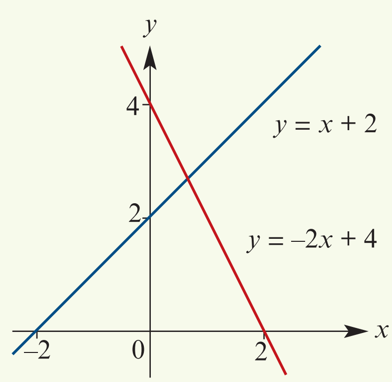
The image of any straight line under an invertible linear transformation is a straight line.

## Example 16

Find a matrix that transforms the line $y = x + 2$ to the line $y = -2x + 4$.

### Solution

Let's find the matrix that maps the $x$-axis intercept of the first line to the $x$-axis intercept of the second line, and likewise for the $y$-axis intercepts.




We want

$$(-2, 0) \to (2, 0) \quad \text{and} \quad (0, 2) \to (0, 4)$$

This can be achieved by a reflection in the $y$-axis and then a dilation of factor 2 parallel to the $y$-axis.

This transformation has the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.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## Transformations of other graphs

The method for finding the image of a straight line can be used for other graphs.

### Example 17

Find the image of the graph of $y = x^2 + 1$ under a translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the $y$-axis.

**Solution**

Let $(x', y')$ be the image of $(x, y)$. Then the transformation is given by

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x+2 \\ y-1 \end{bmatrix} = \begin{bmatrix} -x-2 \\ y-1 \end{bmatrix}\end{aligned}$$

Therefore $x' = -x - 2$ and $y' = y - 1$.

This gives $x = -x' - 2$ and $y = y' + 1$.

The equation $y = x^2 + 1$ becomes

$$\begin{aligned} y' + 1 &= (-x' - 2)^2 + 1 \\ y' &= (-x' - 2)^2 \\ &= (x' + 2)^2 \end{aligned}$$

The equation of the image is $y = (x + 2)^2$.

### Example 18

Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 2 parallel to the $x$-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

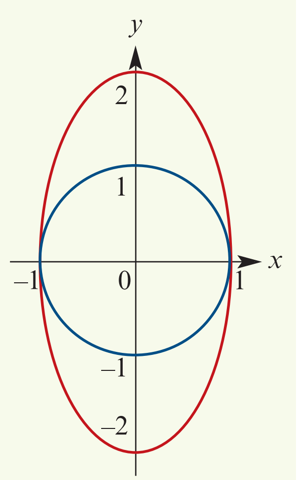
**Solution**

The dilation matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

The rotation matrix is $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let $(x', y')$ be the image of $(x, y)$. Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 2x \end{bmatrix}$$




Thus $x' = -y$ and $y' = 2x$, giving $y = -x'$ and $x = \dfrac{y'}{2}$.

The equation $x^2 + y^2 = 1$ becomes $\left(\dfrac{y'}{2}\right)^2 + (-x')^2 = 1$.

Hence the image is the ellipse with equation $x^2 + \dfrac{y^2}{2^2} = 1$.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## Exercise 12F

Example 14

**1** Find the equation of the image of the graph of $y = 3x + 1$ under:

**a** a reflection in the $x$-axis

**b** a dilation of factor 2 parallel to the $x$-axis

**c** a dilation of factor 3 parallel to the $y$-axis and factor 2 parallel to the $x$-axis

**d** a reflection in the $x$-axis and then in the $y$-axis

**e** a reflection in the $y$-axis and then a dilation of factor 3 parallel to the $y$-axis

**f** a rotation about the origin by 90° anticlockwise

**g** a rotation about the origin by 90° clockwise and then a reflection in the $x$-axis.

Example 15

**2** Find the image of $y = 2 - 3x$ under each of the following transformations:

**a** $(x, y) \to (2x, 3y)$

**b** $(x, y) \to (-y, x)$

**c** $(x, y) \to (x - 2y, y)$

**d** $(x, y) \to (3x + 5y, x + 2y)$

Example 16

**3** Find a matrix that transforms the line $x + y = 1$ to the line $x + y = 2$.

**4** Find a matrix that transforms the line $y = x + 1$ to the line $y = 6 - 2x$.

Example 17

**5** Find the equation of the image of the graph of $y = x^2 - 1$ under a translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and then a reflection in the $x$-axis.

**6** Find the equation of the image of the graph of $y = (x - 1)^2$ under a reflection in the $y$-axis and then a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Example 18

**7** Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 3 parallel to the $y$-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

**8** Consider any invertible linear transformation

$$(x, y) \to (ax + by, cx + dy)$$

Show that the image of the straight line $px + qy = r$ is a straight line.

**9** Rotate the graph of $y = \dfrac{1}{x}$ by 45° anticlockwise. Show that the equation of the image is $y^2 - x^2 = 2$.

Note: This shows that the two curves are congruent hyperbolas.


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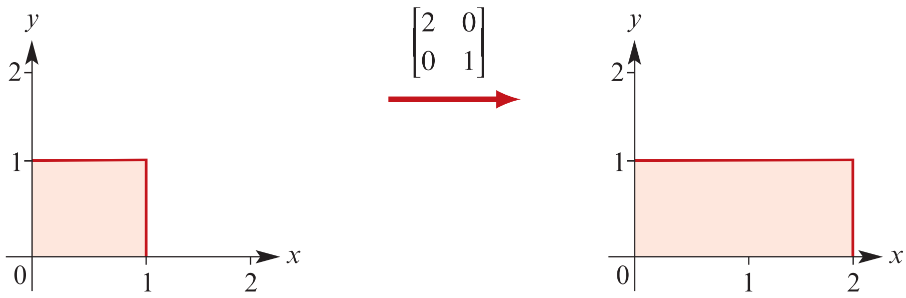
# 12G Area and determinant

If we apply a linear transformation to some region of the plane, then the area may change.

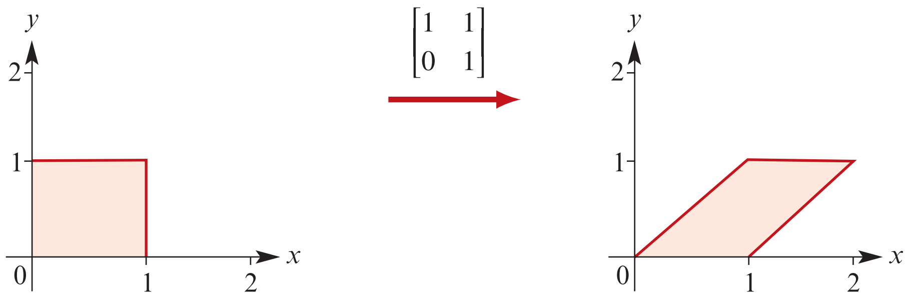
For example, if we dilate the unit square by a factor of 2 parallel to the $x$-axis, then the area increases by a factor of 2.



Notice that this increase corresponds to the determinant of the transformation matrix:

$$\det\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

On the other hand, if we shear the unit square by a factor of 1 parallel to the $x$-axis, then the area is unchanged.



Notice that the determinant of this transformation matrix is

$$\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

More generally, we can prove the following remarkable result.

> If a region of the plane is transformed by matrix **B**, then
>
> Area of image = $|\det(\mathbf{B})| \times$ Area of region

**Proof** We will prove the result when the unit square is transformed by matrix

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The result can be extended to other regions by approximating them by squares.

We will assume that $a$, $b$, $c$ and $d$ are all positive and that $\det(\mathbf{B}) > 0$. The proof can easily be adapted if we relax these assumptions.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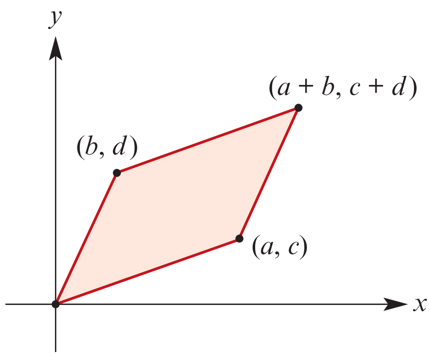
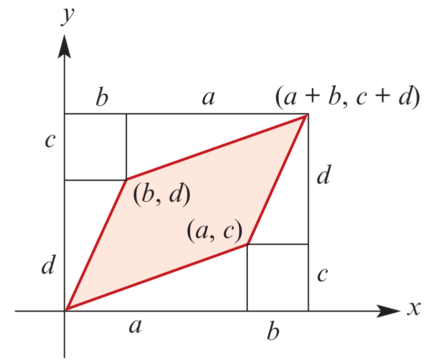
The image of the unit square under transformation **B** is a parallelogram.




To find the area of the image, we draw a rectangle around it as shown, and subtract the area of the two small rectangles and four triangles from the total area:

$$\begin{aligned}\text{Area of image} &= (a+b)(c+d) - bc - bc - \frac{ac}{2} - \frac{ac}{2} - \frac{bd}{2} - \frac{bd}{2}\\ &= (a+b)(c+d) - 2bc - ac - bd\\ &= ac + ad + bc + bd - 2bc - ac - bd\\ &= ad - bc\end{aligned}$$

This is equal to the determinant of matrix **B**.

## Example 19

The triangular region with vertices (1, 1), (2, 1) and (1, 2) is transformed by the rule $(x, y) \to (-x + 2y, 2x + y)$.

**a** Find the matrix of the linear transformation.

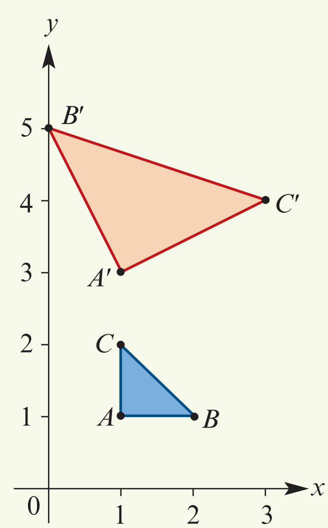
**b** On the same set of axes, sketch the region and its image.

**c** Find the area of the image.

### Solution

**a** The matrix is given by $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

**b** The region is shown in blue and its image in red.




**c** The area of the original region is $\frac{1}{2}$.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = (-1) \times 1 - 2 \times 2 = -5$$

Therefore

$$\begin{aligned}\text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region}\\ &= |-5| \times \frac{1}{2} = \frac{5}{2}\end{aligned}$$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## Example 20

The unit square is mapped to a parallelogram of area 3 by the matrix

$$\mathbf{B} = \begin{bmatrix} m & 2 \\ m & m \end{bmatrix}$$

Find the possible values of $m$.

### Solution

The original area is 1. Therefore

$$\begin{aligned} \text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region} \\ 3 &= |ad - bc| \times 1 \\ 3 &= |m^2 - 2m| \end{aligned}$$

Therefore either $m^2 - 2m = 3$ or $m^2 - 2m = -3$.

**Case 1:**

$$\begin{aligned} m^2 - 2m &= 3 \\ m^2 - 2m - 3 &= 0 \\ (m+1)(m-3) &= 0 \\ m = -1 &\text{ or } m = 3 \end{aligned}$$

**Case 2:**

$$\begin{aligned} m^2 - 2m &= -3 \\ m^2 - 2m + 3 &= 0 \end{aligned}$$

This quadratic equation has no solutions, since the discriminant is

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(3) = 4 - 12 < 0 \end{aligned}$$

The connection between area and determinant has many important applications. In the next example, we see how it can be used to find the area of an ellipse. Alternative approaches to finding this area are much more sophisticated.

## Example 21

The circle with equation $x^2 + y^2 = 1$ is mapped to an ellipse by the rule $(x, y) \to (ax, by)$, where both $a$ and $b$ are positive.

**a** Find the equation of the ellipse and sketch its graph.

**b** Find the area of the ellipse.

### Solution

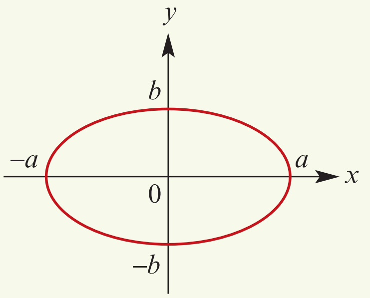
**a** We have $x' = ax$ and $y' = by$.

This gives $x = \dfrac{x'}{a}$ and $y = \dfrac{y'}{b}$.

The equation $x^2 + y^2 = 1$ becomes

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

Hence the equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.





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**b** The area of the original circle of radius 1 is π.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \times b - 0 \times 0 = ab$$

Therefore the area of the ellipse is $\pi ab$.

Note: When $a = b = r$, this formula gives the area of a circle of radius $r$.

## Summary 12G

If a region of the plane is transformed by matrix **B**, then

Area of image = $|\det(\mathbf{B})| \times$ Area of region

## Exercise 12G

**1** Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

**a** $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

Example 19

**2** The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ maps the triangle with vertices (0, 1), (1, 1) and (0, 0) to a new triangle.

**a** Sketch the original triangle and its image.

**b** Find the areas of both triangles.

**3** The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ maps the triangle with vertices (−1, 1), (1, 1) and (1, 0) to a new triangle.

**a** Sketch the original triangle and its image.

**b** Find the areas of both triangles.

Example 20

**4** The matrix $\begin{bmatrix} m & 2 \\ -1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 6 square units. Find the value(s) of $m$.

**5** The matrix $\begin{bmatrix} m & m \\ 1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 2 square units. Find the value(s) of $m$.


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**6** **a** By evaluating a determinant, show that each of the following transformations will not change the area of any region:

**i** a shear of factor $k$ parallel to the $x$-axis

**ii** an anticlockwise rotation about the origin by angle $\theta$

**iii** a reflection in any straight line through the origin

Note: We say that each of these transformations **preserves area**.

**b** Let $k > 0$. A linear transformation has matrix $\begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$.

**i** Describe the geometric effect of the transformation.

**ii** Show that this transformation preserves area.

**7** For each $x \in \mathbb{R}$, the matrix $\begin{bmatrix} x & 1 \\ -2 & x+2 \end{bmatrix}$ maps the unit square to a parallelogram.

**a** Show that the area of the parallelogram is $(x+1)^2 + 1$.

**b** For what value of $x$ is the area of the parallelogram a minimum?

**8** For what values of $m$ does the matrix $\begin{bmatrix} m & 2 \\ 3 & 4 \end{bmatrix}$ map the unit square to a parallelogram of area greater than 2?

**9** Find all matrices that will map the unit square to rhombus of area $\frac{1}{2}$ with one vertex at $(0, 0)$ and another at $(1, 0)$.

**10** **a** Find a matrix that transforms the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ to the triangle with vertices $(0, 0)$, $(a, c)$ and $(b, d)$.

**b** Hence show that the area of the triangle with vertices $(0, 0)$, $(a, c)$ and $(b, d)$ is given by the formula

$$A = \frac{1}{2}|ad - bc|$$

**c** Hence prove that if $a$, $b$, $c$ and $d$ are rational numbers, then the area of this triangle is rational.

**d** A **rational point** has coordinates $(x, y)$ such that both $x$ and $y$ are rational numbers. Prove that no equilateral triangle can be drawn in the Cartesian plane so that all three of its vertices are rational points.

Hint: You can assume that the vertices of the triangle are $(0, 0)$, $(a, c)$ and $(b, d)$. Find another expression for the area of the triangle using Pythagoras' theorem. You can also assume that $\sqrt{3}$ is irrational.


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# 12H General transformations

Earlier in this chapter we considered rotations about the origin. But what if we want to rotate a figure about a point that is not the origin? In this section we will see how a more complicated transformation can be achieved by a sequence of simpler transformations.

## Rotation about the point $(a, b)$

If we want to rotate the plane about the point $(a, b)$ by $\theta$ degrees anticlockwise, we can do this in a sequence of three steps:

**Step 1** Translate the plane so that the centre of rotation is now the origin, by adding $\begin{bmatrix} -a \\ -b \end{bmatrix}$.

**Step 2** Rotate the plane through angle $\theta$ anticlockwise, by multiplying by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

**Step 3** Translate the plane back to its original position, by adding $\begin{bmatrix} a \\ b \end{bmatrix}$.

Chaining these three transformations together gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

### Example 22

**a** Find the transformation that rotates the plane by 90° anticlockwise about the point (1, 1).

**b** Check your answer by showing that (0, 1) is mapped to the correct point.

**Solution**

**a** We do this in a sequence of three steps, starting with the initial point $(x, y)$:

| Initial point | Translate | Rotate 90° anticlockwise | Translate back |
| --- | --- | --- | --- |
| $\begin{bmatrix} x \\ y \end{bmatrix}$ | $\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ |

This gives the overall transformation

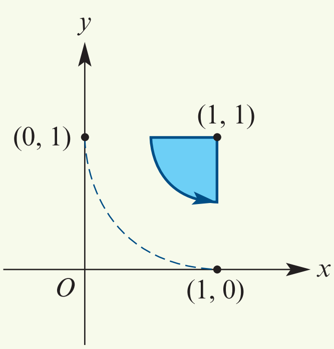
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -y+2 \\ x \end{bmatrix}$$

**b** We check our answer by finding the image of (0, 1).

Let $x = 0$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1+2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore $(0, 1) \to (1, 0)$, as expected.





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## Reflection in the line $y = x \tan\theta + c$

To reflect the plane in a line $y = x\tan\theta + c$ that does not go through the origin, we can also do this in a sequence of three steps:

**Step 1** Translate the plane so that the line passes through the origin, by adding $\begin{bmatrix} 0 \\ -c \end{bmatrix}$.

**Step 2** Reflect the plane in the line $y = x\tan\theta$, by multiplying by $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

**Step 3** Translate the plane back to its original position, by adding $\begin{bmatrix} 0 \\ c \end{bmatrix}$.

### Example 23

**a** Find the transformation that reflects the plane in the line $y = -x + 1$.

**b** Check your answer by finding the image of the point $(1, 1)$.

**Solution**

**a** We do this in a sequence of three steps, starting with the initial point $(x, y)$. The first step translates the line $y = -x + 1$ so that it passes through the origin.

| Initial point | Translate | Reflect in line $y = -x$ | Translate back |
|---|---|---|---|
| $\begin{bmatrix} x \\ y \end{bmatrix}$ | $\begin{bmatrix} x \\ y-1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ |

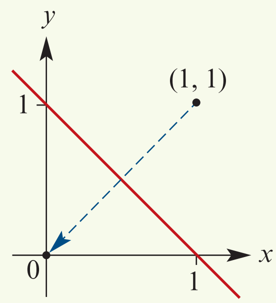
This gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -y+1 \\ -x+1 \end{bmatrix}$$

**b** We check our answer by finding the image of $(1, 1)$.
Let $x = 1$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $(1, 1) \to (0, 0)$, as expected.




### Summary 12H

More difficult transformations can be achieved by combining simpler transformations.

- To rotate the plane about the point $(a, b)$:
  1. Translate the plane so that the origin is the centre of rotation.
  2. Rotate the plane about the origin.
  3. Translate the plane back to its original position.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- To reflect the plane in the line $y = mx + c$:
  1. Translate the plane so that the line passes through the origin.
  2. Reflect the plane in the line $y = mx$.
  3. Translate the plane back to its original position.

## Exercise 12H

Example 22

1 Find the transformation that rotates the plane by 90° clockwise about the point $(2, 2)$. Check your answer by showing that the point $(2, 1)$ is mapped to the correct point.

2 Find the transformation that rotates the plane by 180° anticlockwise about the point $(-1, 1)$. Check your answer by showing that the point $(-1, 0)$ is mapped to the correct point.

Example 23

3 Find the transformation that reflects the plane in each of the following lines. Check your answer by showing that the point $(0, 0)$ is mapped to the correct point.

a $y = x - 1$

b $y = -x - 1$

c $y = 1$

d $x = -2$

4 a Write down the matrix **A** for a rotation about the origin by angle $\theta$ clockwise.

b Write down the matrix **B** for a dilation of factor $k$ parallel to the $y$-axis.

c Write down the matrix **C** for a rotation about the origin by angle $\theta$ anticlockwise.

d Hence find the matrix that increases the perpendicular distance from the line $y = x \tan\theta$ by a factor of $k$.

5 Find the transformation matrix that projects the point $(x, y)$ onto the line $y = x \tan\theta$.
Hint: First rotate the plane clockwise by angle $\theta$.

6 Consider these two transformations:

- $T_1$: A reflection in the line $y = x + 1$.
- $T_2$: A reflection in the line $y = x$.

Show that $T_1$ followed by $T_2$ is a translation.


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## Chapter summary

Assign-ment

Nrich

- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The point $(x', y')$ is called the **image** of the point $(x, y)$.

- The matrix of a composition of two linear transformations can be found by multiplying the two transformation matrices in the correct order.
- If **A** is the matrix of a linear transformation, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.
- If a region of the plane is transformed by matrix **B**, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

- Difficult transformations can be achieved by combining simpler transformations.

| Transformation | Matrix | Transformation | Matrix |
|---|---|---|---|
| Reflection in the $x$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | Reflection in the $y$-axis | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Reflection in the line $y = x$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | Reflection in the line $y = -x$ | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ |
| Dilation parallel to the $x$-axis | $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ | Dilation parallel to the $y$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$ |
| Shear parallel to the $x$-axis | $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ | Shear parallel to the $y$-axis | $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ |
| Projection onto the $x$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ | Projection onto the $y$-axis | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Rotation by $\theta$ anticlockwise | $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ | Reflection in the line $y = x\tan\theta$ | $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ |

## Short-answer questions

**1** The rule for a transformation is $(x, y) \to (2x + y, -x + 2y)$.

**a** Find the image of the point $(2, 3)$.

**b** Find the matrix of this transformation.

**c** Sketch the image of the unit square and find its area.

**d** Find the rule for the inverse transformation.


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Review

**2** Find the matrix corresponding to each of the following linear transformations:

**a** reflection in the $y$-axis
**b** dilation of factor 5 parallel to the $y$-axis
**c** shear of factor $-3$ parallel to the $x$-axis
**d** projection onto the $x$-axis
**e** rotation by $30°$ anticlockwise
**f** reflection in the line $y = x$

**3** **a** Find the matrix that will reflect the plane in the line $y = 3x$.
**b** Find the image of the point $(2, 4)$ under this transformation.

**4** Find the transformation matrix that corresponds to:

**a** a reflection in the $x$-axis and then a reflection in the line $y = -x$
**b** a rotation about the origin by $90°$ anticlockwise and then a dilation of factor 2 parallel to the $y$-axis
**c** a reflection in the line $y = x$ and then a shear of factor 2 parallel to the $y$-axis.

**5** **a** Find the rule for the transformation that will reflect $(x, y)$ in the $x$-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
**b** Find the rule for the transformation if the translation takes place before the reflection.

**6** **a** Write down the matrix for a shear of factor $k$ parallel to the $y$-axis.
**b** Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the $y$-axis.

**7** Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

**a** $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$  **b** $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

**8** **a** Find the rule for the transformation that rotates the plane about the point $(1, -1)$ by $90°$ anticlockwise. (Hint: Translate the point $(1, -1)$ to the origin, rotate the plane, and then translate the point back to its original position.)
**b** Find the image of the point $(2, -1)$ under this transformation.
**c** Sketch the unit square and its image under this transformation.


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## Extended-response questions

**1** **a** Find the matrix that will rotate the plane by 45° anticlockwise.

**b** Find the matrix that will rotate the plane by 30° anticlockwise.

**c** Hence find the matrix that will rotate the plane by 75° anticlockwise.

**d** Hence deduce exact values for cos 75° and sin 75°.

**2** The triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$ is transformed by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

**a** Sketch the triangle and its image on the same set of axes.

**b** Find the area of the triangle and its image.

**c** The image of the triangle is revolved around the $y$-axis to create a three-dimensional solid. Find the volume of this solid.

**3** Consider the transformation with rule $(x, y) \to (x + y, y)$.

**a** Write down the matrix of this transformation.

**b** What name is given to this type of transformation?

**c** Find the images of the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ under this transformation.

**d** Hence sketch the graph of $y = x^2$ and its image under this transformation.

**4** A square with vertices $(\pm 1, \pm 1)$ is rotated about the origin by 45° anticlockwise.

**a** Find the coordinates of the vertices of its image.

**b** Sketch the square and its image on the same set of axes.

**c** When these two squares are combined, the resulting figure is called a Star of Lakshmi. Find its area.

**5** In this chapter we investigated two important transformation matrices. These were the rotation and reflection matrices, which we will now denote by

$$\text{Rot}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \text{Ref}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

**a** Using matrix multiplication and an application of trigonometric identities, prove the following four matrix equations:

**i** $\text{Rot}(\theta)\,\text{Rot}(\varphi) = \text{Rot}(\theta + \varphi)$

**ii** $\text{Ref}(\theta)\,\text{Ref}(\varphi) = \text{Rot}(2\theta - 2\varphi)$

**iii** $\text{Rot}(\theta)\,\text{Ref}(\varphi) = \text{Ref}\left(\varphi + \frac{1}{2}\theta\right)$

**iv** $\text{Ref}(\theta)\,\text{Rot}(\varphi) = \text{Ref}\left(\theta - \frac{1}{2}\varphi\right)$

**b** Explain in words what each of the above four equations shows.

**c** Using these identities, find the matrix $\text{Rot}(60°)\,\text{Ref}(60°)\,\text{Ref}(60°)\,\text{Rot}(60°)$.


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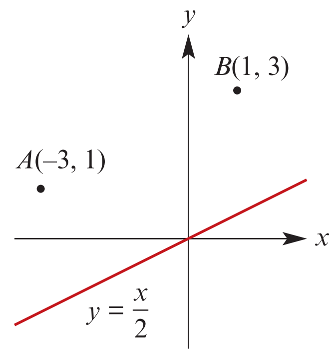
**6** An ant is at point $A(-3, 1)$. His friend is at point $B(1, 3)$. The ant wants to walk from $A$ to $B$, but first wants to visit the straight line $y = \frac{1}{2}x$. Being an economical ant, he wants the total length of his path to be as short as possible.




**a** Find the matrix that will reflect the plane in the line $y = \frac{1}{2}x$.

**b** Find the image $A'$ of the point $A$ when reflected in the line $y = \frac{1}{2}x$.

**c** Find the distance from point $A'$ to point $B$.

**d** The straight line $A'B$ intersects the line $y = \frac{1}{2}x$ at the point $C$. What type of triangle is $ACA'$?

**e** Suppose that $D$ is any other point on the line $y = \frac{1}{2}x$. Show that

$$AD + DB \geq AC + CB$$

**f** Hence find the shortest possible distance travelled by the ant.

**7** A rectangle $R_1$ has vertices $(0, 0)$, $(a, 0)$, $(0, b)$ and $(a, b)$, where $a$ and $b$ are positive real numbers.

**a** Sketch the rectangle $R_1$.

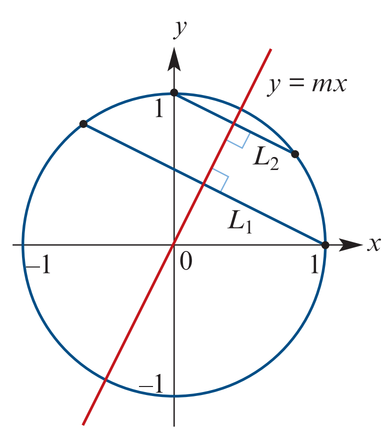
**b** The rectangle $R_1$ is rotated about the origin by angle $\theta$ anticlockwise, where $0° \leq \theta \leq 90°$. The image is another rectangle $R_2$. Find the coordinates of the vertices of rectangle $R_2$ in terms of $a$, $b$ and $\theta$.

**c** The vertices of $R_2$ lie on another rectangle $R_3$ that has edges parallel to the coordinate axes. Show that the area of rectangle $R_3$ is

$$A = \frac{1}{2}(a^2 + b^2)\sin(2\theta) + ab$$

**d** Hence show that the maximum area of rectangle $R_3$ is $\frac{1}{2}(a + b)^2$, which occurs when $\theta = 45°$.

**8** The graphs of the unit circle $x^2 + y^2 = 1$ and the line $y = mx$ are shown. Lines $L_1$ and $L_2$ are perpendicular to the line $y = mx$ and go through the points $(1, 0)$ and $(0, 1)$ respectively.




**a** Find the equation of the line $L_1$, and find where it intersects the unit circle in terms of $m$.

**b** Find the equation of the line $L_2$, and find where it intersects the unit circle in terms of $m$.

**c** Hence deduce the formula for the matrix that reflects the point $(x, y)$ in the line $y = mx$.

Hint: Recall that the columns of the matrix will be the images of the standard unit vectors.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# 13 Number and proof 2

## In this chapter



## Syllabus references

**Topics:** The nature of proof; Proofs involving numbers; Rational and irrational numbers; An introduction to proof by mathematical induction

**Subtopics:** 1.1.2, 2.3.1, 2.3.3 – 2.3.6

In Chapter 6 we introduced the concept of mathematical proof and looked at direct proof, proof by contrapositive, proving equivalent statements, and disproving statements.

In this short chapter we will revisit proof and learn two very useful types of proofs. In 13A we will look at proof by contradiction, where we prove something by first assuming that it is false and then showing that this leads to a mathematical absurdity. In 13B we introduce the powerful technique of mathematical induction, where we follow a specific set of steps to prove facts about the natural numbers.


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# 13A Proof by contradiction

There are various instances when we want to prove mathematically that something cannot be done. To do this, we assume that it can be done, and then show that something goes horribly wrong. Let's first look at a familiar example from geometry.

## Example 1

An angle is called **reflex** if it exceeds $180^\circ$. Prove that no quadrilateral has more than one reflex angle.

### Solution

If there is more than one reflex angle, then the angle sum must exceed $2 \times 180^\circ = 360^\circ$. This contradicts the fact that the angle sum of any quadrilateral is exactly $360^\circ$. Therefore there cannot be more than one reflex angle.

The example above is a demonstration of a **proof by contradiction**. The basic outline of a proof by contradiction is:

1. Assume that the statement we want to prove is false.
2. Show that this assumption leads to mathematical nonsense.
3. Conclude that we were wrong to assume that the statement is false.
4. Conclude that the statement must be true.

## Example 2

A **Pythagorean triple** consists of three natural numbers $(a, b, c)$ satisfying

$$a^2 + b^2 = c^2$$

Show that if $(a, b, c)$ is a Pythagorean triple, then $a$, $b$ and $c$ cannot all be odd numbers.

### Solution

This will be a proof by contradiction.

Let $(a, b, c)$ be a Pythagorean triple. Then $a^2 + b^2 = c^2$.

Suppose that $a$, $b$ and $c$ are all odd numbers.

$\Rightarrow$ $a^2$, $b^2$ and $c^2$ are all odd numbers.

$\Rightarrow$ $a^2 + b^2$ is even and $c^2$ is odd.

Since $a^2 + b^2 = c^2$, this gives a contradiction.

Therefore $a$, $b$ and $c$ cannot all be odd numbers.


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Possibly the most well-known proof by contradiction is the following.

**Theorem**

$\sqrt{2}$ is irrational.

**Proof** This will be a proof by contradiction.

Suppose that $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}$$

We can assume that $p$ and $q$ have no common factors (or else they could be cancelled). Then, squaring both sides and rearranging gives

$$\begin{aligned}
& p^2 = 2q^2 && (1) \\
\Rightarrow\ & p^2 \text{ is divisible by } 2 \\
\Rightarrow\ & p \text{ is divisible by } 2 \\
\Rightarrow\ & p = 2n \text{ for some } n \in \mathbb{Z} \\
\Rightarrow\ & (2n)^2 = 2q^2 && \text{(substituting into (1))} \\
\Rightarrow\ & q^2 = 2n^2 \\
\Rightarrow\ & q^2 \text{ is divisible by } 2 \\
\Rightarrow\ & q \text{ is divisible by } 2
\end{aligned}$$

Therefore both $p$ and $q$ are divisible by 2, which contradicts the fact that they have no common factors.

Hence $\sqrt{2}$ is irrational.

## Example 3

Suppose $x$ satisfies $5^x = 2$. Show that $x$ is irrational.

### Solution

Suppose that $x$ is rational. Since $x$ must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned}
5^x = 2 \quad \Rightarrow \quad & 5^{\frac{m}{n}} = 2 \\
\Rightarrow \quad & \left(5^{\frac{m}{n}}\right)^n = 2^n && \text{(raise both sides to the power } n\text{)} \\
\Rightarrow \quad & 5^m = 2^n
\end{aligned}$$

The left-hand side of this equation is odd and the right-hand side is even. This gives a contradiction, and so $x$ is not rational.

We finish on a remarkable result, which is attributed to Euclid some 2300 years ago.


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**Theorem**

There are infinitely many prime numbers.

**Proof** This is a proof by contradiction, so we will suppose that there are only finitely many primes. This means that we can create a list that contains *every* prime number:

$$2,\ 3,\ 5,\ 7,\ \dots,\ p$$

where $p$ is the largest prime number.

Now for the trick. We a create a new number $N$ by multiplying each number in the list and then adding 1:

$$N = 2 \times 3 \times 5 \times 7 \times \cdots \times p + 1$$

The number $N$ is not divisible by any of the primes $2, 3, 5, 7, \dots, p$, since it leaves a remainder of 1 when divided by any of these numbers.

However, every natural number greater than 1 is divisible by a prime number. (This is proved in Question 13 of Exercise 13B.) Therefore $N$ is divisible by some prime number $q$. But this prime number $q$ is not in the list $2, 3, 5, 7, \dots, p$, contradicting the fact that our list contains every prime number.

Hence there are infinitely many prime numbers.

## Summary 13A

- A **proof by contradiction** is used to prove that something cannot be done.
- These proofs always follow the same basic structure:
  1. Assume that the statement we want to prove is false.
  2. Show that this assumption leads to mathematical nonsense.
  3. Conclude that we were wrong to assume that the statement is false.
  4. Conclude that the statement must be true.

Skill-sheet

## Exercise 13A

Example 1

**1** Prove that every triangle has some interior angle with a magnitude of at least 60°.

**2** Prove that there is no smallest positive rational number.

**3** Let $p$ be a prime number. Show that $\sqrt{p}$ is not an integer.

Example 3

**4** Suppose that $3^x = 2$. Prove that $x$ is irrational.

**5** Prove that $\log_2 5$ is irrational.

**6** Suppose that $x > 0$ is irrational. Prove that $\sqrt{x}$ is also irrational.

**7** Suppose that $a$ is rational and $b$ is irrational. Prove that $a + b$ is irrational.

**8** Suppose that $c^2 - b^2 = 4$. Prove that $b$ and $c$ cannot both be natural numbers.


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**9** Let $a$, $b$ and $c$ be real numbers with $a \neq 0$. Prove by contradiction that there is only one solution to the equation $ax + b = c$.

**10** **a** Prove that all primes $p > 2$ are odd.
**b** Hence, prove that there are no two primes whose sum is 1001.

**11** **a** Prove that there are no integers $a$ and $b$ for which $42a + 7b = 1$.
Hint: The left-hand side is divisible by 7.
**b** Prove that there are no integers $a$ and $b$ for which $15a + 21b = 2$.

**12** **a** Prove that if $n^2$ is divisible by 3, then $n$ is divisible by 3.
Hint: Prove the contrapositive by considering two cases.
**b** Hence, prove that $\sqrt{3}$ is irrational.

**13** **a** Prove that if $n^3$ is divisible by 2, then $n$ is divisible by 2.
Hint: Prove the contrapositive.
**b** Hence, prove that $\sqrt[3]{2}$ is irrational.

**14** Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

**15** **a** Let $a, b, n \in \mathbb{N}$. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
**b** Hence, show that 97 is a prime number.

**16** **a** Let $m$ be an integer. Prove that $m^2$ is divisible by 4 or leaves a remainder of 1.
Hint: Suppose that $m = 4n + r$ and consider $m^2$ for $r = 0, 1, 2, 3$.
**b** Let $a, b, c \in \mathbb{Z}$. Prove by contradiction: If $a^2 + b^2 = c^2$, then $a$ is even or $b$ is even.

**17** **a** Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.
**b** Hence, find $c, d \in \mathbb{Z}$ if $\sqrt{3 + 2\sqrt{2}} = c + d\sqrt{2}$. Hint: Square both sides.

**18** Let $a, b, c \in \mathbb{Z}$. Prove that if $a$, $b$ and $c$ are all odd, then the equation $ax^2 + bx + c = 0$ cannot have a rational solution.

## 13B Mathematical induction

Consider the sum of the first $n$ odd numbers:

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1 + 3 &= 4 = 2^2 \\ 1 + 3 + 5 &= 9 = 3^2 \\ 1 + 3 + 5 + 7 &= 16 = 4^2 \end{aligned}$$

From this limited number of examples, we could make the following proposition $P(n)$ about the number $n$: the sum of the first $n$ odd numbers is $n^2$. Since the $n$th odd number is $2n - 1$, we can write this proposition as

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

However, we have to be careful here: Just because something looks true does not mean that it is true. In this section, we will learn how to prove statements like the one above.


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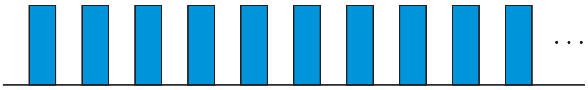
## The principle of mathematical induction

Imagine a row of dominoes extending infinitely to the right. Each of these dominoes can be knocked over provided two conditions are met:

**1** The first domino is knocked over.

**2** Each domino is sufficiently close to the next domino.



This scenario provides an accurate physical model of the following proof technique.

### Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number $n$.

We can prove that $P(n)$ is true for every natural number $n$ as follows:

**a** Show that $P(1)$ is true.

**b** Show that, for every natural number $k$, if $P(k)$ is true, then $P(k+1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1) \text{ is true} \quad \Rightarrow \quad P(2) \text{ is true} \quad \Rightarrow \quad P(3) \text{ is true} \quad \Rightarrow \quad \cdots$$

Let's see how mathematical induction is used in practice.

### Example 4

Prove that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all $n \in \mathbb{N}$.

#### Solution

For each natural number $n$, let $P(n)$ be the proposition:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

**Step 1** $P(1)$ is the proposition $1 = 1^2$, that is, $1 = 1$. Therefore $P(1)$ is true.

**Step 2** Let $k$ be any natural number, and assume $P(k)$ is true. That is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

**Step 3** We now have to prove that $P(k+1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.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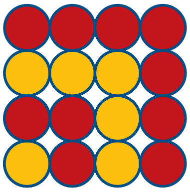
We have

$$\begin{aligned}\text{LHS of } P(k+1) &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1)\\ &= k^2 + (2k+1) \qquad \text{(using } P(k)\text{)}\\ &= (k+1)^2\\ &= \text{RHS of } P(k+1)\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k$.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number $n$.

While mathematical induction is good for proving that formulas are true, it rarely indicates why they should be true in the first place. The formula $1 + 3 + 5 + \cdots + (2n-1) = n^2$ can be discovered in the diagram shown on the right.



## Example 5

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

### Solution

For each natural number $n$, let $P(n)$ be the proposition:

$7^n - 4$ is divisible by 3

**Step 1** $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. Clearly, $P(1)$ is true.

**Step 2** Let $k$ be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

**Step 3** We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3. We have

$$\begin{aligned}7^{k+1} - 4 &= 7 \times 7^k - 4\\ &= 7(3m+4) - 4 \qquad \text{(using } P(k)\text{)}\\ &= 21m + 28 - 4\\ &= 21m + 24\\ &= 3(7m+8)\end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k$.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.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## Proving inequalities

Induction can be used to prove certain inequalities.

For example, consider this table of values:

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $3^n$ | 3 | 9 | 27 | 81 | 243 |
| $3 \times 2^n$ | 6 | 12 | 24 | 48 | 96 |

From the table, it certainly looks as though

$$3^n > 3 \times 2^n \quad \text{for all } n \geq 3$$

We will prove this formally using induction; this time starting with the proposition $P(3)$ instead of $P(1)$.

### Example 6

Prove that

$$3^n > 3 \times 2^n$$

for every natural number $n \geq 3$.

**Solution**

For each natural number $n \geq 3$, let $P(n)$ be the proposition:

$$3^n > 3 \times 2^n$$

**Step 1** $P(3)$ is the proposition $3^3 > 3 \times 2^3$, that is, $27 > 24$. Therefore $P(3)$ is true.

**Step 2** Let $k$ be a natural number with $k \geq 3$, and assume $P(k)$ is true. That is,

$$3^k > 3 \times 2^k$$

**Step 3** We now have to prove that $P(k+1)$ is true, that is,

$$3^{k+1} > 3 \times 2^{k+1}$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= 3^{k+1} \\ &= 3 \times 3^k \\ &> 3 \times 3 \times 2^k \qquad \text{(using } P(k)\text{)} \\ &> 3 \times 2 \times 2^k \\ &= 3 \times 2^{k+1} \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k \geq 3$.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number $n \geq 3$.


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## Applications to sequences

Sequences are studied in Mathematics Methods Units 1 & 2. A **sequence** is a list of numbers, with order being important. An example is the sequence of odd numbers:

$$1, 3, 5, 7, 9, \dots$$

The $n$th term of a sequence is denoted by $t_n$. So, for the sequence of odd numbers, we have $t_1 = 1$, $t_2 = 3$ and $t_3 = 5$. In general, we have $t_n = 2n - 1$ for all $n \in \mathbb{N}$.

Some sequences can be defined by a rule that enables each subsequent term to be found from previous terms. We can define the sequence of odd numbers by $t_1 = 1$ and $t_{n+1} = t_n + 2$. This type of rule is called a **recurrence relation**.

Induction proofs are frequently used in the study of sequences. As an example, we will consider the sequence defined by the recurrence relation

$$t_1 = 11 \quad \text{and} \quad t_{n+1} = 10t_n - 9$$

The first five terms of this sequence are listed in the following table.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $t_n$ | 11 | 101 | 1001 | 10 001 | 100 001 |

Notice that each of these terms is one more than a power of 10. Let's see if we can prove that this is true for *every* term in the sequence.

### Example 7

Given $t_1 = 11$ and $t_{n+1} = 10t_n - 9$, prove that $t_n = 10^n + 1$.

**Solution**

For each natural number $n$, let $P(n)$ be the proposition: $t_n = 10^n + 1$.

**Step 1** Since $t_1 = 11$ and $10^1 + 1 = 11$, it follows that $P(1)$ is true.

**Step 2** Let $k$ be any natural number, and assume $P(k)$ is true. That is,

$$t_k = 10^k + 1$$

**Step 3** We now have to prove that $P(k + 1)$ is true, that is,

$$t_{k+1} = 10^{k+1} + 1$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= t_{k+1} \\ &= 10t_k - 9 \\ &= 10 \times (10^k + 1) - 9 \qquad (\text{using } P(k)) \\ &= 10^{k+1} + 1 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k + 1)$ is true, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.


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## Tower of Hanoi

You have three pegs and a collection of $n$ discs of different sizes. Initially, all the discs are stacked in size order on the left-hand peg. Discs can be moved one at a time from one peg to any other peg, provided that a larger disc never rests on a smaller one. The aim of the puzzle is to transfer all the discs to another peg using the smallest possible number of moves.

### Example 8

Let $a_n$ be the minimum number of moves needed to solve the Tower of Hanoi with $n$ discs.

**a** Find a formula for $a_{n+1}$ in terms of $a_n$.

**b** Evaluate $a_n$ for $n = 1, 2, 3, 4, 5$. Guess a formula for $a_n$ in terms of $n$.

**c** Confirm your formula for $a_n$ using mathematical induction.

**d** If $n = 20$, how many days are needed to transfer all the discs to another peg, assuming that one disc can be moved per second?

#### Solution

**a** Suppose there are $n + 1$ discs on the left-hand peg.

If we want to be able to move the largest disc to the right-hand peg, then first we must transfer the other $n$ discs to the centre peg. This takes a minimum of $a_n$ moves.

It takes 1 move to transfer the largest disc to the right-hand peg. Now we can complete the puzzle by transferring the $n$ discs on the centre peg to the right-hand peg. This takes a minimum of $a_n$ moves.

Hence the minimum number of moves required to transfer all the discs is

$$\begin{aligned} a_{n+1} &= a_n + 1 + a_n \\ &= 2a_n + 1 \end{aligned}$$

**b** We have $a_1 = 1$, since one disc can be moved in one move. Using the recurrence relation from part **a**, we find that

$$a_2 = 2a_1 + 1 = 2 \times 1 + 1 = 3$$

$$a_3 = 2a_2 + 1 = 2 \times 3 + 1 = 7$$

Continuing in this way, we obtain the following table.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | 1 | 3 | 7 | 15 | 31 |

It seems as though every term is one less than a power of 2. We guess that

$$a_n = 2^n - 1$$


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**c** For each natural number $n$, let $P(n)$ be the proposition:

$$a_n = 2^n - 1$$

**Step 1** The minimum number of moves required to solve the Tower of Hanoi puzzle with one disc is 1. Since $2^1 - 1 = 1$, it follows that $P(1)$ is true.

**Step 2** Let $k$ be any natural number, and assume $P(k)$ is true. That is,

$$a_k = 2^k - 1$$

**Step 3** We now wish to prove that $P(k+1)$ is true, that is,

$$a_{k+1} = 2^{k+1} - 1$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= a_{k+1} \\ &= 2a_k + 1 && \text{(using part } \mathbf{a}) \\ &= 2 \times (2^k - 1) + 1 && \text{(using } P(k)) \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$. Hence we have shown that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

**d** A puzzle with 20 discs requires a minimum of $2^{20} - 1$ seconds.

Since there are $60 \times 60 \times 24 = 86\,400$ seconds in a day, it will take

$$\frac{2^{20} - 1}{86\,400} \approx 12.14 \text{ days}$$

to complete the puzzle.

## Summary 13B

The basic outline of a proof by mathematical induction is:

**0** Define the proposition $P(n)$ for $n \in \mathbb{N}$.
**1** Show that $P(1)$ is true.
**2** Assume that $P(k)$ is true for some $k \in \mathbb{N}$.
**3** Show that $P(k+1)$ is true.
**4** Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.


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Skill-sheet

## Exercise 13B

Example 4

**1** Prove each of the following by mathematical induction:

**a** $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

**b** $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$, where $x \neq 1$

**c** $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

**d** $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$

**e** $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

**f** $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\cdots\left(1 - \dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$, for $n \geq 2$

Example 5

**2** Prove each of the following divisibility statements by mathematical induction:

**a** $11^n - 1$ is divisible by 10 for all $n \in \mathbb{N}$

**b** $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$

**c** $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$

**d** $5^n + 6 \times 7^n + 1$ is divisible by 4 for all $n \in \mathbb{N}$

Example 6

**3** Prove each of the following inequalities by mathematical induction:

**a** $4^n > 10 \times 2^n$ for all integers $n \geq 4$

**b** $3^n > 5 \times 2^n$ for all integers $n \geq 5$

**c** $2^n > 2n$ for all integers $n \geq 3$

**d** $n! > 2^n$ for all integers $n \geq 4$

Example 7

**4** Prove each of the following statements by mathematical induction:

**a** If $a_{n+1} = 2a_n - 1$ and $a_1 = 3$, then $a_n = 2^n + 1$.

**b** If $a_{n+1} = 5a_n + 4$ and $a_1 = 4$, then $a_n = 5^n - 1$.

**c** If $a_{n+1} = 2a_n - n + 1$ and $a_1 = 3$, then $a_n = 2^n + n$.

**5** Prove that $3^n$ is odd for every $n \in \mathbb{N}$.

**6** **a** Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$.

**b** Find an easier proof by factorising $n^2 - n$.

**7** **a** Prove by mathematical induction that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$.

**b** Find an easier proof by factorising $n^3 - n$.


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**8** Consider the sequence defined by $a_{n+1} = 10a_n + 9$ where $a_1 = 9$.

**a** Find $a_n$ for $n = 1, 2, 3, 4, 5$.

**b** Guess a formula for $a_n$ in terms of $n$.

**c** Confirm that your formula is valid by using mathematical induction.

**9** The Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$.

**a** Find $f_n$ for $n = 1, 2, \dots, 10$.

**b** Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

**c** Evaluate $f_1 + f_3 + \cdots + f_{2n-1}$ for $n = 1, 2, 3, 4$.

**d** Try to find a formula for the above expression.

**e** Confirm that your formula works using mathematical induction.

**f** Using induction, prove that every third Fibonacci number, $f_{3n}$, is even.

**10** Prove that $4^n + 5^n$ is divisible by 9 for all odd integers $n$.

**11** Prove by induction that, for all $n \in \mathbb{N}$, every set of numbers $S$ with exactly $n$ elements has a largest element.

**12** Standing around a circle, there are $n$ friends and $n$ thieves. You begin with no money, but as you go around the circle clockwise, each friend will give you \$1 and each thief will steal \$1. Prove that no matter where the friends and thieves are placed, it is possible to walk once around the circle without going into debt, provided you start at the correct point.

**13** Prove by induction that every natural number $n \geq 2$ is divisible by some prime number.
Hint: Let $P(n)$ be the statement that every integer $j$ such that $2 \leq j \leq n$ is divisible by some prime number.

**14** If $n$ straight lines are drawn across a sheet of paper, they will divide the paper into regions. Show that it is always possible to colour each region black or white, so that no two adjacent regions have the same colour.


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## Chapter summary

Assignment

Nrich

- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- **Mathematical induction** is used to prove that a statement is true for all natural numbers.

## Short-answer questions

**1** Assume that $n$ is even. Prove that $n^2 - 3n + 1$ is odd.

**2** Let $n \in \mathbb{Z}$. Consider the statement: If $n^3$ is even, then $n$ is even.

**a** Write down the contrapositive of this statement.

**b** Prove the contrapositive.

**c** Hence, prove by contradiction that $\sqrt[3]{6}$ is irrational.

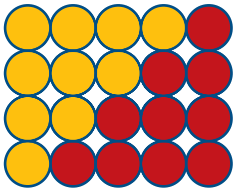
**3** Prove by mathematical induction that:

**a** $6^n + 4$ is divisible by 10 for all $n \in \mathbb{N}$

**b** $1^2 + 3^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \in \mathbb{N}$

## Extended-response questions

**1** **a** Use the diagram on the right to deduce the equation



$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad (1)$$

**b** Using equation (1), prove that the sum $1 + 2 + \cdots + 99$ is divisible by 99.

**c** Using equation (1), prove that if $n$ is odd, then the sum of any $n$ consecutive odd natural numbers is divisible by $n$.

**d** With the help of equation (1) and mathematical induction, prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 \quad \text{for all } n \in \mathbb{N}$$

**2** We call $(a, b, c)$ a Pythagorean triple if $a, b, c$ are natural numbers such that $a^2 + b^2 = c^2$.

**a** Let $n \in \mathbb{N}$. Prove that if $(a, b, c)$ is a Pythagorean triple, then so is $(na, nb, nc)$.

**b** Prove that there is only one Pythagorean triple $(a, b, c)$ of consecutive natural numbers.

**c** Prove that there is no Pythagorean triple $(a, b, c)$ containing the numbers 1 or 2.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**3** Let $a$ be an integer that is not divisible by 3. We know that $a = 3k + 1$ or $a = 3k + 2$, for some $k \in \mathbb{Z}$.

**a** Show that $a^2$ must leave a remainder of 1 when divided by 3.

**b** Hence, prove that if $(a, b, c)$ is any Pythagorean triple, then $a$ or $b$ is divisible by 3.

**4** **a** Prove by mathematical induction that $n^2 + n$ is even for all $n \in \mathbb{N}$.

**b** Find an easier proof by factorising $n^2 + n$.

**c** Hence, prove that if $n$ is odd, then there exists an integer $k$ such that $n^2 = 8k + 1$.

**5** Goldbach's conjecture is that every even integer greater than 2 can be expressed as the sum of two primes. To date, no one has been able to prove this, although it has been verified for all integers less than $4 \times 10^{18}$.

**a** Express 100 and 102 as the sum of two prime numbers.

**b** Prove that 101 cannot be written as the sum of two prime numbers.

**c** Express 101 as the sum of three prime numbers.

**d** Assuming that Goldbach's conjecture is true, prove that every odd integer greater than 5 can be written as the sum of three prime numbers.

**6** **a** Simplify the expression $\dfrac{1}{n-1} - \dfrac{1}{n}$.

**b** Hence, show that

$$\frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \frac{1}{4 \times 3} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$$

**c** Give another proof of the above equation using mathematical induction.

**d** Using the above equation, prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 \quad \text{for all } n \in \mathbb{N}$$


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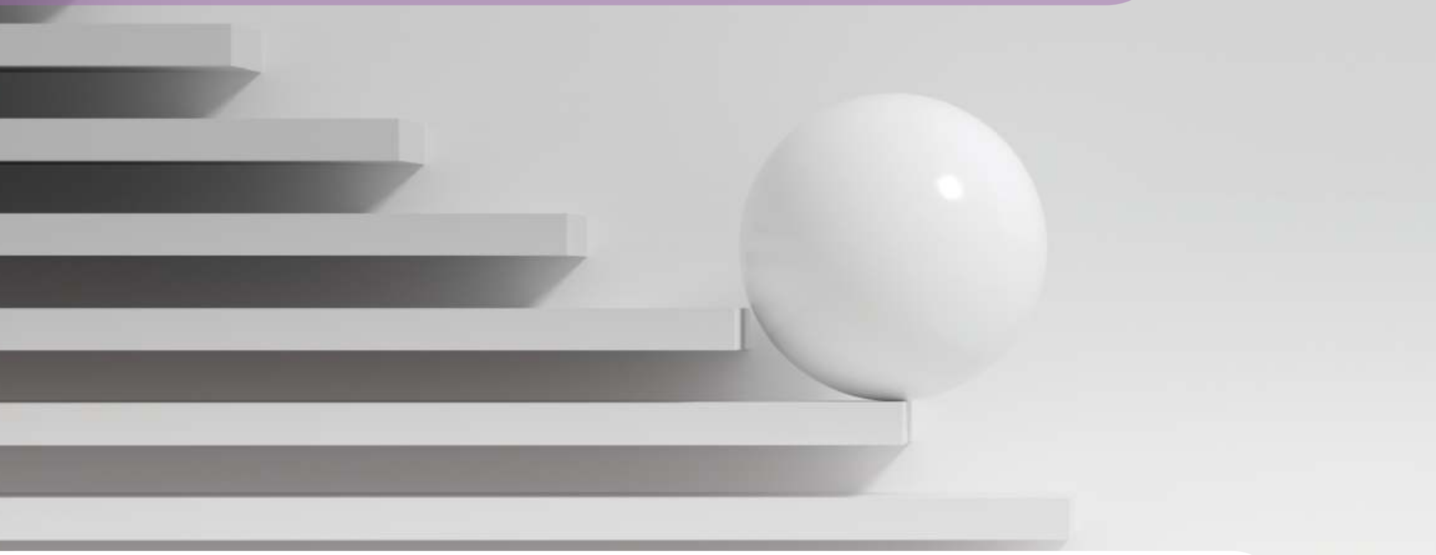
# 14 Complex numbers



## In this chapter



## Syllabus references

**Topics:** Complex numbers; The complex plane; Roots of equations

**Subtopics:** 2.3.7 – 2.3.16

In this chapter we introduce a new set of numbers, called *complex numbers*. These numbers first arose in the search for solutions to polynomial equations.

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as ‘impossible’, they arose in calculations to find real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano’s formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which you can show equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.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# 14A Starting to build the complex numbers

Mathematicians in the eighteenth century introduced the imaginary number $i$ with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely $i$ and $-i$.

By declaring that $i = \sqrt{-1}$, we can find square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)} \\ &= \sqrt{4} \times \sqrt{-1} \\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers $a$ and $b$, but does not hold when both $a$ and $b$ are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

Now consider the equation $x^2 + 2x + 3 = 0$. Using the quadratic formula gives

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 12}}{2} \\ &= \frac{-2 \pm \sqrt{-8}}{2} \\ &= -1 \pm \sqrt{-2}\end{aligned}$$

This equation has no real solutions. However, using complex numbers we obtain solutions

$$x = -1 \pm \sqrt{2}i$$

## The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where $a$ and $b$ are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$$

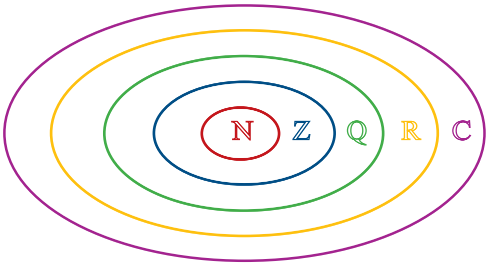
The letter often used to denote a complex number is $z$.

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

We can now extend the diagram from Chapter 2 to include the complex numbers.





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### Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b$$

where $\text{Re}(z)$ is called the **real part** of $z$ and $\text{Im}(z)$ is called the **imaginary part** of $z$.

For example, for the complex number $z = 2 + 5i$, we have $\text{Re}(z) = 2$ and $\text{Im}(z) = 5$.

Note: Both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers.

### Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

**Example 1**

If $4 - 3i = 2a + bi$, find the real values of $a$ and $b$.

**Solution**

$$2a = 4 \quad \text{and} \quad b = -3$$

$$\therefore \quad a = 2 \quad \text{and} \quad b = -3$$

**Example 2**

Find the real values of $a$ and $b$ such that $(2a + 3b) + (a - 2b)i = -1 + 3i$.

**Solution**

$$2a + 3b = -1 \qquad (1)$$

$$a - 2b = 3 \qquad (2)$$

Multiply (2) by 2:

$$2a - 4b = 6 \qquad (3)$$

Subtract (3) from (1):

$$7b = -7$$

Therefore $b = -1$ and $a = 1$.

## Operations on complex numbers

### Addition and subtraction

**Addition of complex numbers**

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.


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If $z = a + bi$, then we define $-z = -a - bi$.

**Subtraction of complex numbers**

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$$

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z + 0 = z$
- $z + (-z) = 0$

**Example 3**

If $z_1 = 2 - 3i$ and $z_2 = -4 + 5i$, find:

**a** $z_1 + z_2$

**b** $z_1 - z_2$

**Solution**

**a** $z_1 + z_2 = (2 - 3i) + (-4 + 5i)$
$= (2 + (-4)) + (-3 + 5)i$
$= -2 + 2i$

**b** $z_1 - z_2 = (2 - 3i) - (-4 + 5i)$
$= (2 - (-4)) + (-3 - 5)i$
$= 6 - 8i$

## Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = k(a + bi) = ka + kbi$$

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

## Powers of *i*

Successive multiplication by $i$ gives the following:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \dots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

**Example 4**

Simplify:

**a** $i^{13}$

**b** $3i^4 \times (-2i)^3$

**Solution**

**a** $i^{13} = i^{4\times3+1}$
$= i$

**b** $3i^4 \times (-2i)^3 = 3 \times (-2)^3 \times i^4 \times i^3$
$= -24i^7$
$= 24i$


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## Summary 14A

- The imaginary number $i$ satisfies $i^2 = -1$.
- If $a$ is a positive real number, then $\sqrt{-a} = i\sqrt{a}$.
- The set of **complex numbers** is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- For a complex number $z = a + bi$:
  - the **real part** of $z$ is $\text{Re}(z) = a$
  - the **imaginary part** of $z$ is $\text{Im}(z) = b$.
- Equality of complex numbers:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

- If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i \quad \text{and} \quad z_1 - z_2 = (a - c) + (b - d)i$$

- When simplifying powers of $i$, remember that $i^4 = 1$.

## Exercise 14A

**1** State the values of $\text{Re}(z)$ and $\text{Im}(z)$ for each of the following:

**a** $z = 2 + 3i$ **b** $z = 4 + 5i$ **c** $z = \frac{1}{2} - \frac{3}{2}i$

**d** $z = -4$ **e** $z = 3i$ **f** $z = \sqrt{2} - 2\sqrt{2}i$

Example 1 Example 2

**2** Find the real values of $a$ and $b$ in each of the following:

**a** $2a - 3bi = 4 + 6i$ **b** $a + b - 2abi = 5 - 12i$

**c** $2a + bi = 10$ **d** $3a + (a - b)i = 2 + i$

Example 3

**3** Simplify:

**a** $(2 - 3i) + (4 - 5i)$ **b** $(4 + i) + (2 - 2i)$

**c** $(-3 - i) - (3 + i)$ **d** $(2 - \sqrt{2}i) + (5 - \sqrt{8}i)$

**e** $(1 - i) - (2i + 3)$ **f** $(2 + i) - (-2 - i)$

**g** $4(2 - 3i) - (2 - 8i)$ **h** $-(5 - 4i) + (1 + 2i)$

**i** $5(i + 4) + 3(2i - 7)$ **j** $\frac{1}{2}(4 - 3i) - \frac{3}{2}(2 - i)$

Example 4

**4** Simplify:

**a** $\sqrt{-16}$ **b** $2\sqrt{-9}$ **c** $\sqrt{-2}$

**d** $i^3$ **e** $i^{14}$ **f** $i^{20}$

**g** $-2i \times i^3$ **h** $4i^4 \times 3i^2$ **i** $\sqrt{8}i^5 \times \sqrt{-2}$

**5** Simplify:

**a** $i(2 - i)$ **b** $i^2(3 - 4i)$

**c** $\sqrt{2}i(i - \sqrt{2})$ **d** $-\sqrt{3}(\sqrt{-3} + \sqrt{2})$


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# 14B Multiplication and division of complex numbers

In the previous section, we defined addition and subtraction of complex numbers. We begin this section by defining multiplication.

## Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (ad + bc)i \qquad \text{(since } i^2 = -1\text{)} \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that the following is a *definition* of multiplication for $\mathbb{C}$.

### Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$.

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $z \times 1 = z$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

**Example 5**

If $z_1 = 3 - 2i$ and $z_2 = 1 + i$, find $z_1 z_2$.

| Solution | Explanation |
|---|---|
| $z_1 z_2 = (3 - 2i)(1 + i)$ | Expand the brackets in the usual way. |
| $= 3 - 2i + 3i - 2i^2$ | Remember that $i^2 = -1$. |
| $= 5 + i$ | |

## The conjugate of a complex number

Let $z = a + bi$. The **conjugate** of $z$ is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi$$

For example, the conjugate of $-4 + 3i$ is $-4 - 3i$, and vice versa.


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For a complex number $z = a + bi$, we have

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 + abi - abi - b^2 i^2 \\ &= a^2 + b^2 \qquad \text{where } a^2 + b^2 \text{ is a real number} \end{aligned}$$

The **modulus** of the complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$

The calculation above shows that

$$z\bar{z} = |z|^2$$

**Example 6**

If $z_1 = 2 - 3i$ and $z_2 = -1 + 2i$, find:

**a** $\overline{z_1 + z_2}$ and $\overline{z_1} + \overline{z_2}$

**b** $\overline{z_1 \cdot z_2}$ and $\overline{z_1} \cdot \overline{z_2}$

**Solution**

We have $\overline{z_1} = 2 + 3i$ and $\overline{z_2} = -1 - 2i$.

**a**

$$\begin{aligned} z_1 + z_2 &= (2 - 3i) + (-1 + 2i) \\ &= 1 - i \end{aligned}$$

$$\overline{z_1 + z_2} = 1 + i$$

$$\begin{aligned} \overline{z_1} + \overline{z_2} &= (2 + 3i) + (-1 - 2i) \\ &= 1 + i \end{aligned}$$

**b**

$$\begin{aligned} z_1 \cdot z_2 &= (2 - 3i)(-1 + 2i) \\ &= 4 + 7i \end{aligned}$$

$$\overline{z_1 \cdot z_2} = 4 - 7i$$

$$\begin{aligned} \overline{z_1} \cdot \overline{z_2} &= (2 + 3i)(-1 - 2i) \\ &= 4 - 7i \end{aligned}$$

- The conjugate of a sum is equal to the sum of the conjugates:

  $$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

- The conjugate of a product is equal to the product of the conjugates:

  $$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

## Division of complex numbers

### Multiplicative inverse

We begin with some familiar algebra that will motivate the definition:

$$\frac{1}{a + bi} = \frac{1}{a + bi} \times \frac{a - bi}{a - bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2}$$

We can see that

$$(a + bi) \times \frac{a - bi}{a^2 + b^2} = 1$$


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Although we have carried out this arithmetic, we have not yet defined what $\dfrac{1}{a+bi}$ means.

### Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\overline{z}}{|z|^2}$$

Note: We can check that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$.

## Division

The formal definition of division in the complex numbers is via the multiplicative inverse:

### Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2} \qquad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di}$$

Multiply the numerator and denominator by the conjugate of $z_2$:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2}\end{aligned}$$

Complete the division by simplifying. This process is demonstrated in the next example.

### Example 7

If $z_1 = 2 - i$ and $z_2 = 3 + 2i$, find $\dfrac{z_1}{z_2}$.

#### Solution

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2 - i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i} \\ &= \frac{6 - 3i - 4i + 2i^2}{3^2 + 2^2} \\ &= \frac{4 - 7i}{13} \\ &= \frac{1}{13}(4 - 7i)\end{aligned}$$

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## Example 8

Solve for $z$ in the equation $(2 + 3i)z = -1 - 2i$.

### Solution

$$
\begin{aligned}
(2 + 3i)z &= -1 - 2i \\
\therefore \quad z &= \frac{-1 - 2i}{2 + 3i} \\
&= \frac{-1 - 2i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i} \\
&= \frac{-8 - i}{13} \\
&= -\frac{1}{13}(8 + i)
\end{aligned}
$$

There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

## Example 9

If $z = 2 - 5i$, find $z^{-1}$ and express with a real denominator.

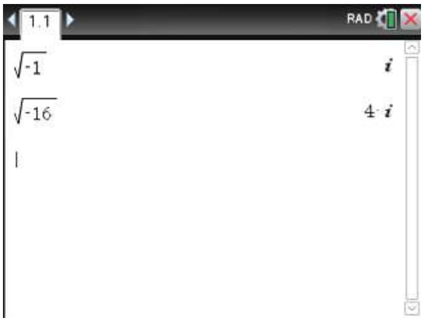
### Solution

$$
\begin{aligned}
z^{-1} &= \frac{1}{z} \\
&= \frac{1}{2 - 5i} \\
&= \frac{1}{2 - 5i} \times \frac{2 + 5i}{2 + 5i} \\
&= \frac{2 + 5i}{29} \\
&= \frac{1}{29}(2 + 5i)
\end{aligned}
$$

### Using the TI-Nspire

Set to complex mode using (on) > **Settings** > **Document Settings**. Select **Rectangular** from the **Real or Complex** field.

Note: The square root of a negative number can be found only in complex mode. But most computations with complex numbers can also be performed in real mode.





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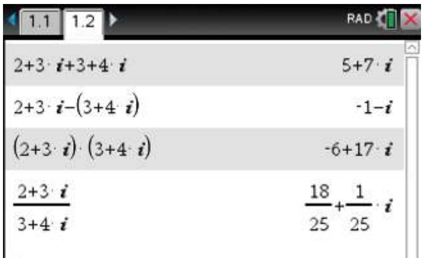
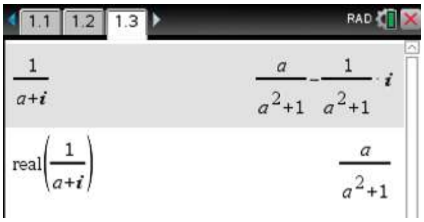
- The results of the arithmetic operations +, −, × and ÷ are illustrated using the two complex numbers $2 + 3i$ and $3 + 4i$.



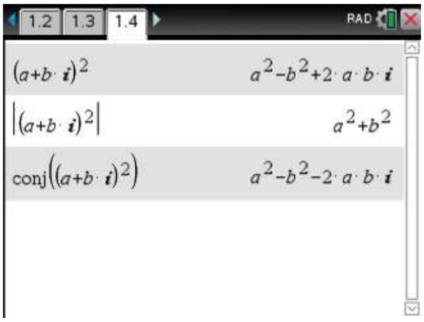

Note: Do not use the text $i$ for the imaginary constant. The symbol $i$ is found using [π▸] or the Symbols palette ( [ctrl] [📖] ).

- To find the real part of a complex number, use [menu] > **Number** > **Complex Number Tools** > **Real Part** as shown.



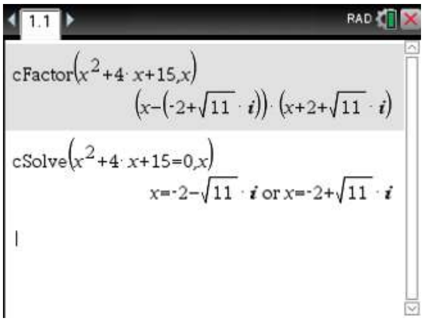

Hint: Type real(.

- To find the modulus of a complex number, use [menu] > **Number** > **Complex Number Tools** > **Magnitude** as shown.
  Alternatively, use |□| from the 2D-template palette [⊞] or type abs(.
- To find the conjugate of a complex number, use [menu] > **Number** > **Complex Number Tools** > **Complex Conjugate** as shown.




Hint: Type conj(.

There are also commands for factorising polynomials over the complex numbers and for solving polynomial equations over the complex numbers. These are available from [menu] > **Algebra** > **Complex**.

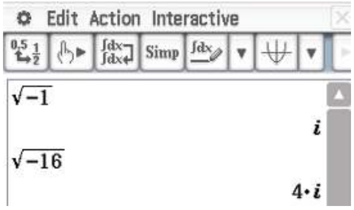



Note: You must use this menu even if the calculator is in complex mode. When using **cFactor**, you must include the variable as shown.

## Using the Casio ClassPad

In Main $\sqrt{\alpha}$, tap **Real** in the status bar at the bottom of the screen to change to **Cplx** mode.

- Enter $\sqrt{-1}$ and tap [EXE] to obtain the answer $i$.
- Enter $\sqrt{-16}$ to obtain the answer $4i$.




Note: The symbol $i$ is found in both the [Math2] and the [Math3] keyboards.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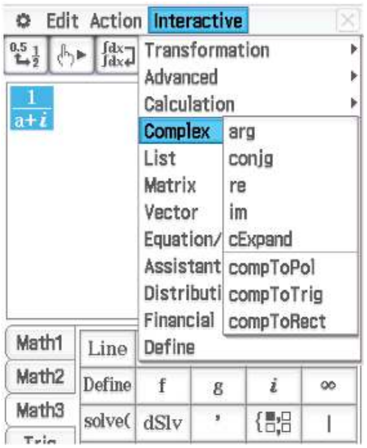
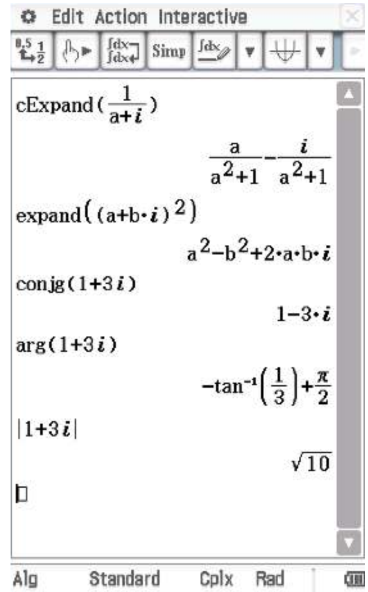
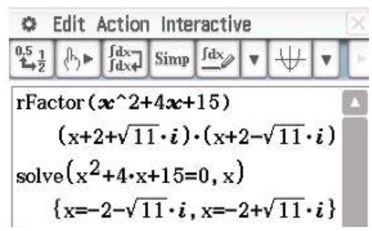
With the calculator set to complex mode, various operations on complex numbers can be carried out using options from **Interactive** > **Complex**.




Polynomials can be factorised over the complex numbers (**Interactive** > **Transformation** > **factor** > **rFactor**).

Equations can be solved over the complex numbers (**Interactive** > **Equation/Inequality** > **solve**).




## Summary 14B

- **Multiplication** To find a product $(a + bi)(c + di)$, expand the brackets in the usual way, remembering that $i^2 = -1$.
- **Conjugate** If $z = a + bi$, then $\bar{z} = a - bi$.
- **Division** To perform a division, start with

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \times \frac{c-di}{c-di}\\ &= \frac{(a+bi)(c-di)}{c^2+d^2}\end{aligned}$$

  and then simplify.
- **Multiplicative inverse** To find $z^{-1}$, calculate $\dfrac{1}{z}$.

## Exercise 14B

Example 5

**1** Expand and simplify:

**a** $(4 + i)^2$ **b** $(2 - 2i)^2$ **c** $(3 + 2i)(2 + 4i)$

**d** $(-1 - i)^2$ **e** $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i)$ **f** $(5 - 2i)(-2 + 3i)$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**2** Write down the conjugate of each of the following complex numbers:

**a** $2 - 5i$ **b** $-1 + 3i$ **c** $\sqrt{5} - 2i$ **d** $-5i$

Example 6

**3** If $z_1 = 2 - i$ and $z_2 = -3 + 2i$, find:

**a** $\overline{z_1}$ **b** $\overline{z_2}$ **c** $z_1 \cdot z_2$ **d** $\overline{z_1 \cdot z_2}$

**e** $\overline{z_1} \cdot \overline{z_2}$ **f** $z_1 + z_2$ **g** $\overline{z_1 + z_2}$ **h** $\overline{z_1} + \overline{z_2}$

**4** If $z = 2 - 4i$, express each of the following in the form $x + yi$:

**a** $\overline{z}$ **b** $z\overline{z}$ **c** $z + \overline{z}$ **d** $z(z + \overline{z})$

Example 9

**e** $z - \overline{z}$ **f** $i(z - \overline{z})$ **g** $z^{-1}$ **h** $\dfrac{z}{i}$

**5** Find the real values of $a$ and $b$ such that $(a + bi)(2 + 5i) = 3 - i$.

Example 7

**6** Express in the form $x + yi$:

**a** $\dfrac{2 - i}{4 + i}$ **b** $\dfrac{3 + 2i}{2 - 3i}$ **c** $\dfrac{4 + 3i}{1 + i}$ **d** $\dfrac{2 - 2i}{4i}$ **e** $\dfrac{1}{2 - 3i}$ **f** $\dfrac{i}{2 + 6i}$

**7** Find the real values of $a$ and $b$ if $(3 - i)(a + bi) = 6 - 7i$.

Example 8

**8** Solve each of the following for $z$:

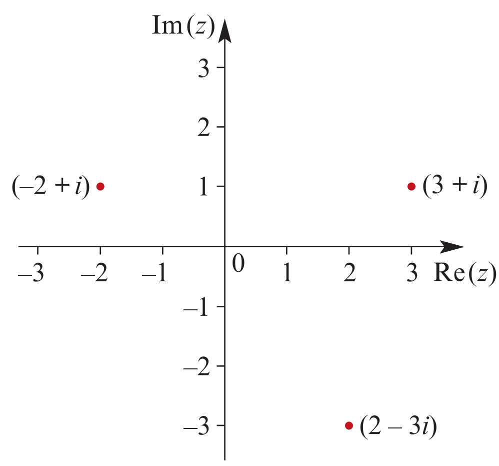
**a** $(2 - i)z = 42i$ **b** $(1 + 3i)z = -2 - i$ **c** $(3i + 5)z = 1 + i$

**d** $2(4 - 7i)z = 5 + 2i$ **e** $z(1 + i) = 4$

# 14C Argand diagrams

An **Argand diagram** is a geometric representation of the set of complex numbers. A complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\mathrm{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\mathrm{Im}(z)$, for $z \in \mathbb{C}$.




Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point $(a, b)$ on the equivalent Cartesian axes, as shown by the examples in this figure.

A complex number written as $a + bi$ is said to be in **Cartesian form**.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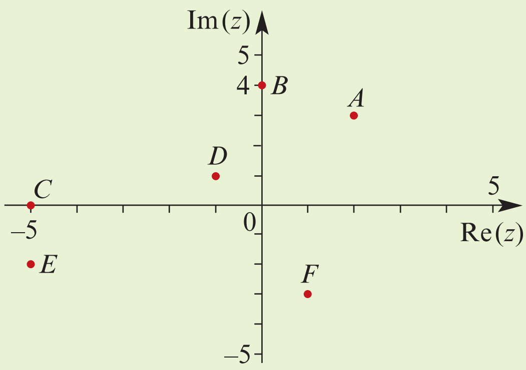
**Example 10**

Write down the complex number represented by each of the points shown on this Argand diagram.



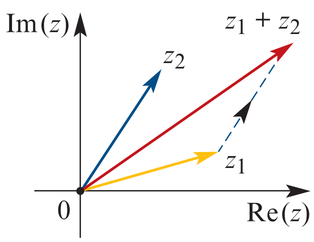

**Solution**

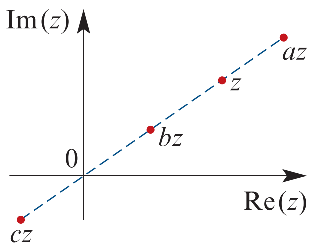
| | | | | | |
|---|---|---|---|---|---|
| $A$ | $2 + 3i$ | $B$ | $4i$ | $C$ | $-5$ |
| $D$ | $-1 + i$ | $E$ | $-5 - 2i$ | $F$ | $1 - 3i$ |

## Geometric representation of the basic operations on complex numbers

In an Argand diagram, the sum of two complex numbers $z_1$ and $z_2$ can be found geometrically by placing the 'tail' of $z_2$ on the 'tip' of $z_1$, as shown in the diagram on the left. We will see in Chapter 3 that this is analogous to vector addition.

When a complex number is multiplied by a real constant, it maintains the same 'direction', but its distance from the origin is scaled. This is shown in the diagram below on the right.




$a > 1$

$0 < b < 1$

$c < 0$

The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

**Example 11**

**a** Represent the following complex numbers as points on an Argand diagram:

**i** $2$ **ii** $-3i$ **iii** $2 - i$ **iv** $-(2 + 3i)$ **v** $-1 + 2i$

**b** Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.
Represent the complex numbers $z_1$, $z_2$, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.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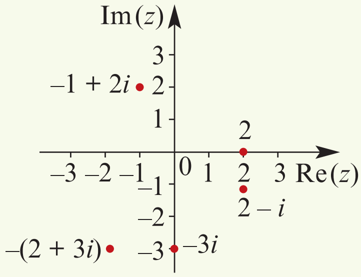
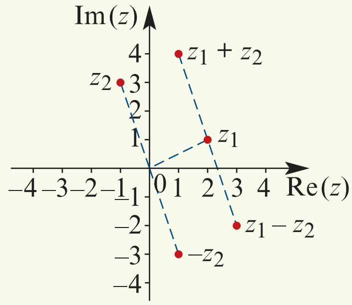
### Solution

**a**




**b**




$z_1 + z_2 = (2 + i) + (-1 + 3i) = 1 + 4i$

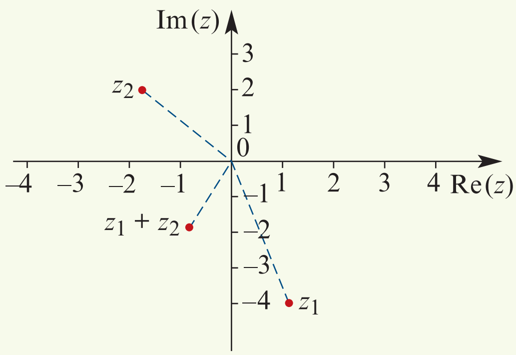
$z_1 - z_2 = (2 + i) - (-1 + 3i) = 3 - 2i$

## Example 12

Let $z_1 = 1 - 4i$ and $z_2 = -2 + 2i$. Find $z_1 + z_2$ algebraically and illustrate $z_1 + z_2$ on an Argand diagram.

### Solution

$$z_1 + z_2 = (1 - 4i) + (-2 + 2i)$$
$$= -1 - 2i$$

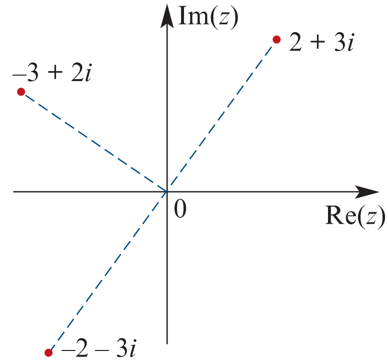



## Rotation about the origin

When the complex number $2 + 3i$ is multiplied by $-1$, the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.

When the complex number $2 + 3i$ is multiplied by $i$, we obtain

$$i(2 + 3i) = 2i + 3i^2$$
$$= 2i - 3$$
$$= -3 + 2i$$




The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by $i$, the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.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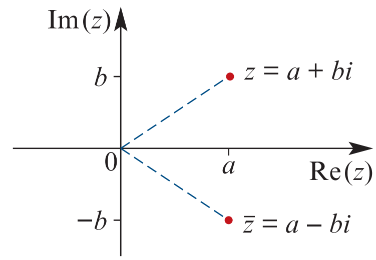
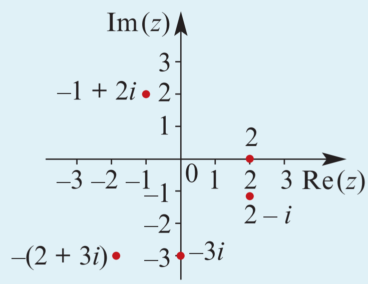
## Reflection in the horizontal axis

The conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$. Therefore $\bar{z}$ is the reflection of $z$ in the horizontal axis of an Argand diagram.




### Summary 14C

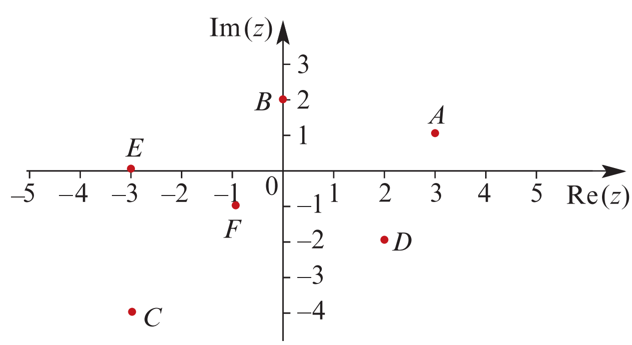
- An **Argand diagram** is a geometric representation of the set of complex numbers.
- The horizontal axis represents Re($z$) and the vertical axis represents Im($z$), for $z \in \mathbb{C}$.
- The operations of addition, subtraction and multiplication by a real constant all have geometric interpretations on an Argand diagram.
- Multiplication of a complex number by $i$ corresponds to a rotation of 90° anticlockwise about the origin.
- Complex conjugate corresponds to reflection in the horizontal axis.




## Exercise 14C

Example 10

**1** Write down the complex numbers represented on this Argand diagram.




Example 11

**2** Represent each of the following complex numbers as points on an Argand diagram:

**a** $3 - 4i$ **b** $-4 + i$ **c** $4 + i$ **d** $-3$ **e** $-2i$ **f** $-5 - 2i$

Example 12

**3** If $z_1 = 6 - 5i$ and $z_2 = -3 + 4i$, represent each of the following on an Argand diagram:

**a** $z_1 + z_2$ **b** $z_1 - z_2$

**4** If $z = 1 + 3i$, represent each of the following on an Argand diagram:

**a** $z$ **b** $\bar{z}$ **c** $z^2$ **d** $-z$ **e** $\dfrac{1}{z}$

**5** If $z = 2 - 5i$, represent each of the following on an Argand diagram:

**a** $z$ **b** $zi$ **c** $zi^2$ **d** $zi^3$ **e** $zi^4$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# 14D Solving equations over the complex numbers

Quadratic equations with a negative discriminant have no real solutions. The introduction of complex numbers enables us to solve such quadratic equations.

## Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

This allows us to solve equations of the form $z^2 + a^2 = 0$.

### Example 13

Solve the equations:

**a** $z^2 + 16 = 0$

**b** $2z^2 + 6 = 0$

**Solution**

**a**

$$\begin{aligned} z^2 + 16 &= 0 \\ z^2 - 16i^2 &= 0 \\ (z + 4i)(z - 4i) &= 0 \\ \therefore \quad z &= \pm 4i \end{aligned}$$

**b**

$$\begin{aligned} 2z^2 + 6 &= 0 \\ z^2 + 3 &= 0 \\ z^2 - 3i^2 &= 0 \\ (z + \sqrt{3}i)(z - \sqrt{3}i) &= 0 \\ \therefore \quad z &= \pm\sqrt{3}i \end{aligned}$$

## Solution of quadratic equations

To solve quadratic equations which have a negative discriminant, we can use the quadratic formula in the usual way.

### Example 14

Solve the equation $3z^2 + 5z + 3 = 0$.

**Solution**

Using the quadratic formula:

$$\begin{aligned} z &= \frac{-5 \pm \sqrt{25 - 36}}{6} \\ &= \frac{-5 \pm \sqrt{-11}}{6} \\ &= \frac{1}{6}(-5 \pm \sqrt{11}i) \end{aligned}$$


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## Factorisation of quadratics

To factorise a quadratic over the complex numbers, we can complete the square in the usual way. We may then need to rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned}(z+a)^2+b^2 &= (z+a)^2-(bi)^2\\ &= (z+a+bi)(z+a-bi)\end{aligned}$$

### Example 15

Factorise:

**a** $z^2+z+3$ **b** $2z^2-z+1$

**Solution**

**a** By completing the square, we have

$$\begin{aligned}z^2+z+3 &= \left(z^2+z+\frac{1}{4}\right)+3-\frac{1}{4}\\ &= \left(z+\frac{1}{2}\right)^2+\frac{11}{4}\\ &= \left(z+\frac{1}{2}\right)^2-\frac{11}{4}i^2\\ &= \left(z+\frac{1}{2}+\frac{\sqrt{11}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{11}}{2}i\right)\end{aligned}$$

**b** By completing the square, we have

$$\begin{aligned}2z^2-z+1 &= 2\left(z^2-\frac{1}{2}z+\frac{1}{2}\right)\\ &= 2\left(\left(z^2-\frac{1}{2}z+\frac{1}{16}\right)+\frac{1}{2}-\frac{1}{16}\right)\\ &= 2\left(\left(z-\frac{1}{4}\right)^2+\frac{7}{16}\right)\\ &= 2\left(\left(z-\frac{1}{4}\right)^2-\frac{7}{16}i^2\right)\\ &= 2\left(z-\frac{1}{4}+\frac{\sqrt{7}}{4}i\right)\left(z-\frac{1}{4}-\frac{\sqrt{7}}{4}i\right)\end{aligned}$$

Note: From this example, we can see that given any quadratic $p(z)=az^2+bz+c$, where the coefficients $a$, $b$ and $c$ are real numbers, we can rewrite $p(z)$ as a product of two linear factors over the complex numbers:

$$az^2+bz+c=a(z-\alpha)(z-\beta)$$

for some complex numbers $\alpha$ and $\beta$. Moreover, if $\alpha, \beta$ both have a non-zero imaginary component, then $\alpha$ and $\beta$ will be conjugates of each other.


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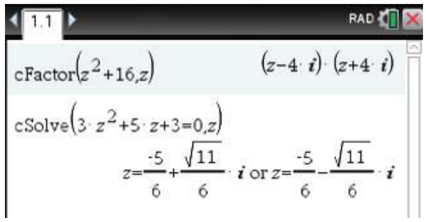
## Using the TI-Nspire

- To factorise polynomials over the complex numbers, use (menu) > **Algebra** > **Complex** > **Factor** as shown.
- To solve polynomial equations over the complex numbers, use (menu) > **Algebra** > **Complex** > **Solve** as shown.



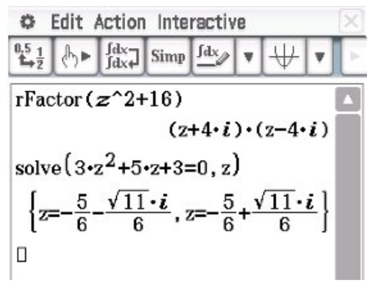

## Using the Casio ClassPad

To factorise:

- Ensure the mode is set to **Cplx**.
- Enter and highlight the expression $z^2 + 16$.
- Select **Interactive** > **Transformation** > **factor** > **rFactor**.

To solve:

- Enter and highlight $3z^2 + 5z + 3$.
- Select **Interactive** > **Equation/Inequality** > **solve**.
- Ensure the variable selected is $z$.




## Summary 14D

- Quadratic equations can be solved over the complex numbers using the same techniques as for the real numbers.
- Two properties of complex numbers that are useful when solving equations:
  - $z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$
  - $\sqrt{-a} = i\sqrt{a}$, where $a$ is a positive real number.

Skill-sheet

## Exercise 14D

Example 13, Example 14

**1** Solve each of the following equations over $\mathbb{C}$:

**a** $z^2 + 4 = 0$ **b** $2z^2 + 18 = 0$ **c** $3z^2 = -15$

**d** $(z - 2)^2 + 16 = 0$ **e** $(z + 1)^2 = -49$ **f** $z^2 - 2z + 3 = 0$

**g** $z^2 + 3z + 3 = 0$ **h** $2z^2 + 5z + 4 = 0$ **i** $3z^2 = z - 2$

**j** $2z = z^2 + 5$ **k** $2z^2 - 6z = -10$ **l** $z^2 - 6z = -14$

Example 15

**2** Factorise each of the following quadratics over $\mathbb{C}$:

**a** $z^2 + 9$ **b** $z^2 + 3$ **c** $3z^2 + 12$

**d** $z^2 + 2z + 5$ **e** $z^2 - 3z + 6$ **f** $2z^2 + 2z + 1$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Review

Assignment

Nrich

## Chapter summary

- The imaginary number $i$ has the property $i^2 = -1$.
- The set of **complex numbers** is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- For a complex number $z = a + bi$:
  - the **real part** of $z$ is $\text{Re}(z) = a$
  - the **imaginary part** of $z$ is $\text{Im}(z) = b$.
- Complex numbers $z_1$ and $z_2$ are equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of $z$, denoted by $|z|$, is the distance from the origin to the point representing $z$ in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The **complex conjugate** of $z = a + bi$ is given by $\overline{z} = a - bi$. Note that $z\overline{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

## Short-answer questions

**1** For $z_1 = m + ni$ and $z_2 = p + qi$, express each of the following in the form $a + bi$:

**a** $2z_1 + 3z_2$ **b** $\overline{z_2}$

**c** $z_1\overline{z_2}$ **d** $\dfrac{z_1}{z_2}$

**e** $z_1 + \overline{z_1}$ **f** $(z_1 + z_2)(z_1 - z_2)$

**g** $\dfrac{1}{z_1}$ **h** $\dfrac{z_2}{z_1}$

**i** $\dfrac{3z_1}{z_2}$

**2** Let $z = 1 - \sqrt{3}i$. For each of the following, express in the form $a + bi$ and mark on an Argand diagram:

**a** $z$ **b** $z^2$

**c** $z^3$ **d** $\dfrac{1}{z}$

**e** $\overline{z}$ **f** $\dfrac{1}{\overline{z}}$


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## Extended-response questions

**1** **a** Find the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$.

**b** **i** Plot the two solutions from part **a** on an Argand diagram.

**ii** Find the equation of the circle, with centre the origin, which passes through these two points.

**iii** Find the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.

**2** Let $z$ be a complex number with $|z| = 6$. Let $A$ be the point representing $z$ and let $B$ be the point representing $(1 + i)z$.

**a** Find:

**i** $|(1 + i)z|$

**ii** $|(1 + i)z - z|$

**b** Prove that $OAB$ is a right-angled isosceles triangle.

**3** Let $z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$.

**a** On an Argand diagram, the points $O$, $A$, $Z$, $P$ and $Q$ represent the complex numbers $0$, $1$, $z$, $1 + z$ and $1 - z$ respectively. Show these points on a diagram.

**b** Prove that the magnitude of $\angle POQ$ is $\dfrac{\pi}{2}$. Find the ratio $\dfrac{OP}{OQ}$.

**4** Let $z_1$ and $z_2$ be two complex numbers. Prove the following:

**a** $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$

**b** $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$

**c** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

State a geometric theorem from the result of **c**.

**5** Let $z_1$ and $z_2$ be two complex numbers.

**a** Prove the following:

**i** $\overline{\overline{z_1}z_2} = z_1\overline{z_2}$

**ii** $z_1\overline{z_2} + \overline{z_1}z_2$ is a real number

**iii** $z_1\overline{z_2} - \overline{z_1}z_2$ is an imaginary number

**iv** $(z_1\overline{z_2} + \overline{z_1}z_2)^2 - (z_1\overline{z_2} - \overline{z_1}z_2)^2 = 4|z_1z_2|^2$

**b** Use the results from part **a** and Question 4 to prove that $|z_1 + z_2| \le |z_1| + |z_2|$.
Hint: Show that $(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 \ge 0$.

**c** Hence prove that $|z_1 - z_2| \ge |z_1| - |z_2|$.


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**6** Assume that $z = a + ib$, for some $a,\ b \in \mathbb{R}$ such that $|z| = 1$. Find the modulus of:

**a** $z + 1$

**b** $z - 1$

**c** $\dfrac{z - 1}{z + 1}$

**7** The quadratic expression $ax^2 + bx + c$ has real coefficients.

**a** Find the discriminant of $ax^2 + bx + c$.

**b** Find the condition in terms of $a$, $b$ and $c$ for which the equation $ax^2 + bx + c = 0$ has no real solutions.

**c** If this condition is fulfilled, let $z_1$ and $z_2$ be the complex solutions of the equation and let $P_1$ and $P_2$ the corresponding points on an Argand diagram.

**i** Find $z_1 + z_2$ and $|z_1|$ in terms of $a$, $b$ and $c$.

**ii** Find $\cos(\angle P_1OP_2)$ in terms of $a$, $b$ and $c$.

**8** Let $z_1$ and $z_2$ be the solutions of the quadratic equation $z^2 + z + 1 = 0$.

**a** Find $z_1$ and $z_2$.

**b** Prove that $z_1 = z_2^2$ and $z_2 = z_1^2$.

**c** Let $P_1$ and $P_2$ be the points on an Argand diagram corresponding to $z_1$ and $z_2$. Find the area of triangle $P_1OP_2$.


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# 15 Revision of Chapters 11–14

Revision

## 15A Short-answer questions

**1** Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & -1 & -2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$.

**a** State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**

**b** Evaluate **DA** and $\mathbf{A}^{-1}$.

**2** If $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$, find:

**a** $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$

**b** $\mathbf{A}^2 - \mathbf{B}^2$

**3** If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ is non-invertible, find the value of $x$.

**4** Find all possible matrices **A** which satisfy the equation $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.

**5** If $\mathbf{A} = \begin{bmatrix} -1 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -4 \\ -1 & -6 \\ -3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, evaluate **AB** and $\mathbf{C}^{-1}$.

**6** A transformation has rule $(x, y) \to (2x + y, -x - 2y)$.

**a** Find the image of the point $(2, 3)$.

**b** Find the matrix of this transformation.

**c** Sketch the image of the unit square and find its area.

**d** Find the rule for the inverse transformation.


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**7** Find the matrix corresponding to each of the following linear transformations:

**a** reflection in the $x$-axis
**b** dilation of factor 3 parallel to the $x$-axis
**c** shear of factor 2 parallel to the $y$-axis
**d** projection onto the $y$-axis
**e** rotation by 45° anticlockwise
**f** rotation by 30° clockwise
**g** reflection in the line $y = -x$
**h** reflection in the line $y = x \tan 30°$

**8** **a** Find the matrix that will reflect the plane in the line $y = 4x$.
**b** Find the image of the point $(2, 4)$ under this transformation.

**9** Find the transformation matrix that corresponds to:

**a** a reflection in the $y$-axis and then a dilation of factor 2 parallel to the $y$-axis
**b** a rotation by 90° anticlockwise and then a reflection in the line $y = x$
**c** a reflection in the line $y = -x$ and then a shear of factor 2 parallel to the $x$-axis

**10** **a** Find the rule for the transformation that will reflect $(x, y)$ in the $y$-axis then translate the result by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
**b** Find the rule for the transformation if the translation takes place before the reflection.

**11** Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

**a** $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$
**b** $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

**12** **a** Find the rule for the transformation that will reflect the plane in the line $y = x - 1$.
Hint: Translate the plane 1 unit in the $y$-direction, reflect in the line $y = x$, and then translate the plane back to its original position.
**b** Find the image of the point $(0, 0)$ under this transformation.
**c** Sketch the unit square and its image under this transformation.

**13** Suppose the number $x$ is irrational. Prove by contradiction that $x + 1$ is also irrational.

**14** Prove by contradiction that 6 cannot be written as the difference of two perfect squares.

**15** Prove by mathematical induction that:

**a** $2 + 4 + \cdots + 2n = n(n + 1)$
**b** $11^n - 6$ is divisible by 5, for all $n \in \mathbb{N}$

**16** If $w = 3 + 2i$ and $z = 3 - 2i$, express each of the following in the form $a + bi$, where $a$ and $b$ are real numbers:

**a** $w + z$
**b** $w - z$
**c** $wz$
**d** $w^2 + z^2$
**e** $(w + z)^2$
**f** $(w - z)^2$
**g** $w^2 - z^2$
**h** $(w - z)(w + z)$


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**17** If $w = 1 - 2i$ and $z = 2 - 3i$, express each of the following in the form $a + bi$, where $a$ and $b$ are real numbers:

**a** $w + z$ **b** $w - z$ **c** $wz$ **d** $\frac{w}{z}$ **e** $iw$ **f** $\frac{i}{w}$

**g** $\frac{w}{i}$ **h** $\frac{z}{w}$ **i** $\frac{w}{w + z}$ **j** $(1 + i)w$ **k** $\frac{w}{1 + i}$ **l** $w^2$

**18** Write each polynomial as a product of linear factors:

**a** $z^2 + 49$ **b** $z^2 - 2z + 10$ **c** $9z^2 - 6z + 5$ **d** $4z^2 + 12z + 13$

**19** **a** Find the two square roots of $3 + 4i$.

**b** Use the quadratic formula to solve $(2 - i)z^2 + (4 + 3i)z + (-1 + 3i) = 0$ for $z$.

## 15B Extended-response questions

**1** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $b \neq 0$ and $c \neq 0$.

**a** **i** Find $\mathbf{A}^2$. **ii** Find $3\mathbf{A}$.

**b** If $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$, show that:

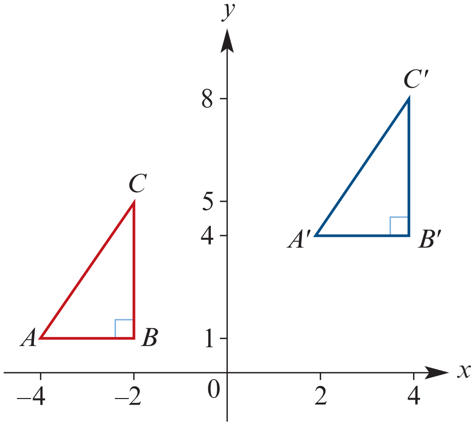
**i** $a + d = 3$ **ii** $\det(\mathbf{A}) = 1$

**c** Assume that $\mathbf{A}$ has the properties:

- $a + d = 3$
- $\det(\mathbf{A}) = 1$

Show that $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$.

**2** The coordinates of $A$, $B$ and $C$ are $(-4, 1)$, $(-2, 1)$ and $(-2, 5)$ respectively.




**a** Find the rule of the transformation that maps triangle $ABC$ to triangle $A'B'C'$.

**b** On graph paper, draw triangle $ABC$ and its image under a reflection in the $x$-axis.

**c** On the same set of axes, draw the image of $ABC$ under a dilation of factor 2 parallel to the $x$-axis.

**d** Find the image of the parabola $y = x^2$ under a dilation of factor 2 parallel to the $y$-axis followed by a translation by the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

**e** Find the rule for the transformation that maps the graph of $y = x^2$ to the graph of $y = -2(x - 3)^2 + 4$.

**f** Let $f(x) = x^3 - 2x$. Use a calculator to help sketch the graph of $y = 3f(x - 2) + 4$.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**3** Let transformation $D$ be a dilation of factor 4 from the $y$-axis.

**a** Find the image of the point $(1, 1)$ under dilation $D$.

**b** **i** Describe the image of the square with vertices $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, $E(1, 0)$ under the dilation $D$.

**ii** Find the area of the square $ABCE$.

**iii** Find the area of the image of $ABCE$.

**iv** If the dilation had been of factor $k$, what would be the area of the image?

**c** State the rule for the dilation $D$.

**d** **i** Find the equation of the image of the graph of $y = x^2$ under the dilation $D$.

**ii** Find the equation of the image of the graph of $y = x^2$ under the dilation $D$ followed by the translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

**iii** Sketch the graph of $y = x^2$ and its image defined in part **d ii** on the one set of axes. State the coordinates of the vertex and the axis intercepts of the image.

**e** State the rule for the transformation that maps the graph of $y = 5(x + 2)^2 - 3$ to the graph of $y = x^2$.

**4** A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

**a** Show that this transformation is a rotation.

**b** Let $C$ be the circle that passes through the origin and has its centre at $(0, 1)$.

**i** Find the equation of $C$.

**ii** Find the equation of $C'$, the image of $C$ under the transformation defined by $\mathbf{M}$.

**c** Find the coordinates of the points of intersection of $C$ and $C'$.

**5** A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

**a** Find the image of the point $(-2, 5)$ under this transformation.

**b** Find the inverse of $\mathbf{M}$.

**c** Given that the point $(11, 13)$ is the image of the point $(a, b)$, find the values of $a$ and $b$.

**d** Find the coordinates of the image of the point $(a, a)$ in terms of $a$.

**e** Given that $a \neq 0$ and $b \neq 0$ with

$$\mathbf{M}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}$$

find the possible values of $\lambda$ and the relationship between $a$ and $b$ for each of these values.


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**6** Let **R** be the transformation matrix for a rotation about the origin by $\frac{\pi}{4}$ anticlockwise.

**a** Give the $2 \times 2$ matrix **R**.

**b** Find the inverse of this matrix.

**c** If the image of $(a, b)$ is $(1, 1)$, find the values of $a$ and $b$.

**d** If the image of $(c, d)$ is $(1, 2)$, find the values of $c$ and $d$.

**e** **i** If $(x, y) \to (x', y')$ under this transformation, use the result of part **b** to find $x$ and $y$ in terms of $x'$ and $y'$.

**ii** Find the image of $y = x^2$ under this transformation.

**7** Consider lines $y = x$ and $y = 2x$.

**a** Sketch these two lines on the same set of axes.

**b** The acute angle between the two lines, $\theta$ radians, can be written in the form $\theta = \tan^{-1}(a) - b$. What are the values of $a$ and $b$?

**c** Hence find a rotation matrix that will rotate the line $y = x$ to the line $y = 2x$. You will need to use the addition formulas for sine and cosine.

**8** Let $M$ be the transformation that reflects the plane in the line $y = x$.

**a** **i** Find the image of the point $A(1, 3)$ under this transformation.

**ii** The image of the triangle with vertices $A(1, 3)$, $B(1, 5)$ and $C(3, 3)$ is another triangle. Find the coordinates of the vertices of the image.

**iii** Sketch triangle $ABC$ and its image on a set of axes, with both axes from $-5$ to 5.

**b** **i** Show that the equation of the image of the graph of $y = x^2 - 2$ under the transformation $M$ is $x = y^2 - 2$.

**ii** Find the coordinates of the points of intersection of $y = x^2 - 2$ and the line $y = x$.

**iii** Show that the $x$-coordinates of the points of intersection of $y = x^2 - 2$ and its image may be determined by the equation $x^4 - 4x^2 - x + 2 = 0$.

**iv** Two solutions of the equation $x^4 - 4x^2 - x + 2 = 0$ are

$$x = \frac{1}{2}(-1 + \sqrt{5}) \quad \text{and} \quad x = \frac{1}{2}(-1 - \sqrt{5})$$

Use this result and the result of part **b ii** to find the coordinates of the points of intersection of $y = x^2 - 2$ and its image under $M$.

**9** The **trace** of a square matrix **A** is defined to be the sum of the entries along the main diagonal of **A** (from top-left to bottom-right) and is denoted by Tr(**A**).

For example, if $\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 2 & 2 \end{bmatrix}$, then $\text{Tr}(\mathbf{A}) = 6 + 2 = 8$.

**a** Prove each of the following for all $2 \times 2$ matrices **X** and **Y**:

**i** $\text{Tr}(\mathbf{X} + \mathbf{Y}) = \text{Tr}(\mathbf{X}) + \text{Tr}(\mathbf{Y})$ **ii** $\text{Tr}(-\mathbf{X}) = -\text{Tr}(\mathbf{X})$ **iii** $\text{Tr}(\mathbf{XY}) = \text{Tr}(\mathbf{YX})$

**b** Use the results of part **a** to show that $\mathbf{XY} - \mathbf{YX} \neq \mathbf{I}$ for all $2 \times 2$ matrices **X** and **Y**.

**10** **a** Suppose that $a$ is odd and $b$ is odd. Prove that $ab$ is odd.

**b** Suppose that $a$ is odd and $n \in \mathbb{N}$. Prove by induction that $a^n$ is odd.

**c** Hence, prove that if $x$ satisfies $3^x = 2$, then $x$ is irrational.

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# Glossary

## A

**Addition formulas** [p. 185]

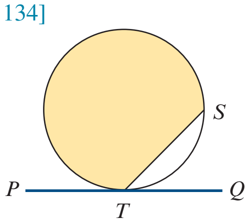
- $\cos(u + v) = \cos u \cos v - \sin u \sin v$
- $\cos(u - v) = \cos u \cos v + \sin u \sin v$
- $\sin(u + v) = \sin u \cos v + \cos u \sin v$
- $\sin(u - v) = \sin u \cos v - \cos u \sin v$
- $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$
- $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

**Addition of complex numbers** [p. 286] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.

**Addition of vectors** [p. 60] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$, then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j}$.

**Addition principle** [p. 3] Suppose there are $m$ ways of performing one task and $n$ ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

**Alternate segment** [p. 134] The alternate segment to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.




**Amplitude of trigonometric functions** [p. 146] The distance between the mean position and the maximum position is called the amplitude. The graph of $y = a \sin x$ has an amplitude of $|a|$.

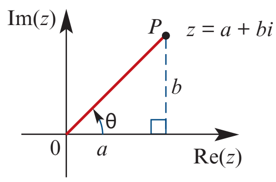
**Angle between two vectors** [p. 73] can be found using the scalar product:
$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$
where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

**Angle sum and difference identities** [p. 185] *see* addition formulas

**Arc** [p. 128] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

**Area of image** [p. 257] If a linear transformation (with matrix $\mathbf{B}$) is applied to a region of the plane, then Area of image $= |\det(\mathbf{B})| \times$ Area of region.

**Argand diagram** [p. 295] a geometric representation of the set of complex numbers




**Arrangement** [p. 6] *see* permutation

## C

$\mathbb{C}$ [p. 285] the set of complex numbers: $\mathbb{C} = \{\, a + bi : a, b \in \mathbb{R} \,\}$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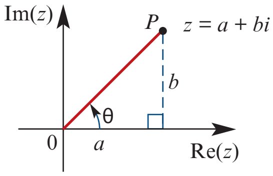
**Cartesian form of a complex number** [p. 285] A complex number is expressed in Cartesian form as $z = a + bi$, where $a$ is the real part of $z$ and $b$ is the imaginary part of $z$.




**Chord** [p. 128] a line segment with endpoints on a circle

**Collinear points** [p. 79] Three or more points are collinear if they all lie on a single line.

**Column vector** [pp. 233, 59] an $n \times 1$ matrix. A column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be used to represent an ordered pair, a point in the Cartesian plane, a vector in the plane or a translation of the plane.

**Combination** [p. 18] a selection where order is not important. The number of combinations of $n$ objects taken $r$ at a time is given by

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for $^nC_r$ is $\binom{n}{r}$.

**Complement of a set** [p. 39] The complement of a set $A$, written $A'$, is the set of all elements of $\xi$ that are not elements of $A$.

**Complementary relationships** [p. 150]

- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

**Complex conjugate, $\bar{z}$** [p. 289]

- If $z = a + bi$, then $\bar{z} = a - bi$.

**Complex conjugate, properties** [p. 290]

- $z + \bar{z} = 2\,\mathrm{Re}(z)$
- $z\bar{z} = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

**Complex number** [p. 285] an expression of the form $a + bi$, where $a$ and $b$ are real numbers

**Complex plane** [p. 295] *see* Argand diagram

**Compound angle formulas** [p. 185] *see* addition formulas

**Concurrent lines** [p. 79] Three or more lines are concurrent if they all pass through a single point.

**Conditional statement** [p. 106] a statement of the form 'If $P$ is true, then $Q$ is true', which can be abbreviated to $P \Rightarrow Q$

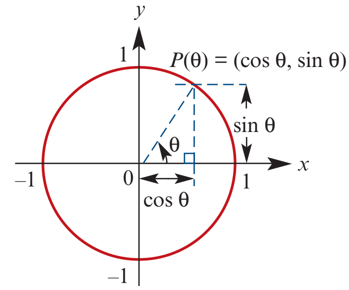
**Contrapositive** [p. 112] The contrapositive of $P \Rightarrow Q$ is the statement (not $Q$) $\Rightarrow$ (not $P$). The contrapositive is equivalent to the original statement.

**Converse** [p. 115] The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

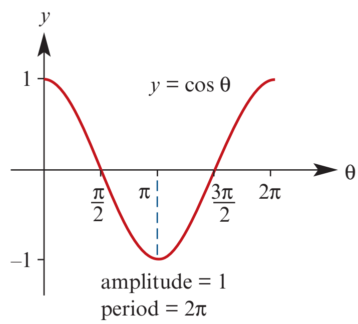
**Cosecant function** [pp. 177, 181]

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad \text{for } \sin\theta \neq 0$$

**Cosine function** [p. 146] cosine θ is defined as the $x$-coordinate of the point $P$ on the unit circle where $OP$ forms an angle of θ radians with the positive direction of the $x$-axis.

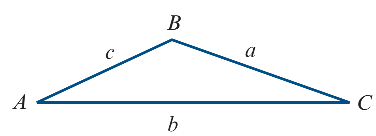



**Cosine function, graph** [p. 146]




**Cosine rule** [p. 71] For triangle $ABC$:

$$a^2 = b^2 + c^2 - 2bc\cos A$$




**Cotangent function** [pp. 177, 182]

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad \text{for } \sin\theta \neq 0$$

**Counterexample** [p. 120] an example that shows that a universal statement is false. For example, the number 2 is a counterexample to the claim 'Every prime number is odd.'

**Cyclic quadrilateral** [p. 130] a quadrilateral such that all its vertices lie on a circle. Opposite angles of a cyclic quadrilateral are supplementary.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# D

**De Morgan's laws** [p. 111]
- 'not ($P$ and $Q$)' is '(not $P$) or (not $Q$)'
- 'not ($P$ or $Q$)' is '(not $P$) and (not $Q$)'

**Determinant of a 2 × 2 matrix** [p. 222]
If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

**Diameter** [p. 129] a chord of a circle that passes through the centre

**Dilation** [p. 238] A dilation scales the $x$- or $y$-coordinate of each point in the plane.
- Dilation parallel to the $y$-axis: $(x, y) \to (x, cy)$
- Dilation parallel to the $x$-axis: $(x, y) \to (cx, y)$

**Direct proof** [p. 107] To give a direct proof of a conditional statement $P \Rightarrow Q$, we assume that $P$ is true and show that $Q$ follows.

**Disjoint sets** [p. 38] Sets $A$ and $B$ are said to be disjoint if they have no elements in common, i.e. if $A \cap B = \emptyset$.

**Division of complex numbers** [p. 290]
$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

**Double angle formulas** [p. 188]
- $\cos(2u) = \cos^2 u - \sin^2 u$
  $= 2\cos^2 u - 1$
  $= 1 - 2\sin^2 u$
- $\sin(2u) = 2 \sin u \cos u$
- $\tan(2u) = \dfrac{2 \tan u}{1 - \tan^2 u}$

# E

**Equality of complex numbers** [p. 286]
$a + bi = c + di$ if and only if $a = c$ and $b = d$

**Equilibrium** [p. 88] A particle is said to be in equilibrium if the resultant force acting on it is zero; the particle will remain at rest or continue moving with constant velocity.

**Equivalence of vectors** [p. 67]
Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$. If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$ and $a_2 = b_2$.

**Equivalent statements** [p. 116]
Statements $P$ and $Q$ are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$; this is abbreviated to $P \Leftrightarrow Q$.

**Existence statement** [pp. 118, 121]
a statement claiming that a property holds for some member of a given set. Such a statement can be written using the quantifier 'there exists'.

# F

**Factorial notation** [p. 6] The notation $n!$ (read as '$n$ factorial') is an abbreviation for the product of all the integers from $n$ down to 1:
$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \cdots \times 2 \times 1$

**Force** [p. 87] causes a change in motion; e.g. gravitational force, tension force, normal reaction force. Force is a vector quantity.

# I

**Imaginary number** $i$ [p. 285] $i^2 = -1$

**Imaginary part of a complex number** [p. 286] If $z = a + bi$, then $\operatorname{Im}(z) = b$.

**Implication** [p. 106] *see* conditional statement

**Inclusion–exclusion principle** [p. 50]
allows us to count the number of elements in a union of sets. In the case of two sets:
$|A \cup B| = |A| + |B| - |A \cap B|$

**Integers** [p. 41] the elements of
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

**Intersection of sets** [p. 39] The intersection of two sets $A$ and $B$, written $A \cap B$, is the set of all elements common to $A$ and $B$.

**Interval** [p. 44] a subset of the real numbers of the form $[a, b]$, $[a, b)$, $(a, \infty)$, etc.

**Irrational number** [p. 41] a real number that is not rational; e.g. $\pi$ and $\sqrt{2}$

# K

**Kilogram weight, kg wt** [p. 88] a unit of force. If an object on the surface of Earth has a mass of 1 kg, then the gravitational force acting on this object is 1 kg wt.

# L

**Linear transformation** [p. 233]
a transformation of the plane with a rule of the form
$(x, y) \to (ax + by, cx + dy)$
Each linear transformation can be represented by a 2 × 2 matrix:
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

**Linear transformation, inverse** [p. 249]
If $\mathbf{A}$ is the matrix of a linear transformation and $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.


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**Linear transformations, composition** [p. 246] If **A** and **B** are the matrices of two different linear transformations, then the product **BA** is the matrix of the transformation **A** followed by **B**.

# M

**Magnitude of a vector** [p. 67] the length of a directed line segment corresponding to the vector.

- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

**Mass** [p. 88] The mass of an object is the amount of matter it contains, and can be measured in kilograms. Mass is not the same as weight.

**Mathematical induction** [p. 273] a proof technique for showing that a statement is true for all natural numbers; uses the *principle of mathematical induction*

**Matrices, addition** [p. 213] Addition is defined for two matrices of the same size (same number of rows and same number of columns). The sum is found by adding corresponding entries. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

**Matrices, equal** [p. 210] Two matrices **A** and **B** are equal, and we write **A** = **B**, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

**Matrices, multiplication** [p. 217] The product of two matrices **A** and **B** is only defined if the number of columns of **A** is the same as the number of rows of **B**. If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row $i$ and column $j$ of **AB**, single out row $i$ in matrix **A** and column $j$ in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

**Matrix** [p. 209] a rectangular array of numbers

**Matrix, identity** [p. 220] For square matrices of a given size (e.g. $2 \times 2$), a multiplicative identity **I** exists.

For $2 \times 2$ matrices, the identity is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for each $2 \times 2$ matrix **A**.

**Matrix, inverse** [p. 221] If **A** is a square matrix and there exists a matrix **B** such that **AB** = **BA** = **I**, then **B** is called the inverse of **A**. When it exists, the inverse of a square matrix **A** is unique and is denoted by $\mathbf{A}^{-1}$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

provided $ad - bc \neq 0$.

**Matrix, invertible** [p. 221] A square matrix is said to be invertible if its inverse exists.

**Matrix, multiplication by a scalar** [p. 213] If **A** is an $m \times n$ matrix and $k$ is a real number, then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are $k$ times the corresponding entries of **A**. For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

**Matrix, non-invertible** [p. 221] A square matrix is said to be non-invertible if it does not have an inverse.

**Matrix, size** [p. 209] A matrix with $m$ rows and $n$ columns is said to be an $m \times n$ matrix.

**Matrix, square** [p. 220] A matrix with the same number of rows and columns is called a square matrix; e.g. a $2 \times 2$ matrix.

**Matrix, zero** [p. 214] The $m \times n$ matrix with all entries equal to zero is called the zero matrix.

**Multiplication of a complex number by *i*** [p. 297] corresponds to a rotation about the origin by 90° anticlockwise. If $z = a + bi$, then $iz = i(a + bi) = -b + ai$.

**Multiplication of a vector by a scalar** [p. 61] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $m \in \mathbb{R}$, then $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j}$.

**Multiplication of complex numbers** [p. 289] If $z_1 = a + bi$ and $z_2 = c + di$, then

$z_1 z_2 = (ac - bd) + (ad + bc)i$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

**Multiplication principle** [p. 2] If there are $m$ ways of performing one task and then there are $n$ ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

# N

***n*!** [p. 6] *see* factorial notation

**Natural numbers** [p. 41] the elements of $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$


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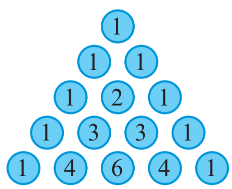
**Negation** [p. 111] The negation of a statement $P$ is the opposite statement, called ‘not $P$’. For example, the negation of $x \geq 1$ is $x < 1$.

**Normal reaction force** [p. 88] A mass placed on a surface (horizontal or inclined) experiences a force perpendicular to the surface, called the normal force.

## P

**Particle model** [p. 88] an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

**Pascal’s triangle** [p. 27] a triangular pattern of numbers formed by the binomial coefficients ${}^nC_r$. Each entry of Pascal’s triangle is the sum of the two entries immediately above.




**Period of a function** [p. 146] A function $f$ with domain $\mathbb{R}$ is periodic if there is a positive constant $a$ such that $f(x + a) = f(x)$ for all $x$. The smallest such $a$ is called the period of $f$. For example, the period of the sine function is $2\pi$, as $\sin(x + 2\pi) = \sin x$.

**Permutation** [p. 6] an ordered arrangement of objects. The number of permutations of $n$ objects taken $r$ at a time is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

**Pigeonhole principle** [p. 30] If $n + 1$ or more objects are placed into $n$ holes, then some hole contains at least two objects.

**Position vector** [p. 63] A position vector, $\overrightarrow{OP}$, indicates the position in space of the point $P$ relative to the origin $O$.

**Principle of mathematical induction** [p. 273] used to prove that a statement is true for all natural numbers

**Product-to-sum identities** [p. 195]

- $2\cos A\cos B = \cos(A - B) + \cos(A + B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$
- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

**Projection** [p. 240] A projection maps each point in the plane onto an axis.

- Projection onto the $x$-axis: $(x, y) \to (x, 0)$
- Projection onto the $y$-axis: $(x, y) \to (0, y)$

**Proof by contradiction** [p. 270] a proof that begins by assuming the negation of what is to be proved

**Pythagorean identity** [pp. 151, 179]

- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

## R

**$\mathbb{R}^2$** [p. 42] $\{(x, y) : x, y \in \mathbb{R}\}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

**Radian** [p. 145] One radian (written $1^c$) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

**Rational number** [p. 41] a number that can be written as $\frac{p}{q}$, for some integers $p$ and $q$ with $q \neq 0$

**Real part of a complex number** [p. 286] If $z = a + bi$, then $\operatorname{Re}(z) = a$.

**Reciprocal trigonometric functions** [pp. 177, 181] the secant, cosecant and cotangent functions

**Reflection** [p. 237] A reflection in a line $\ell$ maps each point in the plane to its mirror image on the other side of the line.

- Reflection in the $x$-axis: $(x, y) \to (x, -y)$
- Reflection in the $y$-axis: $(x, y) \to (-x, y)$
- Reflection in the line $y = x$: $(x, y) \to (y, x)$
- Reflection in the line $y = -x$: $(x, y) \to (-y, -x)$

**Reflection matrix** [p. 244] A reflection in the line $y = mx = x\tan\theta$ is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

**Resultant force** [p. 88] the vector sum of the forces acting at a point

**Rotation matrix** [p. 243] A rotation about the origin by $\theta$ degrees anticlockwise is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$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# S

**Scalar** [p. 61] a real number; name used when working with vectors or matrices

**Scalar product** [p. 71] The scalar product of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$

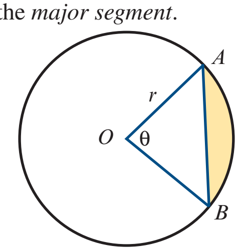
**Scalar product, properties** [p. 72]

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

**Secant function** [pp. 177, 181] $\sec\theta = \dfrac{1}{\cos\theta}$ for $\cos\theta \neq 0$

**Secant of a circle** [p. 128] a line that cuts a circle at two distinct points

**Segment** [p. 128] Every chord divides the interior of a circle into two regions called segments; the smaller is the *minor segment* (shaded), and the larger is the *major segment*.




**Selection** [p. 18] *see* combination

**Set notation** [p. 38]

| | |
|---|---|
| $\in$ | means 'is an element of' |
| $\notin$ | means 'is not an element of' |
| $\subseteq$ | means 'is a subset of' |
| $\cap$ | means 'intersection' |
| $\cup$ | means 'union' |
| $\emptyset$ | is the empty set, containing no elements |
| $\xi$ | is the universal set, containing all elements being considered |
| $A'$ | is the complement of a set $A$ |
| $\lvert A\rvert$ | is the number of elements in a finite set $A$ |

**Sets of numbers** [pp. 41, 285]

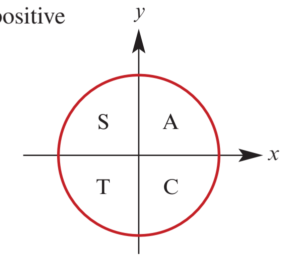
| | |
|---|---|
| $\mathbb{N}$ | is the set of natural numbers |
| $\mathbb{Z}$ | is the set of integers |
| $\mathbb{Q}$ | is the set of rational numbers |
| $\mathbb{R}$ | is the set of real numbers |
| $\mathbb{C}$ | is the set of complex numbers |

**Shear** [p. 239] A shear moves each point in the plane by an amount proportional to its distance from an axis.

- Shear parallel to the $x$-axis: $(x, y) \to (x + cy, y)$
- Shear parallel to the $y$-axis: $(x, y) \to (x, cx + y)$

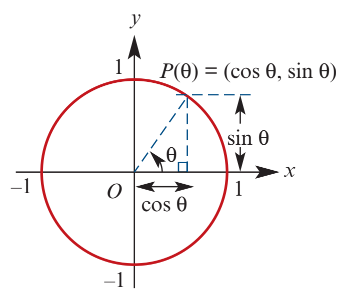
**Signs of trigonometric functions** [p. 149]

| | |
|---|---|
| 1st quadrant | all are positive |
| 2nd quadrant | sin is positive |
| 3rd quadrant | tan is positive |
| 4th quadrant | cos is positive |

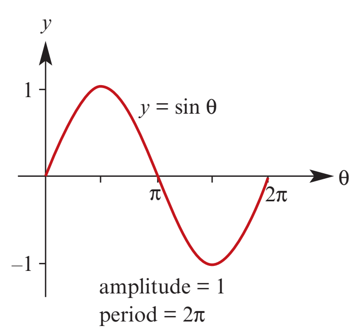



**Simultaneous equations** [p. 225] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

**Sine function** [p. 146] sine θ is defined as the $y$-coordinate of the point $P$ on the unit circle where $OP$ forms an angle of θ radians with the positive direction of the $x$-axis.




**Sine function, graph** [p. 146]





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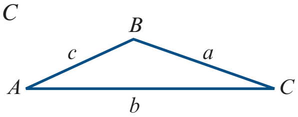
**Sine rule** [p. 71] For triangle $ABC$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$




**Subtraction of complex numbers** [p. 287]
If $z_1 = a + bi$ and $z_2 = c + di$, then
$z_1 - z_2 = (a - c) + (b - d)i$.

**Subtraction of vectors** [p. 61]
If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$, then
$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j}$.

**Sum-to-product identities** [p. 196]

- $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
- $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

# T

**Tangent function** [pp. 147, 157] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$

**Tangent to a circle** [p. 133] a line that touches the circle at exactly one point, called the *point of contact*

**Tension** [p. 89] the pulling force transmitted through a string or cable (or similar) attached to an object.

**Transformation** [p. 233] A transformation of the plane maps each point $(x, y)$ in the plane to a new point $(x', y')$. We say that $(x', y')$ is the *image* of $(x, y)$.

**Translation** [p. 241] a transformation that moves each point in the plane in the same direction and over the same distance: $(x, y) \to (x + a, y + b)$

**Trigonometric functions** [pp. 146, 147] the sine, cosine and tangent functions

**Trigonometric functions, exact values** [p. 148]

| $\theta^c$ | $\theta^\circ$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| 0 | 0° | 0 | 1 | 0 |
| $\frac{\pi}{6}$ | 30° | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ | 45° | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| $\frac{\pi}{3}$ | 60° | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | 90° | 1 | 0 | undefined |

**Trigonometric functions, solving equations** [p. 161]

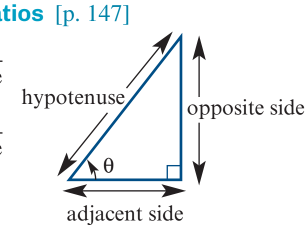
- For $a \in [-1, 1]$, the general solution of the equation $\cos(x) = a$ is $x = 2n\pi \pm \cos^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in [-1, 1]$, the general solution of the equation $\sin(x) = a$ is $x = 2n\pi + \sin^{-1}(a)$ or $x = (2n + 1)\pi - \sin^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in \mathbb{R}$, the general solution of the equation $\tan(x) = a$ is $x = n\pi + \tan^{-1}(a)$, where $n \in \mathbb{Z}$.

**Trigonometric ratios** [p. 147]

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$




# U

**Union of sets** [p. 38] The union of two sets $A$ and $B$, written $A \cup B$, is the set of all elements which are in $A$ or $B$ or both.

**Unit vector** [p. 68] a vector of magnitude 1. The unit vectors in the positive directions of the $x$-, $y$- and $z$-axes are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. The unit vector in the direction of $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

**Universal statement** [pp. 118, 120] a statement claiming that a property holds for all members of a given set. Such a statement can be written using the quantifier 'for all'.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## V

**Vector** [p. 59] a set of equivalent directed line segments

**Vector quantity** [p. 58] a quantity determined by its magnitude and direction; e.g. position, displacement, velocity, acceleration, force

**Vectors, parallel** [p. 62] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = k\boldsymbol{b}$ for some non-zero real number $k$.

**Vectors, perpendicular** [p. 72] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

**Vectors, properties** [pp. 59–61]

- $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ commutative law
- $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$ associative law
- $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$ zero vector
- $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$ additive inverse
- $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$ distributive law

**Vectors, resolution** [p. 75] A vector $\boldsymbol{a}$ is resolved into rectangular components by writing it as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

The *vector resolute* of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$$

## W

**Weight** [p. 88] On Earth's surface, a mass of $m$ kg has a force of $m$ kg wt (or $mg$ newtons) acting on it; this force is known as the weight.

## Z

**Zero vector, 0** [p. 61] a line segment of zero length with no direction


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# Answers

## Chapter 1

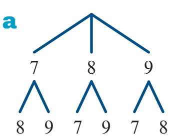
### Exercise 1A

**1** 45

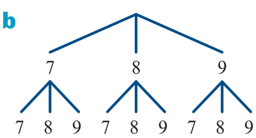
**2** 8

**3** 120

**4 a**




**b**




**5 a** 27 **b** 6

**6** 30

**7 a** 6 **b** 18 **c** 20 **d** 15

**8** BB, BR, BG, RB, RG, GB, GR, GG

**9** 12

**10** 9

**11 a** 6 **b** 13

**12** 16

### Exercise 1B

**1** 1, 1, 2, 6, 24, 120, 720, 5040, 40 320, 362 880, 3 628 800

**2 a** 5 **b** 90 **c** 66 **d** 161 700

**3 a** $n + 1$ **b** $n + 2$ **c** $n(n - 1)$ **d** $\dfrac{n+2}{(n+1)!}$

**4** 1, 4, 12, 24, 24

**5** DOG, DGO, ODG, OGD, GOD, GDO

**6** 120

**7** 362 880

**8** FR, FO, FG, RF, RO, RG, OF, OR, OG, GF, GR, GO

**9 a** 720 **b** 720 **c** 360

**10 a** 120 **b** 120 **c** 60

**11** 20 160

**12 a** 125 **b** 60

**13 a** 120 **b** 360 **c** 720

**14** 60

**15 a** 17 576 000 **b** 11 232 000

**16 a** 384 **b** 3072

**17** $(m, n) = (6, 0), (6, 1), (5, 3)$

**18** $(n^2 - n) \cdot (n - 2)! = n \cdot (n - 1) \cdot (n - 2)! = n!$

**19** 30

### Exercise 1C

**1 a** 120 **b** 72 **c** 24 **d** 96

**2 a** 120 **b** 48 **c** 72 **d** 12

**3 a** 360 **b** 144 **c** 144 **d** 72

**4 a** 1152 **b** 1152

**5 a** 600 **b** 108 **c** 431 **d** 52

**6 a** 720 **b** 48 **c** 144 **d** 96 **e** 48

**7 a** 900 **b** 900

**8** 84

**9** 32

**10 a** 480 **b** 192

**11** 144

### Exercise 1D

**1** 35 **2** 34 650

**3** 4 989 600 **4** 56

**5** 27 720

**6 a** 420 **b** 105 **c** 90 **d** 12 **e** 105

**7** 35

**8 a** 15 **b** $\dfrac{(m+n)!}{m! \cdot n!}$

**9 a** 52! **b** $\dfrac{104!}{(2!)^{52}}$ **c** $\dfrac{(52n)!}{(n!)^{52}}$

**10** 4900

**11** 89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## Exercise 1E

**1** 1, 5, 10, 10, 5, 1

**2 a** 7 **b** 6 **c** 66 **d** 56 **e** 100 **f** 499 500

**3 a** $n$ **b** $\frac{n(n-1)}{2}$ **c** $n$ **d** $n+1$ **e** $\frac{(n+2)(n+1)}{2}$ **f** $\frac{n(n+1)}{2}$

**4 a** 720 **b** 120

**5** 2 598 960

**6 a** 10 **b** 45 **c** 45 **d** 10

**7** 45 379 620

**8** 56

**9 a** 45 **b** 16

**10** 15

**11** $${}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

**12** Each diagonal is obtained by choosing 2 vertices from $n$ vertices. This can be done in ${}^nC_2$ ways. But $n$ of these choices define a side of the polygon, not a diagonal. Therefore there are ${}^nC_2 - n$ diagonals.

**13** There are ${}^{10}C_5$ ways to choose 5 students for team A. The remaining 5 students will belong to team B. However, the labelling of the teams does not matter, so we must divide by 2.

**14** 462

**15** ${}^{n-1}C_{r-1} + {}^{n-1}C_r$

$$= \frac{(n-1)!}{(r-1)!\,(n-1-(r-1))!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right)$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!} \cdot \frac{n}{r(n-r)}$$

$$= \frac{n!}{r!\,(n-r)!} = {}^nC_r$$

**16 a** 2300 **b** 152 **c** 2148

## Exercise 1F

**1** 153 **2** 126

**3** 1176 **4** 140

**5 a** 1716 **b** 700 **c** 980 **d** 1568

**6 a** 25 200 **b** 4200

**7 a** 1 392 554 592 **b** 5 250 960

**8 a** 15 504 **b** 10 800 **c** 15 252

**9 a** 21 **b** 10 **c** 11

**10** 2100

**11 a** 204 490 **b** 7 250 100

**12 a** 48 **b** 210

**13** 1440 **14** 3600

**15** 14 400 **16** 150

**17** 3744

## Exercise 1G

**1** ${}^7C_2 = 21$, ${}^6C_2 = 15$, ${}^6C_1 = 6$

**2** 1, 7, 21, 35, 35, 21, 7, 1; ${}^7C_2 = 21$, ${}^7C_4 = 35$

**3** 1, 8, 28, 56, 70, 56, 28, 8, 1; ${}^8C_4 = 70$, ${}^8C_6 = 28$

**4** $2^6 = 64$

**5** $2^5 = 32$

**6** $2^{10} = 1024$

**7** $2^6 - 1 = 63$

**8** $2^8 - {}^8C_1 - {}^8C_0 = 247$

**9** $2^8 = 256$

**10** $2^4 - 1 = 15$

**11 a** 128 **b** 44

## Exercise 1H

**1** 4

**2** Label 26 holes from A to Z. Put each of the 27 words into the hole labelled by its first letter. Some hole contains at least two words.

**3** Label 4 holes by 0, 1, 2, 3. Put each of the 5 numbers into the hole labelled by its remainder when divided by 4. Some hole contains at least two numbers.

**4 a** 3 **b** 5 **c** 14

**5** Divide [0, 1] into 10 subintervals: [0, 0.1), [0.1, 0.2), . . . , [0.9, 1]. Some interval contains at least two of the 11 numbers.

**6** Divide into 4 equilateral triangles of side length 1 unit as shown. Some triangle contains at least two of the 5 points.

**7** Divide the rectangle into squares of size $2 \times 2$. There are 12 squares and 13 points, so some square contains at least two points.
The distance between two points in the same square cannot exceed the length of the square's diagonal, $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

**8 a** For two-digit numbers, the possible digital sums are $1, 2, \dots, 18$. Since $19 > 18$, some digital sum occurs at least twice.

**b** For three-digit numbers, the possible digital sums are $1, 2, \dots, 27$. Since $82 = 3 \times 27 + 1$, some digital sum occurs at least 4 times.

**9** Label 4 holes by 0, 1, 2, 3. Place each number into the hole labelled by its remainder when divided by 4. Since $13 = 3 \times 4 + 1$, some hole contains at least 4 numbers.


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**10** Two teams can be chosen in ${}^8C_2 = 28$ ways. Since there are 29 games, some pair of teams play each other at least twice.

**11** At least 26 students. To show that 26 numbers suffice, label 25 holes by (1 or 49), (2 or 48), . . . , (24 or 26), (25). To show that 25 numbers do not, consider $1, 2, 3, \dots, 25$.

**12** Label the chairs $1, 2, \dots, 14$. There are 14 groups of three consecutive chairs:

$\{1, 2, 3\}, \{2, 3, 4\}, \dots, \{13, 14, 1\}, \{14, 1, 2\}$

Each of the 10 people belongs to 3 groups, so there are 30 people to be allocated to 14 groups. Since $30 \geq 2 \times 14 + 1$, some group contains at least 3 people.

**13** Draw a diameter through one of the 4 points. This creates 2 half circles. One half circle contains at least two of the 3 remaining points (and the chosen point).

**14** There are 195 possible sums: $3, 4, \dots, 197$. There are ${}^{35}C_2 = 595$ ways to choose a pair of players. Since $595 \geq 3 \times 195 + 1$, at least 4 pairs have the same sum.

**15** Label the chairs $1, 2, \dots, 12$. There are 6 pairs of opposite seats:

$\{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}, \{6, 12\}$

Some pair contains two of the 7 boys.

**16** Label $n$ holes by $0, 1, 2, \dots, n - 1$. Place each guest in the hole labelled by the number of hands they shake. The first or last hole must be empty. (If a guest shakes 0 hands, then no guest shakes $n$ hands. If a guest shakes $n$ hands, then no guest shakes 0 hands.) This leaves $n - 1$ holes, so some hole contains at least two guests.

## Chapter 1 review

### Short-answer questions

**1** **a** 20 **b** 190 **c** 300 **d** 4950

**2** 11

**3** **a** 27 **b** 6

**4** 120 **5** 60

**6** 18 **7** 31

**8** 10 **9** 3

### Extended-response questions

**1** **a** 120 **b** 360 **c** 72 **d** 144

**2** **a** 20 **b** 80 **c** 60

**3** **a** 210 **b** 84 **c** 90 **d** 195

**4** **a** 420 **b** 15 **c** 105 **d** 12

**5** **a** **i** 20 **ii** 10 **iii** 64
**b** 8

**6** **a** 210 **b** 100 **c** 10 **d** 80

**7** **a** 676 **b** 235 **c** 74

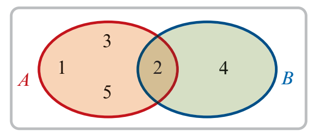
**8** **a** 924
**b** There are at least $365 \times 3 = 1095$ days in three years and there are 924 different paths, so some path is taken at least twice.
**c** **i** 6 **ii** 70 **iii** 420

**9** 196

# Chapter 2

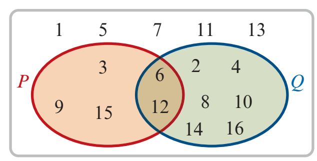
## Exercise 2A

**1** ξ



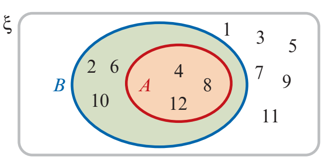

**a** $\{4\}$ **b** $\{1, 3, 5\}$ **c** $\{1, 2, 3, 4, 5\} = \xi$
**d** $\varnothing$ **e** $\varnothing$

**2** ξ



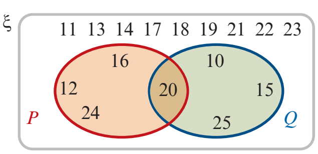

**a** $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16\}$
**b** $\{1, 3, 5, 7, 9, 11, 13, 15\}$
**c** $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}$
**d** $\{1, 5, 7, 11, 13\}$ **e** $\{1, 5, 7, 11, 13\}$

**3** ξ



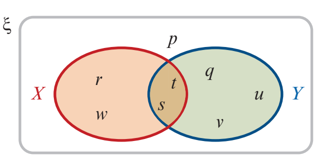

**a** $\{1, 2, 3, 5, 6, 7, 9, 10, 11\}$
**b** $\{1, 3, 5, 7, 9, 11\}$ **c** $\{2, 4, 6, 8, 10, 12\}$
**d** $\{1, 3, 5, 7, 9, 11\}$ **e** $\{1, 3, 5, 7, 9, 11\}$

**4** ξ




**a** $\{10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25\}$
**b** $\{11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24\}$
**c** $\{10, 12, 15, 16, 20, 24, 25\}$
**d** $\{11, 13, 14, 17, 18, 19, 21, 22, 23\}$
**e** $\{11, 13, 14, 17, 18, 19, 21, 22, 23\}$

**5** ξ




**a** $\{p, q, u, v\}$ **b** $\{p, r, w\}$ **c** $\{p\}$
**d** $\{p, q, r, u, v, w\}$ **e** $\{q, r, s, t, u, v, w\}$ **f** $\{p\}$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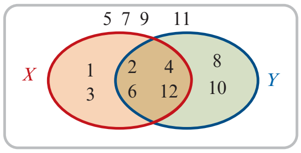
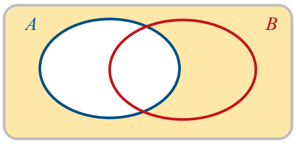
**6** ξ




**a** {5, 7, 8, 9, 10, 11} **b** {1, 3, 5, 7, 9, 11}
**c** {1, 3, 5, 7, 8, 9, 10, 11}
**d** {1, 3, 5, 7, 8, 9, 10, 11}
**e** {1, 2, 3, 4, 6, 8, 10, 12} **f** {5, 7, 9, 11}

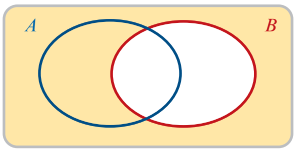
**7 a** ξ

$A'$



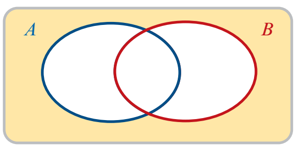

**b** ξ

$B'$



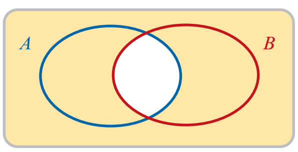

**c** ξ

$A' \cap B'$



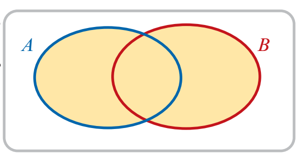

**d** ξ

$A' \cup B'$



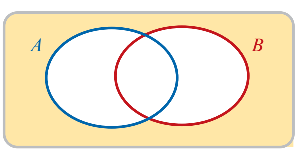

**e** ξ

$A \cup B$

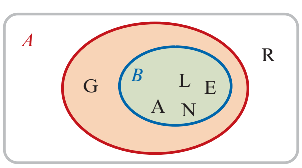



**f** ξ

$(A \cup B)'$

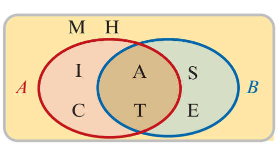



**8** ξ




**a** {R} **b** {G, R} **c** {L, E, A, N}
**d** {A, N, G, E, L} **e** {R} **f** {G, R}

**9** ξ




**a** {E, H, M, S} **b** {C, H, I, M}
**c** {A, T} **d** {H, M} **e** {C, E, H, I, M, S}
**f** {H, M}

## Exercise 2B

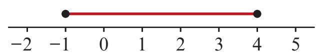
**1 a** Yes **b** Yes **c** Yes

**2 a** No **b** No **c** No

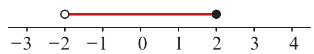
**3 a** $\frac{9}{20}$ **b** $\frac{3}{11}$ **c** $\frac{3}{25}$ **d** $\frac{2}{7}$ **e** $\frac{4}{11}$ **f** $\frac{2}{9}$

**4 a** $0.\dot{2}8571\dot{4}$ **b** $0.\dot{4}\dot{5}$ **c** 0.35
**d** $0.\dot{3}0769\dot{2}$ **e** $0.\dot{0}58823529411764\dot{7}$

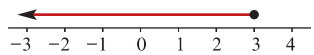
**5 a**



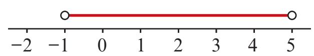

**b**



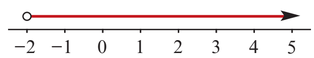

**c**




**d**



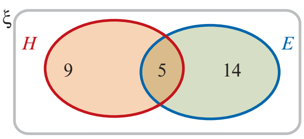

**e**




**6 a** $(-\infty, 3)$ **b** $[-3, \infty)$ **c** $(-\infty, -3]$
**d** $(5, \infty)$ **e** $[-2, 3)$ **f** $[-2, 3]$
**g** $(-2, 3]$ **h** $(-5, 3)$

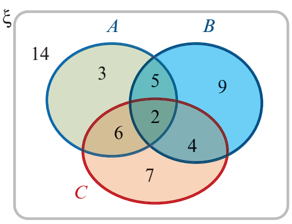
## Exercise 2C

**1 a** ξ




**b i** 19 **ii** 9 **iii** 23

**2 a** ξ




**b i** 23 **ii** 37 **iii** 9

**3** 20%

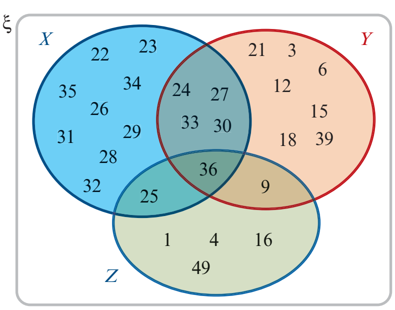
**4** 7

**5 a** 5 **b** 10

**6** 45

**7 a** $x = 5$ **b** 16 **c** 0

**8 a** ξ




**b i** $X \cap Y \cap Z = \{36\}$ **ii** $|X \cap Y| = 5$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**9** 31 students; 15 black, 12 green, 20 red

**10** $|M \cap F| = 11$ **11** 1

**12** $x = 6$; 16 students **13** 102 students

## Exercise 2D

**1 a** $\{1, 3, 4\}$ **b** $\{1, 3, 4, 5, 6\}$ **c** $\{4\}$
**d** $\{1, 2, 3, 4, 5, 6\}$ **e** 3
**f** $\emptyset, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{4, 5, 6\}$

**2** 36

**3** 4

**4** 150

**5 a** 64 **b** 32

**6 a** 72 **b** 72 **c** 36 **d** 108

**7 a** 12 **b** 38

**8** 88 **9** 80

**10** 4

**11 a** 756 **b** 700 **c** 360 **d** 1096

**12** 1 452 555 **13** 3417

**14** 5

## Chapter 2 review

### Short-answer questions

**1 a** $\frac{7}{90}$ **b** $\frac{5}{11}$ **c** $\frac{1}{200}$
**d** $\frac{81}{200}$ **e** $\frac{4}{15}$ **f** $\frac{6}{35}$

**2 a** 15 **b** 15

**3 a** 1 **b** 22 **c** 22

**4** 5 **5** $2\text{ cm}^2$

**6 a** 57 **b** 3 **c** 32

**7** 12

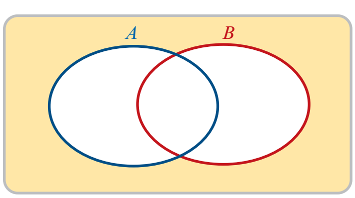
**8** 192

### Extended-response questions

**2 a i** Region 8
**ii** Male, red hair, blue eyes
**iii** Male, not red hair, blue eyes
**b i** 5 **ii** 182

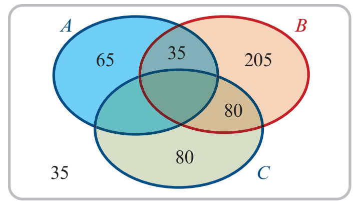
**3 a i** Students shorter than or equal to 180 cm
**ii** Students who are female or taller than 180 cm
**iii** Students who are male and shorter than or equal to 180 cm

**b**




$(A \cup B)' = A' \cap B'$ is shaded

**4 a**




$|A \cap C| = 0$

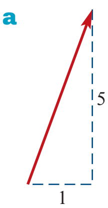
**b** 160 **c** 65 **d** 0

**5 a** 24 **b** 4 **c** 24 **d** $\frac{3}{4}$

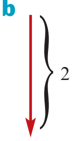
# Chapter 3

## Exercise 3A

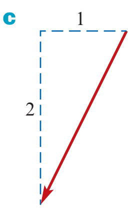
**1 a**



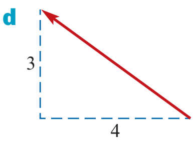

**b**




**c**




**d**




**2** $a = 5,\ b = 1$

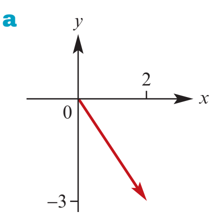
**3** $a = 3,\ b = -15$

**4 a** $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ **d** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **e** $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

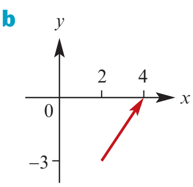
**5 a i** $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **ii** $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

**b** $\boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{c}$

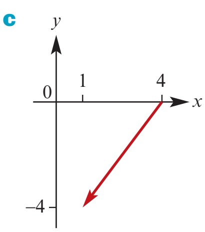
**6 a**



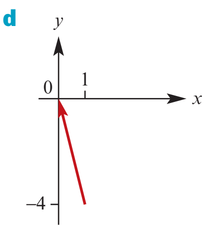

**b**




**c**




**d**





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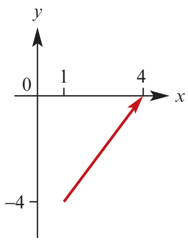
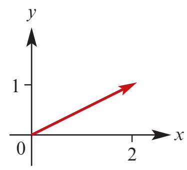
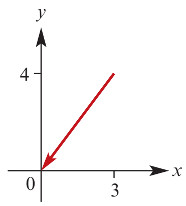
**e**



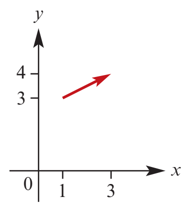

**7 a**



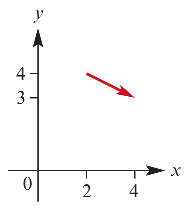

**b**



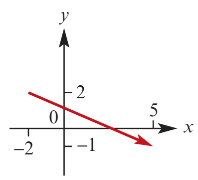

**c**



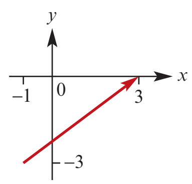

**d**




**e**

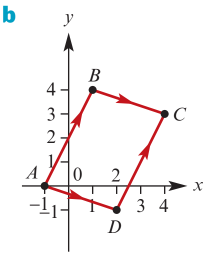



**f**




**8 a** and **c**

**9 a b**




**d** Parallelogram

**10** $m = -11,\ n = 7$

**11 a i** $\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ **ii** $\boldsymbol{b}$

**b** $\overrightarrow{MN} = \overrightarrow{AD}$

**12 a** $\overrightarrow{CB} = \boldsymbol{a} - \boldsymbol{b},\ \overrightarrow{MN} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

**b** $\overrightarrow{CB} = -2\overrightarrow{MN}$

**13 a** $\boldsymbol{a}$ **b** $\boldsymbol{b}$ **c** $2\boldsymbol{a}$ **d** $2\boldsymbol{b}$

**e** $-\boldsymbol{a}$ **f** $\boldsymbol{b} - \boldsymbol{a}$ **g** $\boldsymbol{a} + \boldsymbol{b}$

**14 a** $\boldsymbol{a}$ **b** $-\boldsymbol{b}$ **c** $\boldsymbol{a} + \boldsymbol{b}$

**d** $-\boldsymbol{a} - \boldsymbol{b}$ **e** $\boldsymbol{b} - \boldsymbol{a}$

**15 a** $\boldsymbol{a} - \boldsymbol{b}$ **b** $\frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})$ **c** $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

**d** $\frac{1}{9}(\boldsymbol{a} + 2\boldsymbol{b})$ **e** $\frac{1}{9}(4\boldsymbol{a} - \boldsymbol{b})$

**16 a** $\boldsymbol{u} + \boldsymbol{v}$ **b** $\boldsymbol{v} + \boldsymbol{w}$ **c** $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$

**17 a** $\overrightarrow{OB} = \boldsymbol{u} + \boldsymbol{v},\ \overrightarrow{OM} = \boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$ **b** $\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$

**c** $\frac{2}{3}\left(\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}\right)$

**d** $\overrightarrow{OP} = \frac{2}{3}(\boldsymbol{u} + \boldsymbol{v}) = \frac{2}{3}\overrightarrow{OB}$ **e** 2 : 1

## Exercise 3B

**1** $2\boldsymbol{i} - 7\boldsymbol{j}$

**2 a** $5\boldsymbol{i} + 6\boldsymbol{j}$ **b** $-5\boldsymbol{i} + 6\boldsymbol{j}$ **c** $5\boldsymbol{i} - 6\boldsymbol{j}$

**3 a** 5 **b** 2 **c** 5 **d** 13

**4 a** 13 **b** $x = 2,\ y = -7$

**5** $7\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

**6 a i** $\frac{2}{5}\boldsymbol{i}$ **ii** $\frac{-2}{5}\boldsymbol{i} + \boldsymbol{j}$ **iii** $\frac{1}{6}\left(-\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}\right)$

**iv** $\frac{1}{3}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$ **v** $2\boldsymbol{i} + \boldsymbol{j}$

**b i** $\overrightarrow{ON} = \frac{1}{6}\overrightarrow{OA}$ **ii** 1 : 5

**7** $4\sqrt{2}$ units

**8 a** $k = \frac{3}{2},\ \ell = \frac{1}{2}$ **b** $x = 6,\ y = 2$

**c** $x = 3, y = 3$ **d** $k = -\frac{1}{3},\ \ell = -\frac{5}{3}$

**9** $3\boldsymbol{i} - 2\boldsymbol{j},\ \sqrt{13}$

**10 a** $-2\boldsymbol{i} + 4\boldsymbol{j}$ **b** $-6\boldsymbol{i} + \boldsymbol{j}$ **c** 5

**11 a** $D(-6, 3)$ **b** $F(4, -3)$ **c** $G\left(\frac{3}{2}, -\frac{3}{2}\right)$

**12** $A(-1, -4),\ B(-2, 2),\ C(0, 10)$

**13 a i** $2\boldsymbol{i} - \boldsymbol{j}$ **ii** $-5\boldsymbol{i} + 4\boldsymbol{j}$ **iii** $\boldsymbol{i} + 7\boldsymbol{j}$

**iv** $6\boldsymbol{i} + 3\boldsymbol{j}$ **v** $6\boldsymbol{i} + 3\boldsymbol{j}$

**b** $D(8, 2)$

**14 a** $\overrightarrow{OP} = 12\boldsymbol{i} + 5\boldsymbol{j},\ \overrightarrow{PQ} = 6\boldsymbol{i} + 8\boldsymbol{j}$ **b** 13, 10

**15 a i** $\sqrt{29}$ **ii** $\sqrt{116}$ **iii** $\sqrt{145}$

**b** $(\sqrt{29})^2 + (\sqrt{116})^2 = (\sqrt{145})^2$

**16 a i** $-\boldsymbol{i} - 3\boldsymbol{j}$ **ii** $4\boldsymbol{i} + 2\boldsymbol{j}$ **iii** $-3\boldsymbol{i} + \boldsymbol{j}$

**b i** $\sqrt{10}$ **ii** $2\sqrt{5}$ **iii** $\sqrt{10}$

**17 a i** $-3\boldsymbol{i} + 2\boldsymbol{j}$ **ii** $7\boldsymbol{j}$

**iii** $-3\boldsymbol{i} - 5\boldsymbol{j}$ **iv** $\frac{1}{2}(-3\boldsymbol{i} - 5\boldsymbol{j})$

**b** $M\left(\frac{-3}{2}, \frac{9}{2}\right)$

**18 a** $\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ **b** $\frac{1}{\sqrt{10}}(3\boldsymbol{i} - \boldsymbol{j})$

**c** $\frac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$ **d** $\frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

**e** $\frac{6}{\sqrt{13}}\left(\frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}\right)$ **f** $\frac{1}{\sqrt{13}}(3\boldsymbol{i} - 2\boldsymbol{j})$

## Exercise 3C

**1 a** 17 **b** 13 **c** 8 **d** −10

**e** −4 **f** 3 **g** −58

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**2 a** $5$ **b** $13$ **c** $8$ **d** $-5$ **e** $13$

**3 a** $15\sqrt{2}$ **b** $-15\sqrt{2}$

**4 a** $|\boldsymbol{a}|^2 + 4|\boldsymbol{b}|^2 + 4\boldsymbol{a}\cdot\boldsymbol{b}$ **b** $4\boldsymbol{a}\cdot\boldsymbol{b}$
**c** $|\boldsymbol{a}|^2 - |\boldsymbol{b}|^2$ **d** $|\boldsymbol{a}|$

**5 a** $-3\boldsymbol{i} + \boldsymbol{j}$ **b** $\sqrt{10}$ **c** $116.57°$

**6** $\sqrt{66}$

**7 a** $-\frac{11}{2}$ **b** $\frac{10}{3}$ **c** $-1$ **d** $\frac{-2 \pm \sqrt{76}}{6}$

**8 a** $-\boldsymbol{a} + q\boldsymbol{b}$ **b** $\frac{22}{29}$ **c** $\left(\frac{44}{29}, \frac{110}{29}\right)$

**9 a** $139.40°$ **b** $71.57°$ **c** $26.57°$ **d** $126.87°$

**11 a** $\frac{3}{2}\boldsymbol{i}$ **b** $45°$ **c** $116.57°$

**12 a i** $\frac{3}{2}\boldsymbol{i} + 2\boldsymbol{j}$ **ii** $\frac{1}{2}\boldsymbol{i} + 3\boldsymbol{j}$ **b** $27.41°$ **c** $55.30°$

## Exercise 3D

**1 a** $\frac{1}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$ **b** $\frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j})$ **c** $\frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

**2 a i** $\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ **ii** $\sqrt{2}$
**b** $\frac{\sqrt{2}}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

**3 a i** $\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ **ii** $\frac{1}{13}(5\boldsymbol{i} + 12\boldsymbol{j})$
**b** $\frac{1}{\sqrt{65}}(4\boldsymbol{i} + 7\boldsymbol{j})$

**4 a** $-\frac{11}{17}(\boldsymbol{i} - 4\boldsymbol{j})$ **b** $\frac{13}{17}(\boldsymbol{i} - 4\boldsymbol{j})$ **c** $4\boldsymbol{i}$

**5 a** $2$ **b** $\frac{1}{\sqrt{5}}$ **c** $\frac{2\sqrt{3}}{\sqrt{7}}$ **d** $\frac{-1 - 4\sqrt{5}}{\sqrt{17}}$

**6 a** $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i}$ and $\boldsymbol{w} = \boldsymbol{j}$
**b** $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{w} = \boldsymbol{i} - \boldsymbol{j}$
**c** $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{w} = -\boldsymbol{i} + \boldsymbol{j}$

**7 a** $2\boldsymbol{i} + 2\boldsymbol{j}$ **b** $\frac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$

**8 a** $\frac{3}{2}(\boldsymbol{i} - \boldsymbol{j})$ **b** $\frac{5}{2}(\boldsymbol{i} + \boldsymbol{j})$ **c** $\frac{5\sqrt{2}}{2}$

**9 a i** $\boldsymbol{i} - \boldsymbol{j}$ **ii** $\boldsymbol{i} - 5\boldsymbol{j}$
**b** $\frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$ **c** $\frac{2\sqrt{26}}{13}$ **d** $2$

## Exercise 3E

**1 a i** $\frac{4}{5}\boldsymbol{p}$ **ii** $\frac{1}{5}\boldsymbol{p}$ **iii** $-\boldsymbol{p}$ **iv** $\frac{1}{5}(\boldsymbol{q} - \boldsymbol{p})$ **v** $\frac{1}{5}\boldsymbol{q}$
**b** $RS$ and $OQ$ are parallel
**c** $ORSQ$ is a trapezium
**d** $120\text{ cm}^2$

**2 a i** $\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$ **ii** $\frac{k}{7}\boldsymbol{a} + \frac{6}{7}\boldsymbol{b}$
**b i** $3$ **ii** $\frac{7}{2}$

**3 a i** $\overrightarrow{OD} = 2\boldsymbol{i} - 0.5\boldsymbol{j}$, $\overrightarrow{OE} = \frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}$
**ii** $\frac{\sqrt{170}}{4}$
**b i** $p\left(\frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}\right)$
**ii** $(q + 2)\boldsymbol{i} + (4q - 0.5)\boldsymbol{j}$
**c** $p = \frac{2}{3}$, $q = \frac{1}{2}$

**5 a i** $\overrightarrow{AB} = \boldsymbol{c}$ **ii** $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{c}$ **iii** $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$
**b** $|\boldsymbol{c}|^2 - |\boldsymbol{a}|^2$

**6 a** $\boldsymbol{r} + \boldsymbol{t}$ **b** $\frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

# Chapter 3 review

## Short-answer questions

**1 a** $\frac{12}{7}$ **b** $\pm 9$

**2** $A(2, -1)$, $B(5, 3)$, $C(3, 8)$, $D(0, 4)$

**3** $p = \frac{1}{6}$

**4 a** $\sqrt{26}$ **b** $\frac{1}{\sqrt{26}}(\boldsymbol{i} - 5\boldsymbol{j})$

**5** $6$

**6 a** $\frac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$ **b** $\frac{16}{25}(4\boldsymbol{i} + 3\boldsymbol{j})$

**7 a i** $\boldsymbol{a} + \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$ **iii** $\boldsymbol{b} - \boldsymbol{a}$
**iv** $\frac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$ **v** $\frac{2}{3}(2\boldsymbol{a} - \boldsymbol{b})$
**b** $\overrightarrow{TR} = 2\overrightarrow{PT}$, so $P$, $T$ and $R$ are collinear

**8 a** $s = -2$, $t = 5$
**b** $\sqrt{29}$

**9** $\sqrt{109}$ units

**10 a** $(-1, 10)$ **b** $h = 3$, $k = -2$

**11** $m = 2$, $n = 1$

**12 a** $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c}$ **b** $\boldsymbol{b} = \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$

**13 a** $13$ **b** $10$ **c** $8$ **d** $-11$
**e** $-9$ **f** $0$ **g** $-27$

**15 a** $\frac{6}{5}$ **b** $\pm\frac{3}{\sqrt{2}}$ **c** $\frac{7}{3}$

**16 a i** $\overrightarrow{AB} = -\boldsymbol{i}$ **ii** $\overrightarrow{AC} = -5\boldsymbol{j}$
**b** $\boldsymbol{0}$ **c** $1$

## Extended-response questions

**1 a** $\begin{bmatrix} -31 \\ -32 \end{bmatrix}$ **b** $\begin{bmatrix} -15 \\ -20 \end{bmatrix}$ **c** $|\overrightarrow{OR}| = 25$

**2 a** $\sqrt{34}$ **b** $\sqrt{10} - \sqrt{20}$ **c** $\boldsymbol{r} = \boldsymbol{i} - 9\boldsymbol{j}$

**3 a** $(25, -7)$, $\begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **b** $\begin{bmatrix} -20 \\ 15 \end{bmatrix}$

**4 a** $(12, 4)$ **b** $\begin{bmatrix} k - 12 \\ -4 \end{bmatrix}$
**c** $\sqrt{160}$, $k$, $\sqrt{(k - 12)^2 + 16}$, $k = \frac{40}{3}$
**d** $34.7°$


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# Chapter 4

## Exercise 4A

**1** $T_1 = 3$ kg wt, $T_2 = 7$ kg wt

**2** $T_1 = T_2 = \frac{5\sqrt{2}}{2}$ kg wt

**3** 90°

**4** $T_1 = 14.99$ kg wt, $T_2 = 12.10$ kg wt

**5** 28.34 kg wt, W48.5°S

**6** $T = 40$ kg wt, $N = 96$ kg wt

**7** $F = 6.39$ kg wt

**8** **a** No **b** Yes

**9** 146.88°, 51.32°, 161.8°

**10** **a** 7.5 kg wt **b** 9.64 kg wt **c** 7.62 kg wt

**11** 32.97 kg wt, 26.88 kg wt, 39.29 kg wt, $W = 39.29$ kg

## Exercise 4B

**1** 13.05 kg wt

**2** 5.74 kg wt

**3** 3.73 kg wt, 8.83 kg wt

**4** 4.13 kg wt

**5** 6.93 kg wt

**6** 31.11 kg, 23.84 kg wt

**7** 44.10 kg, 22.48° to the vertical

**8** 6.43 kg wt, 7.66 kg wt, 11.92 kg

**9** 3.24 kg wt

## Chapter 4 review

### Short-answer questions

**1** 9 kg wt, 12 kg wt

**2** $10\sqrt{3}$ kg wt, 150° to the 10 kg wt

**3** $14\sqrt{5}$ kg wt, $28\sqrt{5}$ kg wt

**4** $5\sqrt{3}$ kg wt

**5** $-\frac{7}{8}$

**6** $\frac{40\sqrt{3}}{3}$ kg wt

**7** $\frac{15\sqrt{2}}{2}$ kg wt

**8** 28 kg, $14\sqrt{3}$ kg wt

**9** $4\sqrt{3}$ kg wt

# Chapter 5

### Short-answer questions

**1** 24

**2** 360

**3** **a** 125 **b** 60

**4** **a** 9 **b** 25

**5** **a** 24 **b** 30 **c** 28 **d** 45

**6** **a** 120 **b** 120

**7** **a** 120 **b** 36

**8** **a** 96 **b** 24 **c** 72 **d** 60

**9** 10

**10** **a** 20 **b** 325 **c** 210 **d** 56

**11** **a** 28 **b** 21 **c** $2^8 = 256$

**12** 60

**13** 120

**14** 7

**15** **a** 3 **b** 12 **c** 8

**16** **a** 13 **b** 13 **c** 13 **d** −13 **e** 5 **f** 0 **g** −13

**17** **a** $m = \frac{46}{11}$, $n = -\frac{18}{11}$ **b** $p = -48$ **c** $p = 3, 5$

**18** $F = 7$ kg wt, $\cos\theta = \frac{-31}{49}$

**19** $\cos\theta = \frac{-5}{8}$

**20** $Q^2 = 100 - 48\sqrt{2}$

**21** **a** $T = 5$ kg wt, $N = 5\sqrt{3}$ kg wt
**b** $T = \frac{10\sqrt{3}}{3}$ kg wt, $N = \frac{20\sqrt{3}}{3}$ kg wt

**22** $T = 10$ kg wt, $\tan\theta = \frac{3}{4}$

**23** $\frac{50}{13}$ kg wt, $\frac{120}{13}$ kg wt

### Extended-response questions

**1** **a** 2160 **b** 360 **c** 900 **d** 1260

**2** **a** 70 **b** 30 **c** 15 **d** 55

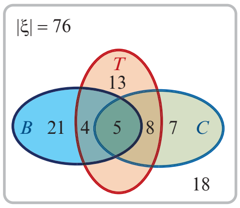
**3** **a** 20 **b** 4 **c** 68

**4** **a** 420 **b** 60 **c** 120 **d** 24

**5** **a** 300 **b** 10 and 15

**6** **a** $|B' \cap C' \cap T| = |C \cap T|$, $|B \cap C' \cap T'| = 3|B' \cap C \cap T'|$, $|B \cap C' \cap T| = 4$

**b**




**c** **i** 5 **ii** 0

**7** **a** $\overrightarrow{AE} = \frac{1}{t+1}(2\boldsymbol{a} + t\boldsymbol{b})$

**b** $\overrightarrow{AE} = \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$ **d** $t = \frac{9}{7}$

**8** **b** $(n-1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c}$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**9 a** $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$, $\overrightarrow{PQ} = \frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

**b i** $n\left(\frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$ **ii** $\left(k + \frac{1}{2}\right)\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

**c** $n = \frac{5}{3}$, $k = \frac{1}{3}$

**10 a** $4\sqrt{2}$ km/h blowing from the south-west

**b** $\sqrt{5}$ km/h; 200 m downstream

**c** 43.1 km/h at bearing 80° **d** 222°

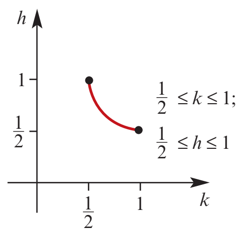
**11 b ii** $\overrightarrow{ZG} = \frac{1}{3h}\overrightarrow{ZH} + \frac{1}{3k}\overrightarrow{ZK}$

**iii** $\frac{1}{h} + \frac{1}{k} = 3$ **iv** $h = \frac{2}{3}$; similarity

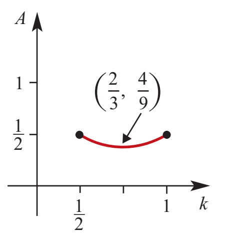
**v** $\frac{4}{9}$ cm$^2$

**vi** $h = \frac{1}{2}$; $H$ is midpoint of $ZX$, $K = Y$

**vii**




**viii**




# Chapter 6

See solutions supplement

# Chapter 7

## Exercise 7A

**1 a** $x = 100$, $y = 50$

**b** $x = 126$, $y = 252$, $z = 54$

**c** $y = 145$, $z = 290$

**d** $x = 180$, $y = 90$

**e** $x = 45$, $y = 90$, $z = 270$

**2 a** $x = 68$, $y = 121$ **b** $x = 112$, $y = 87$

**c** $x = 50$, $y = 110$

**3** 110°, 110°, 140°

**4** $\angle ABC = 98°$, $\angle BCD = 132°$, $\angle CDE = 117°$, $\angle DEA = 110°$, $\angle EAB = 83°$

**7** 60° or 120°

**8** $\angle P = 78°$, $\angle Q = 74°$, $\angle R = 102°$, $\angle S = 106°$

## Exercise 7B

**1 a** $x = 73$, $y = 81$ **b** $x = 57$, $q = 57$

**c** $x = 53$, $y = 74$, $z = 53$

**d** $x = 60$, $y = 60$, $z = 20$, $w = 100$

**e** $w = 54$, $x = 54$, $y = 72$, $z = 54$

**2 a** 40° **b** 40° **c** 80°

**3** 32° and 148°

**4** $\angle ACB = 40°$, $\angle ABC = 70°$, $\angle BAT = 40°$

## Exercise 7C

**1 a** 10 cm **b** 6 cm

**2** 7 cm

**3** $5\sqrt{6}$ cm

## Chapter 7 review

### Short-answer questions

**1** $\angle MCN = 18°$

**2 a** $x = 110$, $y = 70$ **b** $x = 35$, $y = 35$

**c** $x = 47$, $y = 53$, $z = 100$

**d** $x = 40$, $y = 40$, $z = 70$

**6 a** $x = 66$ **b** $x = 116$ **c** $x = 66$, $y = 114$

**8** 3 cm

### Extended-response questions

**5 b** 24 cm$^2$

# Chapter 8

## Exercise 8A

**1 a** $4\pi$ **b** $3\pi$ **c** $-\frac{5\pi}{2}$ **d** $\frac{\pi}{12}$

**e** $-\frac{\pi}{18}$ **f** $-\frac{7\pi}{4}$

**2 a** 225° **b** −120° **c** 105° **d** −330°

**e** 260° **f** −165°

**3 a** 0 **b** −1 **c** 1 **d** −1

**4 a** −1 **b** −1 **c** 1 **d** 1

## Exercise 8B

**1 a** $-\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** 1 **d** 1 **e** $\frac{1}{\sqrt{2}}$

**f** $\frac{1}{\sqrt{2}}$ **g** 0 **h** $\frac{\sqrt{3}}{2}$ **i** 0 **j** 0

**k** 1 **l** 0 **m** $-\frac{1}{2}$ **n** −1 **o** −1

**2 a** $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $\frac{\sqrt{3}}{2}$ **d** $-\frac{1}{2}$

**e** $\frac{1}{\sqrt{2}}$ **f** $\frac{\sqrt{3}}{2}$

**3 a** 0.6 **b** 0.6 **c** 0.3 **d** −0.3

**e** −0.3 **f** 0.6 **g** −0.6 **h** −0.3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4 $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = -\frac{1}{\sqrt{3}}$

5 $\sin x = -\frac{\sqrt{51}}{10}$, $\tan x = \frac{\sqrt{51}}{7}$

6 $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = \frac{1}{\sqrt{3}}$

7 $\cos x = \frac{\sqrt{91}}{10}$, $\tan x = -\frac{3\sqrt{91}}{91}$

## Exercise 8C

1 $2\pi - a,\ 2\pi - b,\ 2\pi - c,\ 2\pi - d$

2 a $\frac{4\pi}{3}, \frac{5\pi}{3}$ b $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ d $\frac{5\pi}{6}, \frac{3\pi}{2}$

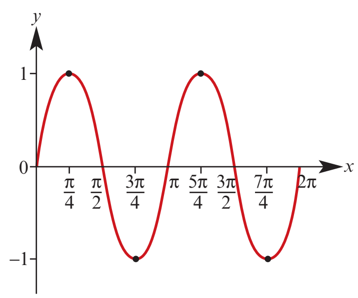
e $0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ f $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

3 a $-\frac{5\pi}{6}, -\frac{\pi}{6}$ b $0, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, -\pi, \pi$

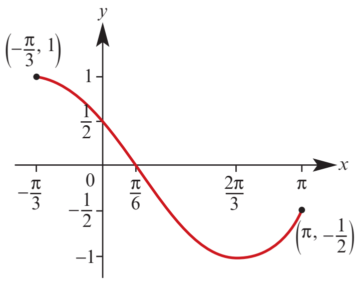
c 0 d $0, -\frac{2\pi}{3}$ e $-\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$

## Exercise 8D

1 a




b

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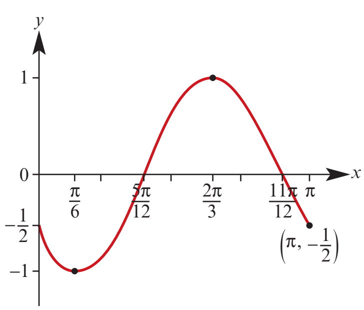
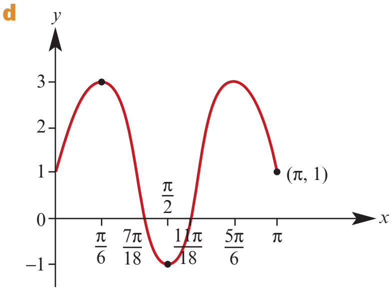



d

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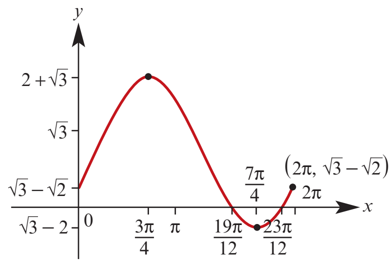
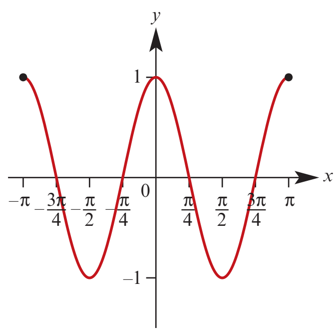
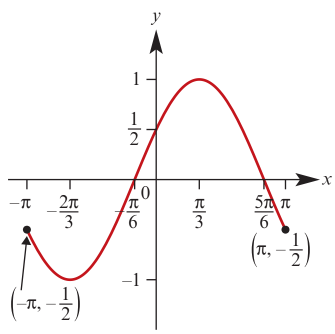


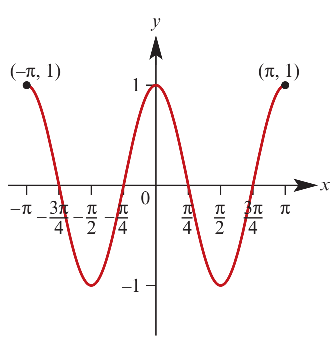

f




2 a




b



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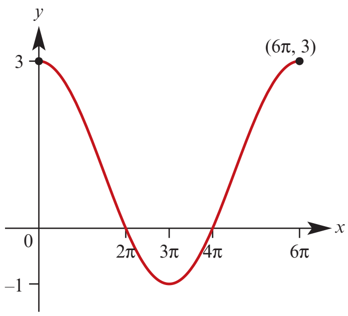
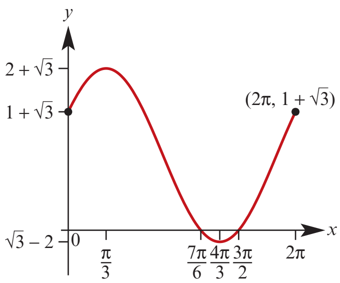
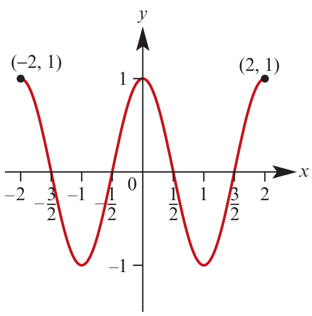
**c**



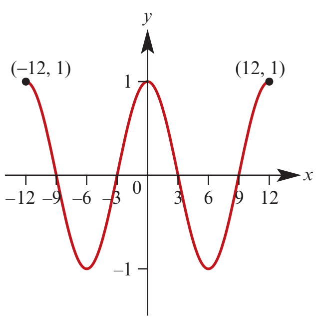

**d**




**e**




**f**




## Exercise 8E

**1 a** $1$ **b** $\sqrt{3}$ **c** $\dfrac{1}{\sqrt{3}}$

**2 a** $\sqrt{3}$ **b** $\dfrac{1}{\sqrt{3}}$ **c** $-1$

**3 a** $\dfrac{-\sqrt{17}}{17}$ **b** $\dfrac{-4\sqrt{17}}{17}$ **c** $\dfrac{-1}{4}$ **d** $\dfrac{-1}{4}$

**4 a** $\dfrac{\sqrt{21}}{7}$ **b** $\dfrac{-2\sqrt{7}}{7}$ **c** $\dfrac{\sqrt{3}}{2}$ **d** $\dfrac{-\sqrt{3}}{2}$

**5 a** $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ **b** $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$ **c** $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$

**d** $-\dfrac{7\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}$ **e** $-\dfrac{5\pi}{6}, -\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{2\pi}{3}$

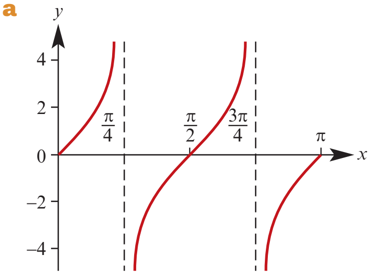
**f** $-\dfrac{7\pi}{12}, -\dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{11\pi}{12}$

**6 a** $\dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{15\pi}{8}$

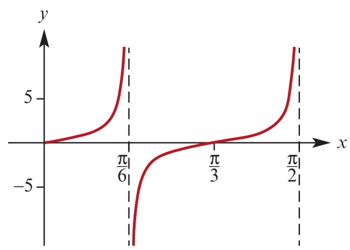
**b** $-\dfrac{7\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}$

**c** $-\dfrac{13\pi}{18}, -\dfrac{7\pi}{18}, -\dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{11\pi}{18}, \dfrac{17\pi}{18}$ **d** $-\dfrac{\pi}{6}$

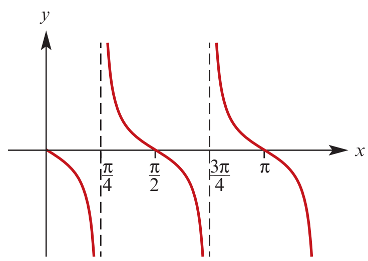
**7 a**



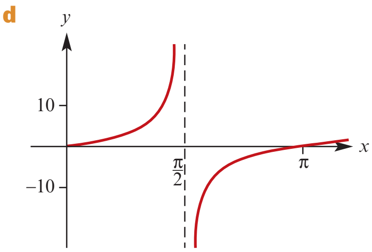

**b**




**c**




**d**





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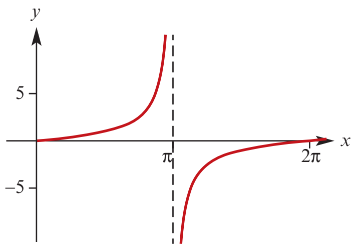
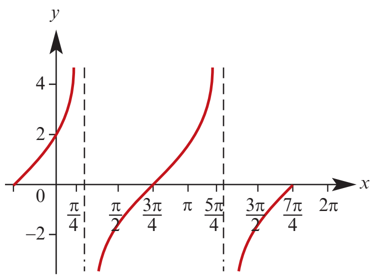
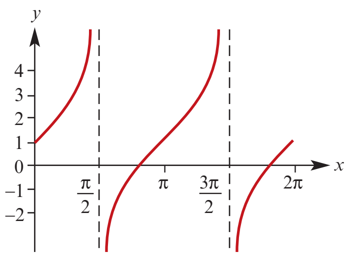
**e**



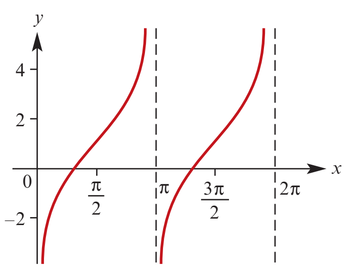

**f**



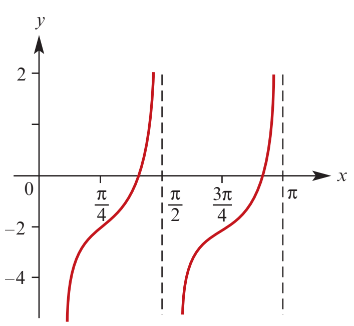

**g**



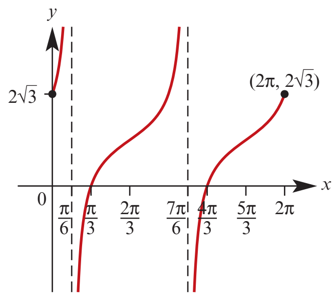

**h**



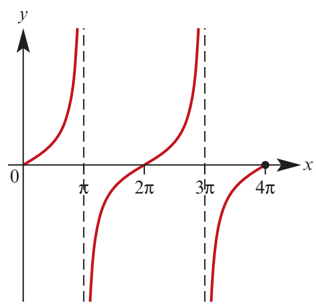

**i**



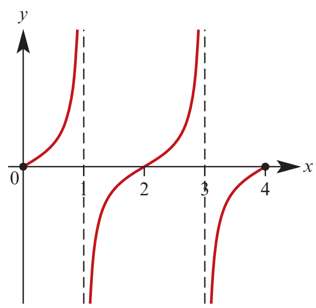

**8 a**




**b**




**c**




## Exercise 8F

**1 a i** $2\pi$ **ii** $4\pi$ **iii** $-4\pi$

**b i** $\frac{4\pi}{3}, \frac{8\pi}{3}$ **ii** $\frac{14\pi}{3}, \frac{10\pi}{3}$ **iii** $\frac{-14\pi}{3}, \frac{-10\pi}{3}$

**2 a** $2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

**b** $\frac{2n\pi}{3} + \frac{\pi}{9}$ or $\frac{2n\pi}{3} + \frac{2\pi}{9}, n \in \mathbb{Z}$

**c** $n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$

**3 a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{12}, \frac{11\pi}{12}$ **c** $\frac{\pi}{3}, \frac{5\pi}{6}$

**4** $\frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

**5** $\frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$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**6 a** $x = n\pi - \frac{\pi}{6}$ or $x = n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

**b** $x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in \mathbb{Z}$

**c** $x = 2n\pi + \frac{5\pi}{6}$ or $x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

**7** $x = \frac{(4n-1)\pi}{4}$ or $x = n\pi, n \in \mathbb{Z}$; $\left\{\frac{-5\pi}{4}, -\pi, \frac{\pi}{4}, 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}\right\}$

**8** $x = \frac{n\pi}{3}, n \in \mathbb{Z}; \left\{-\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0\right\}$

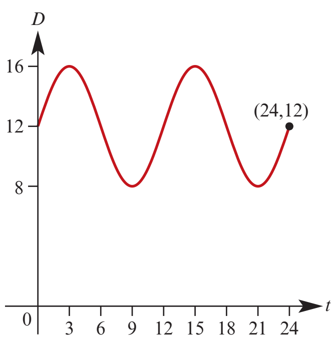
**9** $x = \frac{6n-1}{12}$ or $x = \frac{3n+2}{6}, n \in \mathbb{Z}$; $\left\{\frac{-2}{3}, \frac{-7}{12}, \frac{-1}{6}, \frac{-1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}\right\}$

## Exercise 8G

**1 a i** 0.00 hours **ii** 24.00 hours

**b** 13 February ($t = 1.48$), 24 October ($t = 9.86$)

**2 a**




**b** $t \in [0, 6] \cup [12, 18]$ **c** 15.9 m

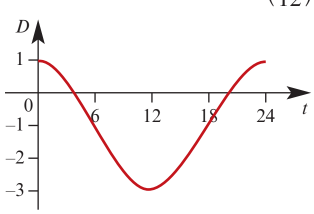
**3 a** 7 metres **b** 1 metre

**c** $t = \frac{1}{4}$ or $t = \frac{5}{4}$ **d** $t = \frac{3}{4}$ or $t = \frac{7}{4}$

**e** Particle oscillates between $x = 1$ and $x = 7$

**4 a** 19.5°C **b** $D = -1 + 2\cos\left(\frac{\pi t}{12}\right)$

**c**




**d** $\{t : 4 < t < 20\}$

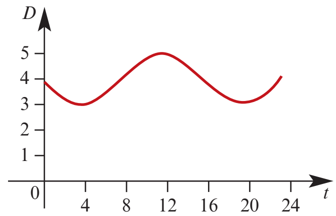
**5 a** 2 a.m. **b** 8 a.m. and 8 p.m.

**6 a i** $\frac{3}{2}$ **ii** 12 **iii** $d(t) = \frac{7}{2} - \frac{3}{2}\cos\left(\frac{\pi}{6}t\right)$

**iv** 1.5 m

**b** Between 9 p.m. and 3 a.m. and between 9 a.m. and 3 p.m. each day

**7 a**



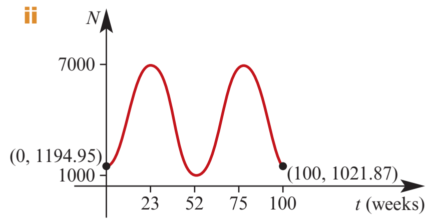

**b** The boat can enter at 8 a.m. and must leave by 4 p.m.

**c** The boat can enter at 6:40 a.m. and must leave by 5:20 p.m.

**8 a i** 52 weeks **ii** 3000 **iii** [1000, 7000]

**b i** $N(0) = 1194.95$, $N(100) = 1021.87$

**ii**




**c i** $t = 23, 75$ **ii** 49

**d** $\left(14\frac{1}{3}, 31\frac{2}{3}\right) \cup \left(66\frac{1}{3}, 83\frac{2}{3}\right)$

**e** $d = 25\,000$, $a = 15\,000$, $b = 10$, $c = 5$

# Chapter 8 review

## Short-answer questions

**1 a** $\frac{13\pi}{6}$ **b** $\frac{14\pi}{3}$ **c** $\frac{37\pi}{6}$ **d** $\frac{71\pi}{12}$ **e** $\frac{11\pi}{12}$

**f** $\frac{5\pi}{2}$ **g** $\frac{7\pi}{3}$ **h** $\frac{13\pi}{6}$ **i** $\frac{2\pi}{9}$

**2 a** 330° **b** 765° **c** 405° **d** 105°

**e** 1530° **f** −495° **g** −225° **h** −585°

**i** 1035°

**3 a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** −1 **d** 0 **e** $\frac{\sqrt{3}}{2}$

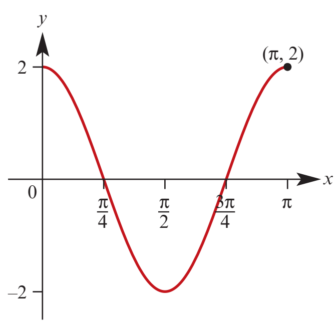
**f** −1 **g** $-\sqrt{3}$ **h** 1

**4 a** $4, 4\pi$ **b** $5, \frac{\pi}{3}$ **c** $\frac{1}{3}, \frac{\pi}{2}$ **d** $2, \frac{2\pi}{5}$

**e** 7, 8 **f** $\frac{2}{3}, 3$

**5 a** 5, 1 **b** 9, −1

**6 a** $y = 2\cos(2x)$





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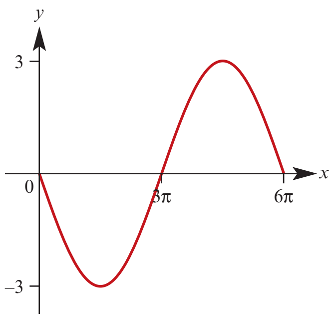
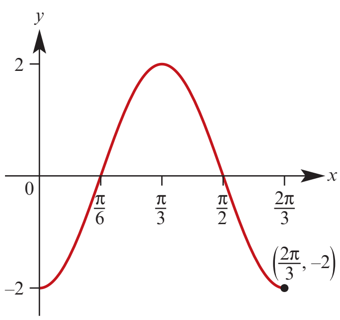
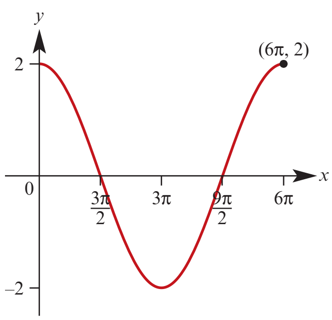
**b** $y = -3\sin\left(\frac{x}{3}\right)$



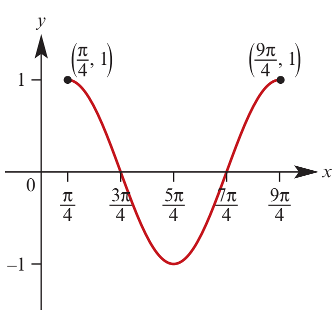

**c** $y = -2\cos(3x)$



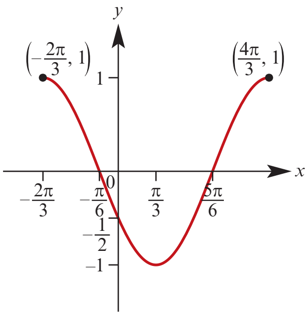

**d** $y = 2\cos\left(\frac{x}{3}\right)$



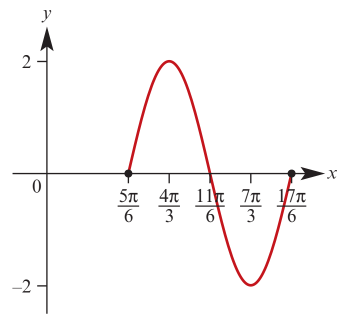

**e** $y = \cos\left(x - \frac{\pi}{4}\right)$



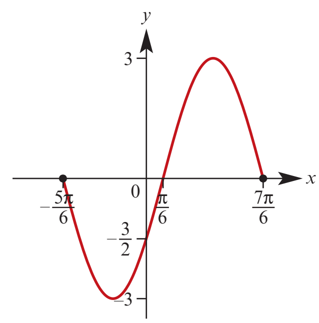

**f** $y = \cos\left(x + \frac{2\pi}{3}\right)$




**g** $y = 2\sin\left(x - \frac{5\pi}{6}\right)$



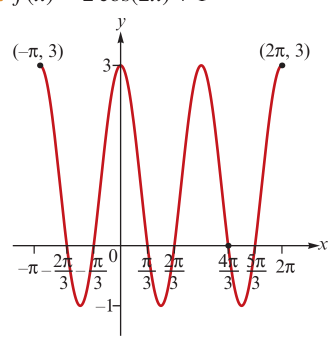

**h** $h = -3\sin\left(x + \frac{5\pi}{6}\right)$




**7 a** $-\frac{5\pi}{6}, \frac{5\pi}{6}$ **b** $-\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$

**c** $\pi, \frac{5\pi}{3}$ **d** $\frac{2\pi}{3}$ **e** $\pi, \frac{5\pi}{3}$

**8 a** $f(x) = 2\cos(2x) + 1$





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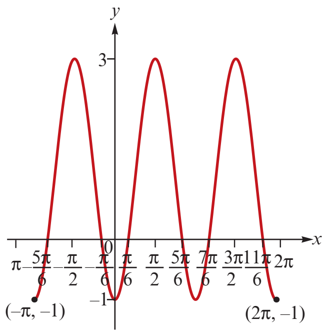
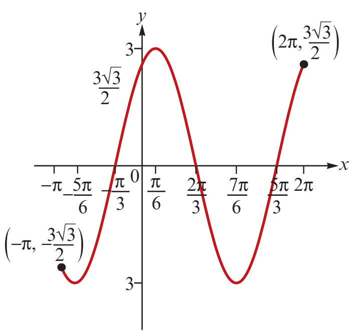
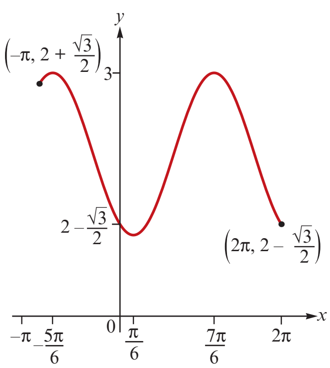
**b** $f(x) = 1 - 2\cos(2x)$



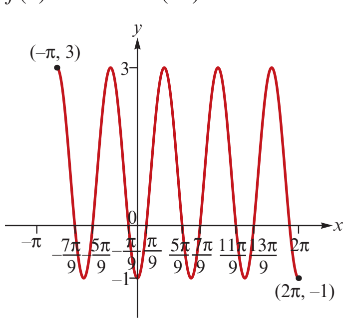

**c** $f(x) = 3\sin\left(x + \frac{\pi}{3}\right)$




**d** $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$



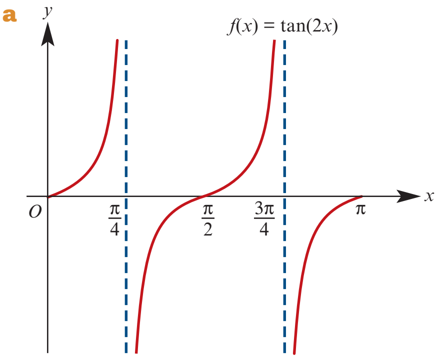

**e** $f(x) = 1 - 2\cos(3x)$



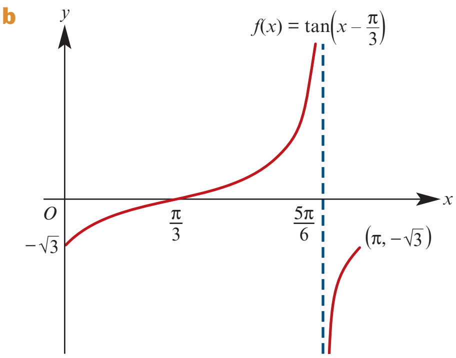

**9 a** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

**c** $\frac{3\pi}{2}$ **d** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

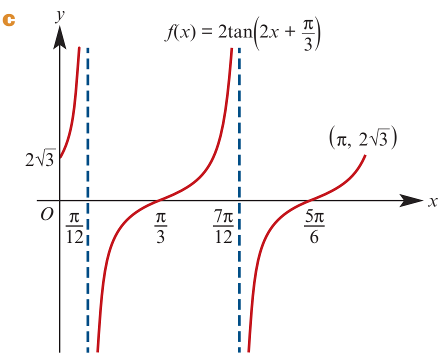
**10 a**



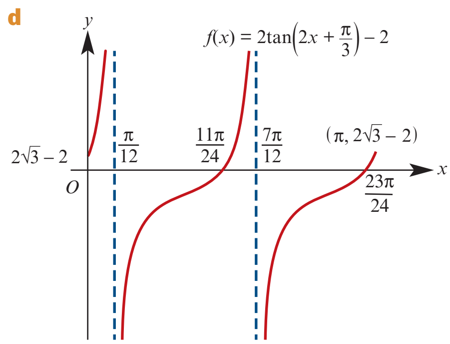

**b**




**c**




**d**




**11 a** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

**c** $\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$

**d** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

**12** $0°, 60°, 180°, 300°, 360°$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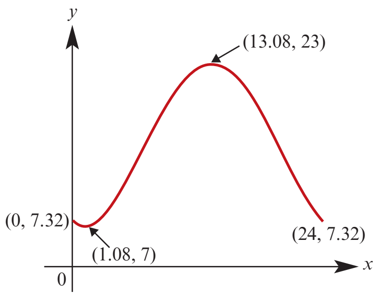
**13 a** $n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$ **b** $\frac{2n\pi}{3}, n \in \mathbb{Z}$
**c** $-\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$

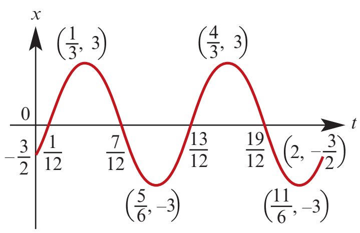
## Extended-response questions

**1 a i** $a = 13.4$ **ii** $b = 2$ **iii** $k = 12$
**b** 3 a.m., 9 a.m., 3 p.m., 9 p.m.
**c** $2 < t < 10,\ 14 < t < 22$

**2 a** 7.3°C **b** Min = 7°C; Max = 23°C
**c** Between 9:40 a.m. and 4:30 p.m.
**d**




**3 a** $a = \frac{\pi}{6}$
**b**




**c** 3 m **d** $\frac{5}{6}$ s **e** 1 s **f** $\frac{1}{4}$ s
**g i** 24 m **ii** 30 m

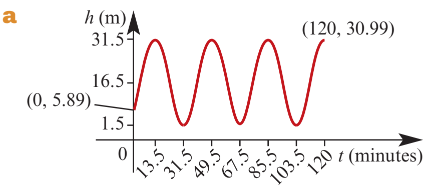
**4 a** $p = 6,\ q = 4.2$ **b** 3 a.m., 3 p.m.
**c** 6 m **d** 7 a.m., 11 a.m., 7 p.m., 11 p.m.
**e** 8 hours

**5 a i** $-1 < k < 1$
**ii** $k = -1$ or $k = 3$
**iii** $k < -1$ or $k > 3$
**b** A translation of 1 unit in the negative direction of the $y$-axis, followed by a dilation of factor $\frac{1}{2}$ from the $x$-axis and a dilation of factor 3 from the $y$-axis
**c i** $h = \frac{\pi}{2}$ **ii** $h = \frac{\pi}{6}$

**6 a** A translation of $\frac{\pi}{2}$ units in the positive direction of the $x$-axis
**b** A dilation of factor $\frac{1}{2}$ from the $y$-axis, followed by a translation of $\frac{\pi}{4}$ units in the negative direction of the $x$-axis, and then a dilation of factor $\frac{1}{4}$ from the $x$-axis
**c i** $y = -\sin\left(\frac{\pi x}{2}\right) + 4$
**ii** Range = [3, 5]; Period = 4

**7 a** For $N$: Max = 7000, occurs in April ($t = 4$); Min = 1000, occurs in October ($t = 10$).
For $M$: Max = 8500, occurs in June ($t = 6$); Min = 2500, occurs at end of January and November ($t = 1$ and $t = 11$)
**b** $t = 4.31$ (April), population is 6961; $t = 0.24$ (January), population is 2836
**c** 145 556 in May ($t = 5.19$)
**d** $t = 7.49$ (July)

**8 a**




**b** 5.89 m **c** 27.51 s **d** 6 times **e** 20 times
**f** 4.21 m **g** 13.9 m

# Chapter 9

## Exercise 9A

**1 a** $-1$ **b** $-\sqrt{2}$ **c** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **d** 1
**e** $-2$ **f** 2 **g** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **h** 2

**2 a** $-1$ **b** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **c** 1
**d** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **e** $-\sqrt{2}$ **f** $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
**g** $-1$ **h** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **i** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

**3 a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{6}, \frac{7\pi}{6}$ **c** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **d** $\frac{\pi}{4}, \frac{5\pi}{4}$

**4 a** $-\frac{8}{17}$ **b** $\frac{15}{17}$ **c** $-\frac{15}{8}$

**5** $\cos\theta = \frac{24}{25},\ \sin\theta = -\frac{7}{25}$

**6** $-\frac{\sqrt{29}}{5}$

**7** $\frac{8}{31}$

**8** $\frac{15}{4(9 + \sqrt{5})} = \frac{15(6 - \sqrt{5})}{124}$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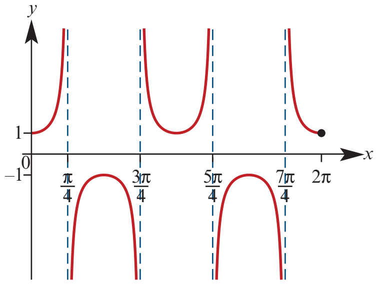
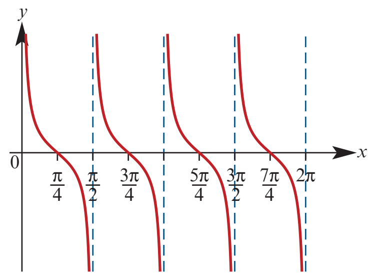
## Exercise 9B

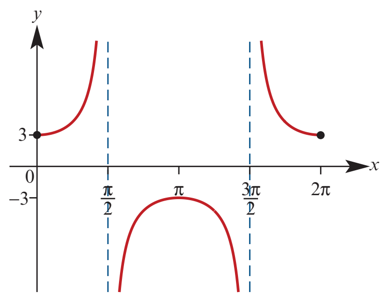
**1 a**



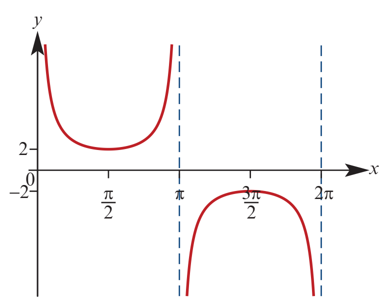

**b**



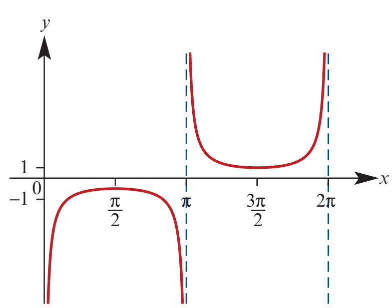

**c**



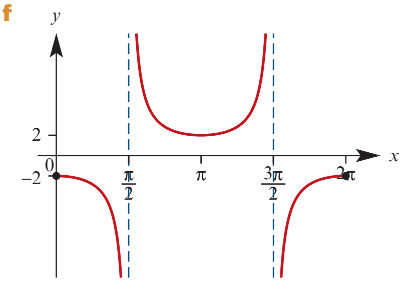

**d**



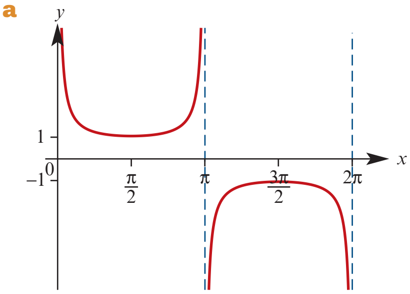

**e**



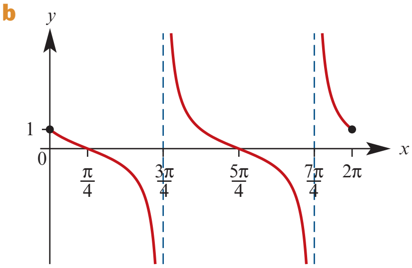

**f**



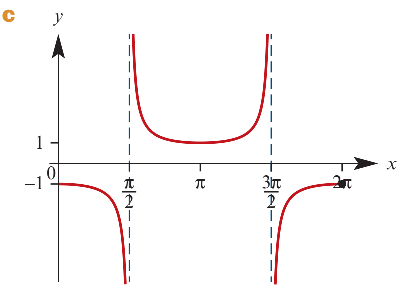

**2 a**



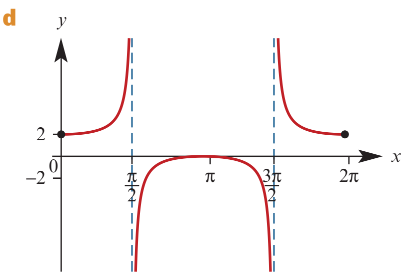

**b**




**c**




**d**





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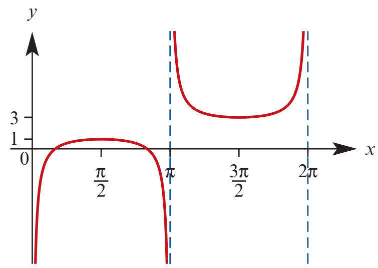
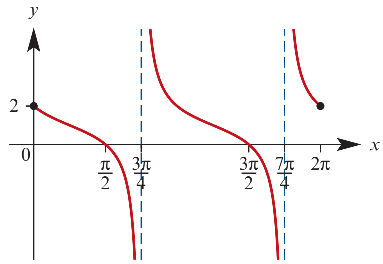
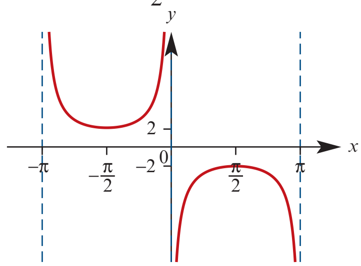
**e**



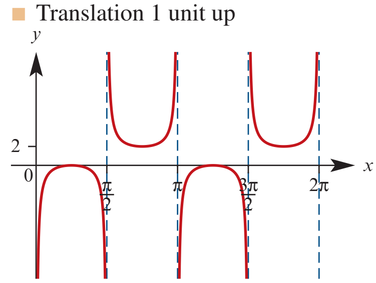

**f**



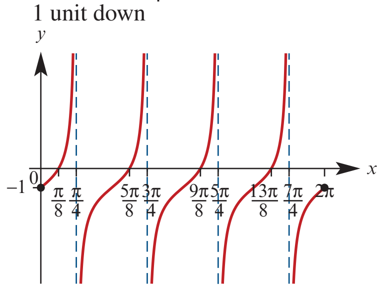

**3**
- Reflection in the $x$-axis
- Dilation of factor 2 from the $x$-axis
- Translation $\frac{\pi}{2}$ units to the right



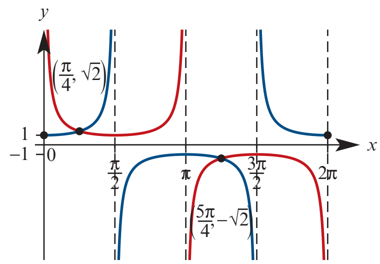

**4**
- Reflection in the $y$-axis
- Dilation of factor $\frac{1}{2}$ from the $y$-axis
- Translation 1 unit up




**5**
- Reflection in the $x$-axis
- Dilation of factor $\frac{1}{2}$ from the $y$-axis
- Translation $\frac{\pi}{4}$ units to the right and 1 unit down




**6**




## Exercise 9C

**1 a** $\frac{\sqrt{2}+\sqrt{6}}{4}$ **b** $\frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

**2 a** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

**3 a** $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

**b** $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

**c** $\frac{1-\sqrt{3}}{1+\sqrt{3}} = -2+\sqrt{3}$

**4** For $u, v \in \left(0, \frac{\pi}{2}\right)$, $\sin(u+v) = \frac{63}{65}$;

For $u, v \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(u+v) = -\frac{63}{65}$;

For $u \in \left(0, \frac{\pi}{2}\right)$, $v \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(u+v) = -\frac{33}{65}$;

For $u \in \left(\frac{\pi}{2}, \pi\right)$, $v \in \left(0, \frac{\pi}{2}\right)$, $\sin(u+v) = \frac{33}{65}$

**5 a** $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$ **b** $\frac{1}{\sqrt{2}}(\cos\varphi + \sin\varphi)$

**c** $\frac{\tan\theta + \sqrt{3}}{1-\sqrt{3}\tan\theta}$ **d** $\frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

**6 a** $\sin u$ **b** $\cos u$

**7 a** $-\frac{119}{169}$ **b** $\frac{24}{25}$ **c** $\frac{24}{7}$ **d** $-\frac{169}{119}$

**e** $-\frac{33}{65}$ **f** $-\frac{16}{65}$ **g** $-\frac{65}{33}$ **h** $\frac{7}{24}$

**8 a** $\frac{63}{16}$ **b** $-\frac{24}{7}$ **c** $\frac{56}{65}$ **d** $\frac{24}{25}$

**9 a** $\frac{7}{25}$ **b** $\frac{3}{5}$ **c** $\frac{117}{44}$ **d** $\frac{-336}{625}$

**10 a** $-\frac{\sqrt{3}}{2}$ for $\theta = \frac{5\pi}{3}$ **b** $-\frac{1}{2}$

**11 a** $1 - \sin(2\theta)$ **b** $\cos(2\theta)$

## Exercise 9D

**1 a** $5, -5$ **b** $2, -2$ **c** $\sqrt{2}, -\sqrt{2}$

**d** $\sqrt{2}, -\sqrt{2}$ **e** $2\sqrt{3}, -2\sqrt{3}$

**f** $2, -2$ **g** $4, 0$ **h** $5+\sqrt{13}, 5-\sqrt{13}$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**2 a** $\frac{\pi}{2}, \pi$ **b** $0, \frac{2\pi}{3}, 2\pi$

**c** $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $0, \frac{5\pi}{3}, 2\pi$

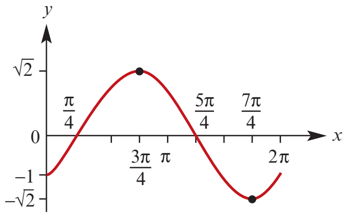
**e** $53.13°$ **f** $95.26°, 155.26°$

**3** $2\cos\left(2x + \frac{\pi}{6}\right)$

**4** $\sqrt{2}\sin\left(3x - \frac{5\pi}{4}\right)$

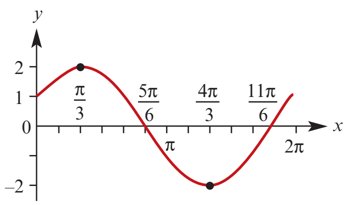
**5 a** $f(x) = \sin x - \cos x = \sqrt{2}\cos\left(x - \frac{3\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{7\pi}{4}\right) = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$



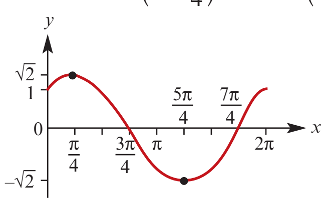

**b** $f(x) = \sqrt{3}\sin x + \cos x$

$= 2\cos\left(x - \frac{\pi}{3}\right) = 2\sin\left(x + \frac{\pi}{6}\right)$



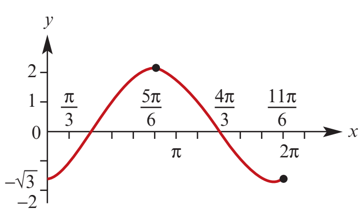

**c** $f(x) = \sin x + \cos x$

$= \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$




**d** $f(x) = \sin x - \sqrt{3}\cos x = 2\cos\left(x - \frac{5\pi}{6}\right)$

$= 2\sin\left(x + \frac{5\pi}{3}\right) = 2\sin\left(x - \frac{\pi}{3}\right)$




## Exercise 9E

**1 a** $\sin(5\pi t) + \sin(\pi t)$ **b** $\frac{1}{2}(\sin 50° - \sin 10°)$

**c** $\sin(\pi x) + \sin\left(\frac{\pi x}{2}\right)$ **d** $\sin(A) + \sin(B + C)$

**2** $\cos(\theta) - \cos(5\theta)$

**3** $\sin A - \sin B$

**5 a** $2\sin 39°\cos 17°$ **b** $2\cos 39°\cos 17°$

**c** $2\cos 39°\sin 17°$ **d** $-2\sin 39°\sin 17°$

**6 a** $2\sin(4A)\cos(2A)$ **b** $2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$

**c** $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{7x}{2}\right)$ **d** $-2\sin(2A)\sin(A)$

**11 a** $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

**b** $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

**c** $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$

**d** $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

**12 a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

**c** $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$

**d** $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

**17** $\frac{1 - \cos(100x)}{2\sin(x)}$

## Chapter 9 review

### Short-answer questions

**1 a** 5, 1 **b** 4, −2 **c** 4, −4 **d** 2, 0 **e** $1, \frac{1}{3}$

**2 a** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

**c** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **d** $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

**4 a** $\frac{140}{221}$ **b** $\frac{-21}{221}$ **c** $\frac{171}{140}$

**5 a** $\frac{1}{2}$ **b** 1

**6 a** 1 **b** 0

**8 a** $\frac{-1}{9}$ **b** $\frac{-4\sqrt{5}}{9}$ **c** $\frac{8\sqrt{5}}{81}$

**10** $2 - \sqrt{3}$

**11 a** $0, \frac{\pi}{2}, 2\pi$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $0, \pi, 2\pi$

**d** $\frac{\pi}{2}, \frac{3\pi}{2}$ **e** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

**f** $\frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$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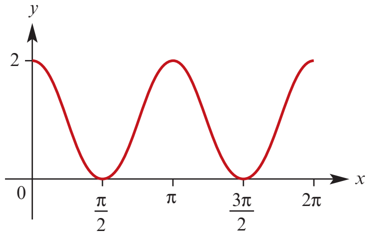
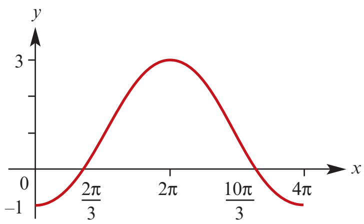
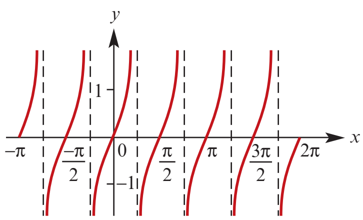
**12 a** $y = 2\cos^2 x$




**b** $y = 1 - 2\sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$




**c** $f(x) = \tan(2x)$




**13** $\frac{2}{9}$

**14 a** $\sqrt{85}\cos(\theta - \alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$

**b i** $\sqrt{85}$ **ii** $\frac{2}{\sqrt{85}}$

**iii** $\theta = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$

**15 a** $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$ **b** $0, \frac{\pi}{3}, \pi$

## Extended-response questions

**1 b** $P = 10\sqrt{5}\cos(\theta - \alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$;

$\theta = 70.88°$

**c** $k = 25$ **d** $\theta = 45°$

**2 a** $AD = \cos\theta + 2\sin\theta$

**b** $AD = \sqrt{5}\cos(\theta - \alpha)$ where

$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63°$

**c** Max length of $AD$ is $\sqrt{5}$ m when $\theta = 63°$

**d** $\theta = 79.38°$

**3 b ii** $a = 1,\ b = 1$

**c** $\frac{1 + \sqrt{2} - \sqrt{3}}{1 + \sqrt{3} + \sqrt{6}} = \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1}$

$= \sqrt{6} + \sqrt{2} - \sqrt{3} - 2$

**4 a ii** $2\cos\left(\frac{\pi}{5}\right)$

**b iii** $4\cos^2\left(\frac{\pi}{5}\right) - 2\cos\left(\frac{\pi}{5}\right) - 1 = 0$

**iv** $\frac{1 + \sqrt{5}}{4}$

**5 b** $-\frac{2}{3}$ or $\frac{1}{2}$

# Chapter 10

## Short-answer questions

**3 a** If $n$ is odd, then $5n + 3$ is even.

**c** If $n$ is even, then $5n + 3$ is odd.

**7 a** 90° **b** 54° **c** 80° **d** 220°

**e** $x = 96°,\ y = 70°$ **f** 46°

**8 a** 4 **b** $\frac{3\sqrt{10}}{2}$ **c** 12

**9 a** $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$

**b** $\frac{-17\pi}{24}, \frac{-11\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}$

**10** $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

**11 a** $\frac{5}{4}$ **b** $\frac{4}{3}$ **c** $-\frac{\sqrt{3}}{3}$ **d** $\frac{2\sqrt{3}}{3}$

**12** $\pm\frac{\sqrt{6}}{3}$

**14 a** $\frac{1}{2}\sin(4x) - \frac{1}{2}\sin(2x)$

**b** $\theta = \frac{(2n+1)\pi}{2}$ or $\theta = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$

## Extended-response questions

**2 b** $\angle BCA = x°,\ \angle BOA = 2x°,\ \angle TAB = x°,$ $\angle TBA = x°$

**4 a** $x = 4,\ y = 9$

**b i** 4 **ii** 2

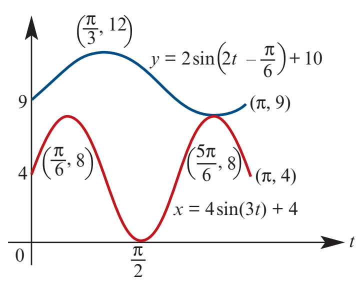
**c i** 8, 0 **ii** 12, 8

**d i** $\frac{2\pi}{3}$ **ii** $\pi$

**e** $\frac{\pi}{6}$ s **f** $\frac{5\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}, \frac{17\pi}{6}$

**g** $\frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$

**h**




**i** $\frac{5\pi}{6}$ s **j** $2\pi$ s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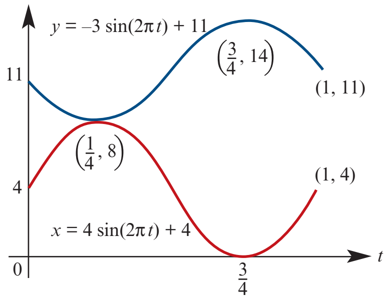
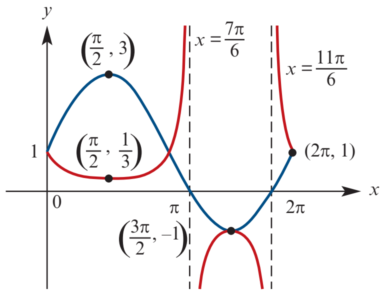
**5 a** One possible set of values is $a = 4$, $b = 4$, $n = 2\pi$ and $c = -3$, $d = 11$, $m = 2\pi$

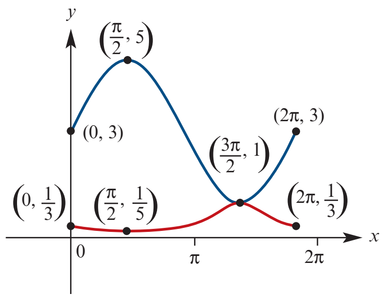
**b**




**6 a i**

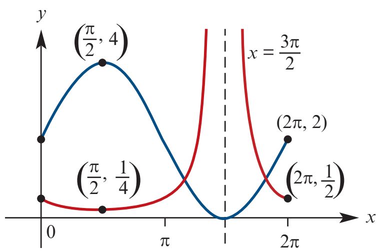



**ii**




**b** $k = 2$

**c**




# Chapter 11

## Exercise 11A

**1 a** $2 \times 2$ **b** $2 \times 3$ **c** $1 \times 4$ **d** $4 \times 1$

**2 a** $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

**3 a** $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

**c** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

**4** $\begin{bmatrix} 200 & 180 & 135 & 110 & 56 & 28 \\ 110 & 117 & 98 & 89 & 53 & 33 \end{bmatrix}$

**5 a** $\begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ if $x = 4$

**b** $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$ if $x = 4$

**c** $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$ if $x = 0$, $y = 2$

**6 a** $x = 2, y = 3$ **b** $x = 3, y = 2$

**c** $x = 4, y = -3$ **d** $x = 3, y = -2$

**7** $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$

## Exercise 11B

**1** $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $2\mathbf{X} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $4\mathbf{Y} + \mathbf{X} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$

$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix}$

$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -7 & -7 \end{bmatrix}$

**2** $2\mathbf{A} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ 0 & -6 \end{bmatrix}$

$-6\mathbf{A} = \begin{bmatrix} -6 & 6 \\ 0 & -12 \end{bmatrix}$

**3 a** Yes **b** Yes

**4 a** $\begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix}$

**c** $\begin{bmatrix} 6 & -5 \\ 8 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} -6 & -13 \\ 16 & 7 \end{bmatrix}$

**5 a** $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 3 \\ 6 & 3 \end{bmatrix}$

**c** $\begin{bmatrix} 3 & 3 \\ -1 & 7 \end{bmatrix}$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**6** $\mathbf{X} = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} \frac{-9}{2} & \frac{-23}{2} \\ \frac{-1}{2} & 11 \end{bmatrix}$

**7** $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 310 & 180 & 220 & 90 \\ 200 & 0 & 125 & 0 \end{bmatrix}$ represents the total production at two factories in two successive weeks

## Exercise 11C

**1** $\mathbf{AX} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\mathbf{AY} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ $\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$ $\mathbf{A}(\mathbf{CA}) = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

$\mathbf{A}^2\mathbf{C} = \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$

**2** Defined: **AY**, **CI**;
Not defined: **YA**, **XY**, $\mathbf{X}^2$, **XI**

**3** $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

**4** No

**5** One possible answer is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

**6** $\mathbf{LX} = \begin{bmatrix} 7 \end{bmatrix}$, $\mathbf{XL} = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

**7** **AB** and **BA** are not defined unless $m = n$

**8 b** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**9** One possible answer is
$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

**10** One possible answer is
$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$,
$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$, $\mathbf{AB} + \mathbf{AC} = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$,
$(\mathbf{B} + \mathbf{C})\mathbf{A} = \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$

**11** For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

**12 a** $\begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$, John took 29 minutes to eat food costing \$8.50

**b** $\begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$,
John's friends took 22 and 12 minutes to eat food costing \$8.00 and \$3.00 respectively

**13** $\mathbf{A}^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}$,
$\mathbf{A}^8 = \begin{bmatrix} -527 & 336 \\ -336 & -527 \end{bmatrix}$

**14** $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$,
$\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

## Exercise 11D

**1 a** 1 **b** $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ **c** 2 **d** $\frac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$

**2 a** $\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{-1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$

**c** $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

**4 a** $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

**b** $\mathbf{AB} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$, $(\mathbf{AB})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

**c** $\mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$,
$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

**5 a** $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$

**c** $\begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$

**6 a** $\begin{bmatrix} \frac{-3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{-11}{16} & \frac{17}{16} \\ \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$

**7** $\begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$




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**9** $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix},$
$\begin{bmatrix}1&0\\k&-1\end{bmatrix}, \begin{bmatrix}-1&0\\k&1\end{bmatrix}, \begin{bmatrix}1&k\\0&-1\end{bmatrix}, \begin{bmatrix}-1&k\\0&1\end{bmatrix}, k \in \mathbb{R},$
$\begin{bmatrix}a&b\\ \frac{1-a^2}{b}&-a\end{bmatrix}, b \neq 0$

**10** $a = \pm\sqrt{2}$

## Exercise 11E

**1 a** $\begin{bmatrix}3\\10\end{bmatrix}$ **b** $\begin{bmatrix}5\\17\end{bmatrix}$

**2 a** $x = -\frac{1}{7},\ y = \frac{10}{7}$ **b** $x = 4,\ y = 1.5$

**3** $(2, -1)$

**4** Book \$12, CD \$18

**5 a** $\begin{bmatrix}2&-3\\4&-6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$

**b** $\begin{bmatrix}2&-3\\4&-6\end{bmatrix}$ is non-invertible

**c** System has solutions (not a unique solution)

**d** Solution set contains infinitely many pairs

**6 a** $\mathbf{A}^{-1}\mathbf{C}$ **b** $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C}$ **c** $\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$

**d** $\mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$ **e** $\mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

**f** $(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1}$

## Chapter 11 review

### Short-answer questions

**1 a** $\begin{bmatrix}0&0\\12&8\end{bmatrix}$ **b** $\begin{bmatrix}0&0\\8&8\end{bmatrix}$

**2** $\begin{bmatrix}a\\2-\frac{3}{4}a\end{bmatrix}, a \in \mathbb{R}$

**3 a** Exist: **AC**, **CD**, **BE**; Does not exist: **AB**

**b** $\mathbf{DA} = \begin{bmatrix}14&0\end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{7}\begin{bmatrix}1&2\\3&-1\end{bmatrix}$

**4** $\mathbf{AB} = \begin{bmatrix}2&0\\2&-2\end{bmatrix}, \quad \mathbf{C}^{-1} = \begin{bmatrix}-2&1\\ \frac{3}{2}&-\frac{1}{2}\end{bmatrix}$

**5** $\begin{bmatrix}-1&2\\-3&5\end{bmatrix}$

**6** $\mathbf{A}^2 = \begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix}\frac{1}{2}&0&0\\0&0&\frac{1}{2}\\0&\frac{1}{2}&0\end{bmatrix}$

**7** 8

**8 a i** $\begin{bmatrix}3&-5\\5&8\end{bmatrix}$ **ii** $\begin{bmatrix}1&-18\\18&19\end{bmatrix}$ **iii** $\frac{1}{7}\begin{bmatrix}3&1\\-1&2\end{bmatrix}$

**b** $x = 2,\ y = 1$

### Extended-response questions

**1 a i** $\begin{bmatrix}2&-3\\4&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$

**ii** $\det(\mathbf{A}) = 14,\ \mathbf{A}^{-1} = \frac{1}{14}\begin{bmatrix}1&3\\-4&2\end{bmatrix}$

**iii** $\frac{1}{7}\begin{bmatrix}9\\-1\end{bmatrix}$

**iv** $\left(\frac{9}{7}, -\frac{1}{7}\right)$ is the point of intersection of the two lines

**b i** $\begin{bmatrix}2&1\\4&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\8\end{bmatrix}$

**ii** $\det(\mathbf{A}) = 0$, so **A** is non-invertible

**c** Equations of two parallel lines

**2 a** $\begin{bmatrix}79&78&80\\80&78&82\end{bmatrix}$ **b** $\begin{bmatrix}0.2\\0.3\\0.5\end{bmatrix}$

**c** Semester 1: 79.2; Semester 2: 80.4

**d** Semester 1: 83.8; Semester 2: 75.2

**e** No, total score is 318.6

**f** 3 marks

**3 a** $\begin{bmatrix}10&2\\8&4\\8&8\\6&10\end{bmatrix}$ **b** $\begin{bmatrix}70\\60\end{bmatrix}$

**c** Term 1: \$820; Term 2: \$800; Term 3: \$1040; Term 4: \$1020

**d** $\begin{bmatrix}2&2&1\\2&2&1\\3&4&2\\3&4&2\end{bmatrix}$ **e** $\begin{bmatrix}60\\55\\40\end{bmatrix}$

**f** Term 1: \$270; Term 2: \$270; Term 3: \$480; Term 4: \$480

**g** Term 1: \$1090; Term 2: \$1070; Term 3: \$1520; Term 4: \$1500

# Chapter 12

## Exercise 12A

**1 a** $(-2, 6)$ **b** $(-8, 22)$

**c** $(26, 2)$ **d** $(-4, -2)$

**2 a** $(3, 2)$ **b** $(-4, 9)$

**c** $(8, 3)$ **d** $(7, 11)$

**3 a** $\begin{bmatrix}2&3\\4&5\end{bmatrix}$ **b** $\begin{bmatrix}11&-3\\3&-8\end{bmatrix}$

**c** $\begin{bmatrix}2&0\\1&-3\end{bmatrix}$ **d** $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$


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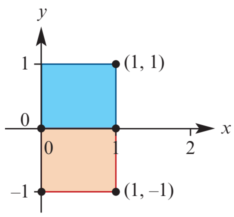
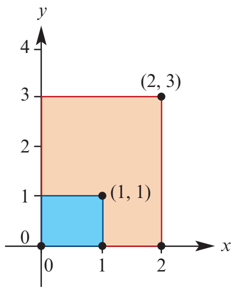
**4** Unit square is blue; image is red

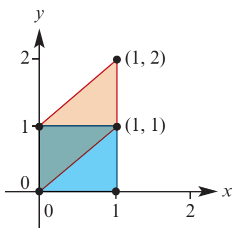
**a**



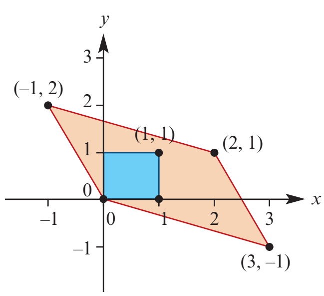

**b**



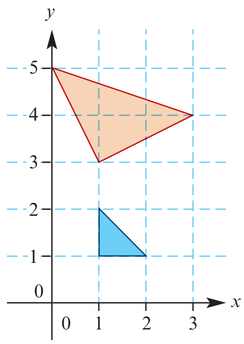

**c**




**d**




**5**



**6** $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$

**7** $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

**8 a** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

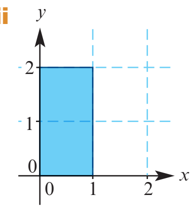
**b** $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

**c** $\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}$

## Exercise 12B

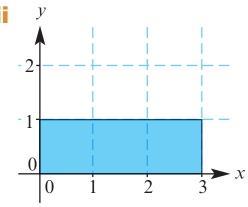
**1 a i** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

**ii**



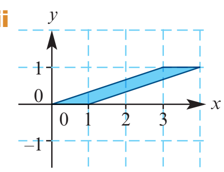
**b i** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

**ii**



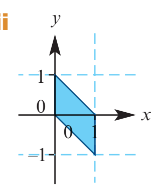
**c i** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

**ii**



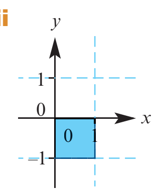
**d i** $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

**ii**



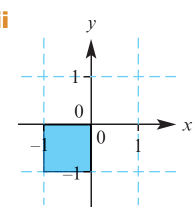
**e i** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

**ii**



**f i** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

**ii**

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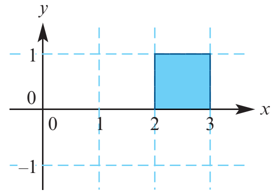
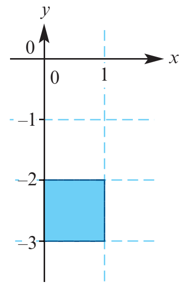
**2 a i** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$

**ii**



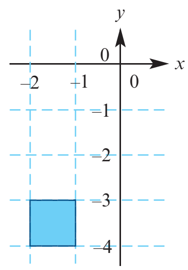
**b i** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y-3 \end{bmatrix}$

**ii**



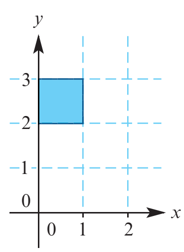
**c i** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x-2 \\ y-4 \end{bmatrix}$

**ii**



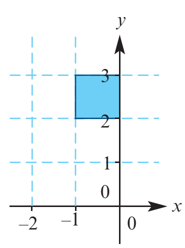
**d i** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y+2 \end{bmatrix}$

**ii**



**e i** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$

**ii**



## Exercise 12C

**1 a** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

**c** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

**2 a** $(-3, 2)$ **b** $\left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

**3 a** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

**c** $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

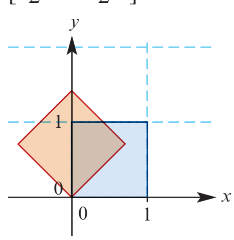
**4 a** $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$

**c** $\begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

**5 a** $\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$ **b** $\left(\frac{-23}{37}, \frac{47}{37}\right)$

**6 a** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

**b**



**c** $\sqrt{2} - 1$

**7 a** $C\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), B\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

**b** Equilateral

**c** $y = -\sqrt{3}x,\ y = 0,\ y = \sqrt{3}x$

## Exercise 12D

**1** $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ **2** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$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**3 a** $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

**b** $\begin{bmatrix}\cos 180^\circ & -\sin 180^\circ\\ \sin 180^\circ & \cos 180^\circ\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

**4 a** $\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$ **b** $\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$ **c** No

**5 a** $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ **b** $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ **c** Yes

**6 a** $(x, y) \to (-x - 3, y + 5)$

**b** $(x, y) \to (-x + 3, y + 5)$ **c** Yes

**7 a** $\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ **b** $\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

**c** $\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ **d** $\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$

**8 a** $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

**b** $\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

**9** $\theta = 180^\circ k$, where $k \in \mathbb{Z}$

**10 a** $2\theta$

**b** $\begin{bmatrix}\cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta\end{bmatrix}$

**c** $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

**11 a** $x' = y + 1$

$y' = x + 2$

**b** $\begin{bmatrix}2\\ 1\end{bmatrix}$

**12 a** $\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$ **b** $\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$

**c** $\begin{bmatrix}\frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4}\\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{2}\end{bmatrix}$

**d** $\cos 15^\circ = \dfrac{\sqrt{2}+\sqrt{6}}{4}$, $\sin 15^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

**13** $\begin{bmatrix}\cos(2\theta - 2\varphi) & -\sin(2\theta - 2\varphi)\\ \sin(2\theta - 2\varphi) & \cos(2\theta - 2\varphi)\end{bmatrix}$,
rotation matrix for angle $2\theta - 2\varphi$

## Exercise 12E

**1 a** $\begin{bmatrix}1 & -1\\ -3 & 4\end{bmatrix}$ **b** $\begin{bmatrix}\frac{2}{7} & \frac{1}{14}\\ \frac{1}{7} & -\frac{3}{14}\end{bmatrix}$

**c** $\begin{bmatrix}\frac{2}{3} & -\frac{1}{2}\\ \frac{1}{3} & 0\end{bmatrix}$ **d** $\begin{bmatrix}\frac{5}{7} & -\frac{3}{7}\\ \frac{4}{7} & -\frac{1}{7}\end{bmatrix}$

**2 a** $(x, y) \to (x - 2y, 2x - 5y)$

**b** $(x, y) \to (y, -x + y)$

**3 a** $(-1, 1)$ **b** $\left(-\frac{1}{2}, 1\right)$

**4** $\begin{bmatrix}-4 & 3\\ -1 & 1\end{bmatrix}$

**5** $(0, 0), (-1, -2), (1, 1), (0, -1)$

**6 a** $\mathbf{A} = \begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix}\frac{1}{k} & 0\\ 0 & 1\end{bmatrix}$

**7 a** $\mathbf{A} = \begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix}1 & -k\\ 0 & 1\end{bmatrix}$

**8 a** $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

**b** Reflecting twice in the same axis will return any point $(x, y)$ to its original position

**9 a** $\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{bmatrix}$

**b** Reflecting twice in the same line will return any point $(x, y)$ to its original position

## Exercise 12F

**1 a** $y = -3x - 1$ **b** $y = \dfrac{x}{2} + 1$ **c** $y = \dfrac{9x}{2} + 3$

**d** $y = 3x - 1$ **e** $y = -9x + 3$ **f** $y = \dfrac{-x-1}{3}$

**g** $y = \dfrac{x-1}{3}$

**2 a** $y = 6 - \dfrac{9x}{2}$ **b** $y = \dfrac{x+2}{3}$

**c** $y = \dfrac{2-3x}{7}$ **d** $y = \dfrac{5x-2}{12}$

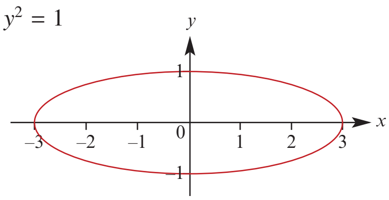
**3** $\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

**4** $\begin{bmatrix}-3 & 0\\ 0 & 6\end{bmatrix}$

**5** $y = -(x + 1)^2 - 1$

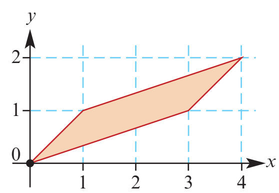
**6** $y = (x - 1)^2 - 3$

**7** $\dfrac{x^2}{3^2} + y^2 = 1$




## Exercise 12G

**1 a** Area = 2





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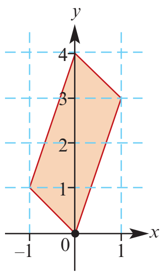
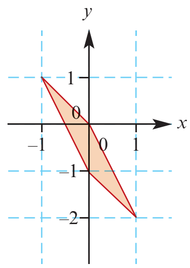
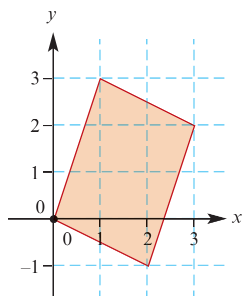
**b** Area = 4



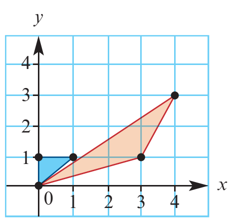
**c** Area = 1



**d** Area = 7

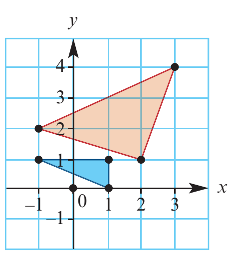


**2 a**



**b** Original area $= \frac{1}{2}$; image area $= \frac{5}{2}$

**3 a**



**b** Original area = 1; image area = 5

**4** $m = \pm 2$

**5** $m = -1, 2$

**6 a i** $\det\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = 1$

**ii** $\det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1$

**iii** $\det\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = -1$

**b i** Dilation of factor $k$ from the $y$-axis and dilation of factor $\frac{1}{k}$ from the $x$-axis

**ii** Determinant of matrix is 1

**7 b** $x = -1$

**8** $m > 2$ or $m < 1$

**9** $\begin{bmatrix} 1 & \pm\frac{\sqrt{3}}{2} \\ 0 & \pm\frac{1}{2} \end{bmatrix}$

**10 a** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

## Exercise 12H

**1** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x+4 \end{bmatrix}$

**2** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-2 \\ -y+2 \end{bmatrix}$

**3 a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}$

**b** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y-1 \\ -x-1 \end{bmatrix}$

**c** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ -y+2 \end{bmatrix}$

**d** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-4 \\ y \end{bmatrix}$

**4 a** $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

**b** $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

**c** $\mathbf{C} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

**d** $\mathbf{CBA} =$

$$\begin{bmatrix} \cos^2\theta + k\sin^2\theta & \cos\theta\sin\theta - k\sin\theta\cos\theta \\ \cos\theta\sin\theta - k\sin\theta\cos\theta & \sin^2\theta + k\cos^2\theta \end{bmatrix}$$

**5** $\begin{bmatrix} \cos^2\theta & \cos\theta\,\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

**6** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$

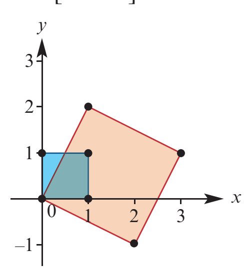
# Chapter 12 review

## Short-answer questions

**1 a** $(7, 4)$

**b** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

**c** Area = 5



**d** $(x, y) \to \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$

**2 a** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

**b** $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

**c** $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

**d** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

**e** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

**f** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**3 a** $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

**b** $\left(\frac{4}{5}, \frac{22}{5}\right)$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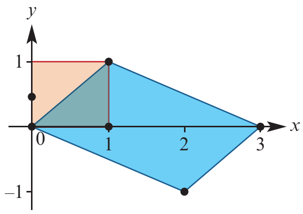
**4 a** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

**5 a** $(x, y) \to (x - 3, -y + 4)$

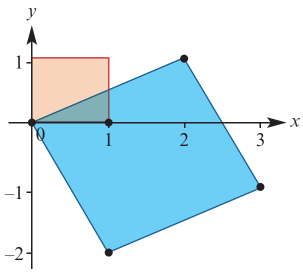
**b** $(x, y) \to (x - 3, -y - 4)$

**6 a** $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$

**7 a** Area of image = 3 square units

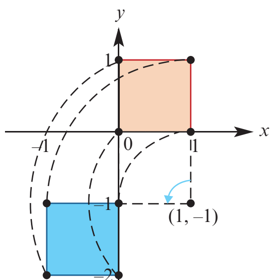



**b** Area of image = 5 square units




**8 a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ x - 2 \end{bmatrix}$ **b** $(1, 0)$

**c**




## Extended-response questions

**1 a** $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

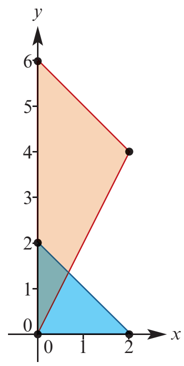
**c** Product of these two matrices:

$$\begin{bmatrix} \frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

**d** $\cos 75° = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4}$

$\sin 75° = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$

**2 a**




**b** Original area = 2 square units;
Image area = 6 square units

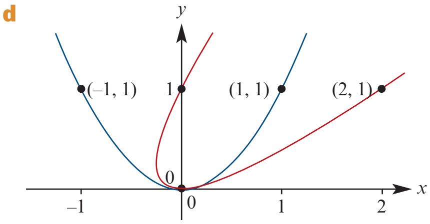
**c** $8\pi$ cubic units

**3 a** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

**b** Shear of factor 1 parallel to the $x$-axis

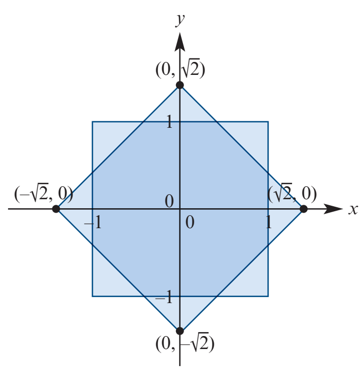
**c** $(0, 0), (2, 1), (0, 1)$

**d**




**4 a** $(0, \sqrt{2}), (\sqrt{2}, 0), (0, -\sqrt{2}), (-\sqrt{2}, 0)$

**b**




**c** $13 - 8\sqrt{2}$ square units

**5 b i** The composition of two rotations is a rotation

**ii** The composition of two reflections is a rotation

**iii** The composition of a reflection followed by a rotation is a reflection

**iv** The composition of a rotation followed by a reflection is a reflection

**c** $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$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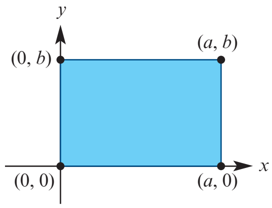
**6 a** $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ **b** $A'(-1, -3)$ **c** $2\sqrt{10}$

**d** Isosceles **f** $2\sqrt{10}$

**7 a**




**b** $O(0, 0)$, $A(a\cos\theta, a\sin\theta)$, $B(-b\sin\theta, b\cos\theta)$, $C(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$

**8 a** $y = \frac{1}{m} - \frac{x}{m}$; $(1, 0)$, $\left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$

**b** $y = 1 - \frac{x}{m}$; $(0, 1)$, $\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right)$

**c** $\begin{bmatrix} \frac{1 - m^2}{1 + m^2} & \frac{2m}{1 + m^2} \\ \frac{2m}{1 + m^2} & \frac{m^2 - 1}{1 + m^2} \end{bmatrix}$

# Chapter 13

See solutions supplement

# Chapter 14

## Exercise 14A

**1**

| | Re(z) | Im(z) |
|---|---|---|
| **a** | 2 | 3 |
| **c** | $\frac{1}{2}$ | $-\frac{3}{2}$ |
| **e** | 0 | 3 |

| | Re(z) | Im(z) |
|---|---|---|
| **b** | 4 | 5 |
| **d** | −4 | 0 |
| **f** | $\sqrt{2}$ | $-2\sqrt{2}$ |

**2 a** $a = 2,\ b = -2$
**b** $a = 3,\ b = 2$ or $a = 2,\ b = 3$
**c** $a = 5,\ b = 0$ **d** $a = \frac{2}{3},\ b = -\frac{1}{3}$

**3 a** $6 - 8i$ **b** $6 - i$ **c** $-6 - 2i$
**d** $7 - 3\sqrt{2}i$ **e** $-2 - 3i$ **f** $4 + 2i$
**g** $6 - 4i$ **h** $-4 + 6i$ **i** $-1 + 11i$
**j** $-1$

**4 a** $4i$ **b** $6i$ **c** $\sqrt{2}i$
**d** $-i$ **e** $-1$ **f** $1$
**g** $-2$ **h** $-12$ **i** $-4$

**5 a** $1 + 2i$ **b** $-3 + 4i$
**c** $-\sqrt{2} - 2i$ **d** $-\sqrt{6} - 3i$

## Exercise 14B

**1 a** $15 + 8i$ **b** $-8i$ **c** $-2 + 16i$
**d** $2i$ **e** $5$ **f** $-4 + 19i$

**2 a** $2 + 5i$ **b** $-1 - 3i$
**c** $\sqrt{5} + 2i$ **d** $5i$

**3 a** $2 + i$ **b** $-3 - 2i$ **c** $-4 + 7i$
**d** $-4 - 7i$ **e** $-4 - 7i$ **f** $-1 + i$
**g** $-1 - i$ **h** $-1 - i$

**4 a** $2 + 4i$ **b** $20$ **c** $4$
**d** $8 - 16i$ **e** $-8i$ **f** $8$
**g** $\frac{1}{10}(1 + 2i)$ **h** $-4 - 2i$

**5** $a = \frac{1}{29}$, $b = -\frac{17}{29}$

**6 a** $\frac{7}{17} - \frac{6}{17}i$ **b** $i$ **c** $\frac{7}{2} - \frac{1}{2}i$
**d** $-\frac{1}{2} - \frac{1}{2}i$ **e** $\frac{2}{13} + \frac{3}{13}i$ **f** $\frac{3}{20} + \frac{1}{20}i$

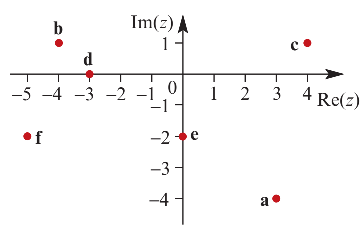
**7** $a = \frac{5}{2}$, $b = -\frac{3}{2}$

**8 a** $-\frac{42}{5}(1 - 2i)$ **b** $-\frac{1}{2}(1 - i)$
**c** $\frac{1}{17}(4 + i)$ **d** $\frac{1}{130}(6 + 43i)$
**e** $2 - 2i$

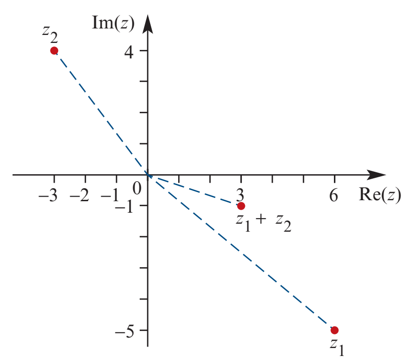
## Exercise 14C

**1** $A = 3 + i$, $B = 2i$, $C = -3 - 4i$
$D = 2 - 2i$, $E = -3$, $F = -1 - i$

**2**




**3 a** $z_1 + z_2 = 3 - i$





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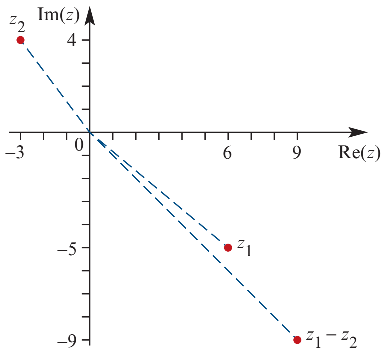
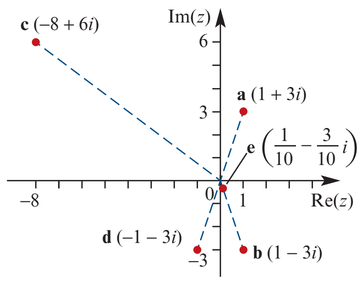
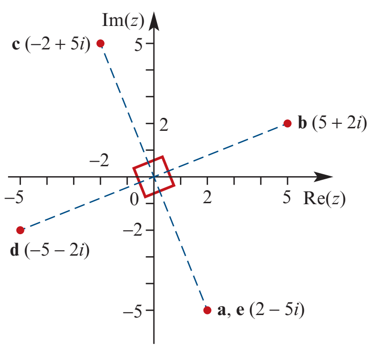
**b** $z_1 - z_2 = 9 - 9i$




**4**




**5**




## Exercise 14D

**1** **a** $\pm 2i$ **b** $\pm 3i$ **c** $\pm\sqrt{5}i$ **d** $2 \pm 4i$
**e** $-1 \pm 7i$ **f** $1 \pm \sqrt{2}i$ **g** $\frac{1}{2}(-3 \pm \sqrt{3}i)$
**h** $\frac{1}{4}(-5 \pm \sqrt{7}i)$ **i** $\frac{1}{6}(1 \pm \sqrt{23}i)$
**j** $1 \pm 2i$ **k** $\frac{1}{2}(3 \pm \sqrt{11}i)$ **l** $3 \pm \sqrt{5}i$

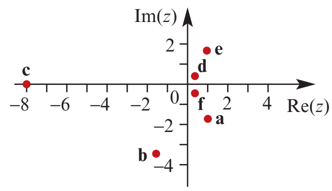
**2** **a** $(z + 3i)(z - 3i)$ **b** $(z + \sqrt{3}i)(z - \sqrt{3}i)$
**c** $3(z + 2i)(z - 2i)$
**d** $(z + 1 + 2i)(z + 1 - 2i)$
**e** $\left(z - \frac{3}{2} + \frac{\sqrt{15}}{2}i\right)\left(z - \frac{3}{2} - \frac{\sqrt{15}}{2}i\right)$
**f** $2\left(z + \frac{1}{2} + \frac{1}{2}i\right)\left(z + \frac{1}{2} - \frac{1}{2}i\right)$

# Chapter 14 review

## Short-answer questions

**1** **a** $(2m + 3p) + (2n + 3q)i$ **b** $p - qi$
**c** $(mp + nq) + (np - mq)i$
**d** $\frac{(mp + nq) + (np - mq)i}{p^2 + q^2}$ **e** $2m$
**f** $(m^2 - n^2 - p^2 + q^2) + (2mn - 2pq)i$
**g** $\frac{m - ni}{m^2 + n^2}$
**h** $\frac{(mp + nq) + (mq - np)i}{m^2 + n^2}$
**i** $\frac{3((mp + nq) + (np - mq)i)}{p^2 + q^2}$

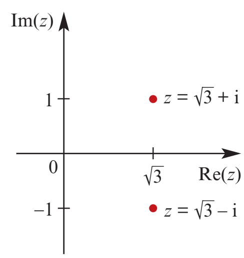
**2**




**a** $1 - \sqrt{3}i$ **b** $-2 - 2\sqrt{3}i$ **c** $-8$
**d** $\frac{1}{4}(1 + \sqrt{3}i)$ **e** $1 + \sqrt{3}i$ **f** $\frac{1}{4}(1 - \sqrt{3}i)$

## Extended-response questions

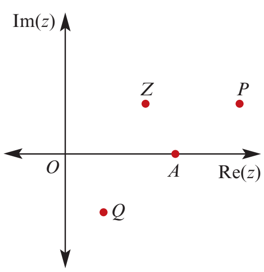
**1** **a** $z = \sqrt{3} + i$ or $z = \sqrt{3} - i$
**b** **i**




**ii** $x^2 + y^2 = 4$ **iii** $a = 2$

**2** **a** **i** $6\sqrt{2}$ **ii** 6

**3** **a**




**b** $\sqrt{2} + 1$

**6** **a** $|z + 1| = \sqrt{2 + 2a}$
**b** $|z - 1| = \sqrt{2 - 2a}$
**c** $\left|\frac{z - 1}{z + 1}\right| = \sqrt{\frac{2 - a}{2 + a}}$

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**7 a** $\Delta = b^2 - 4ac$

**b** $b^2 < 4ac$

**c i** $-\frac{b}{a}, \frac{\sqrt{4ac}}{2a}$ **ii** $\frac{b^2}{2ac} - 1$

**8 a** $z_1 = \frac{1}{2}(-1 + \sqrt{3}i),\ z_2 = \frac{1}{2}(-1 - \sqrt{3}i)$

**c** $\frac{\sqrt{3}}{4}$

# Chapter 15

## Short-answer questions

**1 a** All defined except **AB**

**b** $\mathbf{DA} = \begin{bmatrix} 6 & -12 \end{bmatrix},\quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$

**2 a** $\begin{bmatrix} -2 & 4 \\ 18 & -24 \end{bmatrix}$ **b** $\begin{bmatrix} -10 & -19 \\ 7 & -16 \end{bmatrix}$

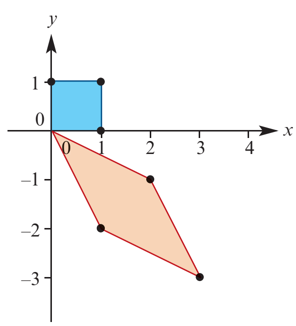
**3** 8

**4** $\mathbf{A} = \begin{bmatrix} t \\ 3t-5 \end{bmatrix}, t \in \mathbb{R}$

**5** $\mathbf{AB} = \begin{bmatrix} -9 & -8 \\ -15 & 10 \end{bmatrix},\quad \mathbf{C}^{-1} = \begin{bmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

**6 a** $(7, -8)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

**c** Area = 3




**d** $(x, y) \to \left(\frac{2}{3}x + \frac{1}{3}y, -\frac{1}{3}x - \frac{2}{3}y\right)$

**7 a** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

**d** $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

**f** $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **g** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

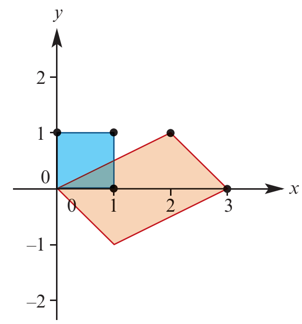
**8 a** $\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}$ **b** $\left(\frac{2}{17}, \frac{76}{17}\right)$

**9 a** $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **c** $\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$

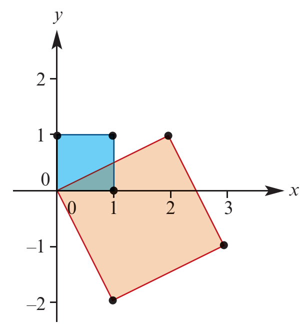
**10 a** $(x, y) \to (-x + 2, y - 1)$

**b** $(x, y) \to (-x - 2, y - 1)$

**11 a** Area = 3

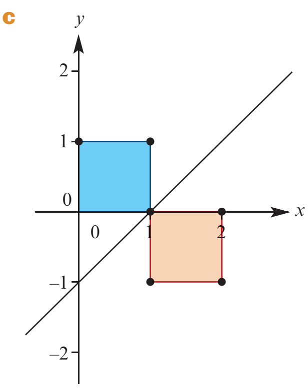



**b** Area = 5




**12 a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}$ **b** $(0, 0) \to (1, -1)$

**c**




**16 a** 6 **b** $4i$ **c** 13 **d** 10
**e** 36 **f** −16 **g** $24i$ **h** $24i$

**17 a** $3 - 5i$ **b** $-1 + i$ **c** $-4 - 7i$ **d** $\frac{8 - i}{13}$
**e** $2 + i$ **f** $\frac{-2 + i}{5}$ **g** $-2 - i$ **h** $\frac{8 + i}{5}$
**i** $\frac{13 - i}{34}$ **j** $3 - i$ **k** $\frac{-1 - 3i}{2}$ **l** $-3 - 4i$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**18 a** $(z-7i)(z+7i)$

**b** $(z-1-3i)(z-1+3i)$

**c** $9\left(z-\frac{1}{3}-\frac{2}{3}i\right)\left(z-\frac{1}{3}+\frac{2}{3}i\right)$

**d** $4\left(z+\frac{3}{2}-i\right)\left(z+\frac{3}{2}+i\right)$

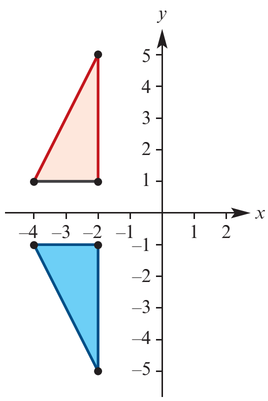
**19 a** $2+i,\ -2-i$ **b** $z=-1-i$ or $z=-i$

## Extended-response questions

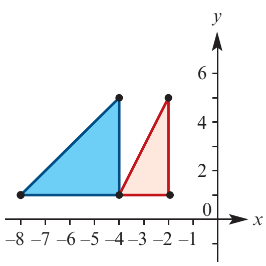
**1 a i** $\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & d^2+bc \end{bmatrix}$ **ii** $\begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$

**2 a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+6 \\ y+3 \end{bmatrix}$

**b**



**c**



**d** $y=2(x+3)^2+2$ **e** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+3 \\ -2y+4 \end{bmatrix}$

**3 a** $(4, 1)$

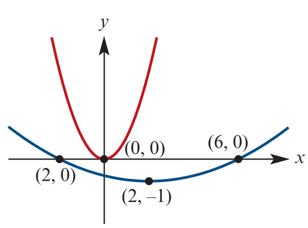
**b i** Rectangle with vertices $A'(0,0)$, $B'(0,1)$, $C'(4,1)$, $E'(4,0)$

**ii** 1 **iii** 4 **iv** $k$

**c** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x \\ y \end{bmatrix}$

**d i** $y=\frac{1}{16}x^2$ **ii** $y=\frac{1}{16}(x-2)^2-1$

**iii**




**e** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+2 \\ \frac{1}{5}(y+3) \end{bmatrix}$

**4 b i** $x^2+(y-1)^2=1$

**ii** $\left(x-\frac{4}{5}\right)^2+\left(y-\frac{3}{5}\right)^2=1$

**c** $(0,0),\ \left(\frac{4}{5},\frac{8}{5}\right)$

**5 a** $(-3, 11)$ **b** $\frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$

**c** $a=2,\ b=3$ **d** $(5a, 5a)$

**e** $\lambda=2,\ b=-2a;\ \lambda=5,\ b=a$

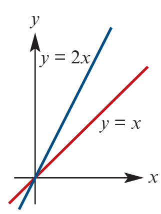
**6 a** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

**c** $a=\sqrt{2},\ b=0$ **d** $c=\frac{3\sqrt{2}}{2},\ d=\frac{\sqrt{2}}{2}$

**e i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x'+\frac{\sqrt{2}}{2}y' \\ -\frac{\sqrt{2}}{2}x'+\frac{\sqrt{2}}{2}y' \end{bmatrix}$

**ii** $\sqrt{2}(y-x)=(x+y)^2$

**7 a**

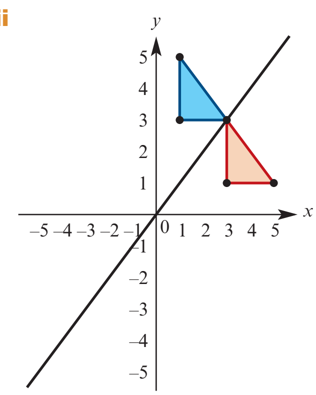



**b** $a=2,\ b=\frac{\pi}{4}$

**c** $\begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

**8 a i** $(3, 1)$ **ii** $A'(3,1)$, $B'(5,1)$, $C'(3,3)$

**iii**



**b ii** $(-1,-1), (2,2)$

**iv** $(-1,-1), (2,2)$, $\left(\frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})\right)$, $\left(\frac{1}{2}(-1-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})\right)$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Appendix

# A

# TI-Nspire CAS CX with OS4.0

## Keystroke actions and short cuts for the TI-Nspire CAS CX

esc : removes menus and dialogue boxes

ctrl + esc : undo last move

⇧shift + esc : redo last move

tab : move to next entry box (field)

ctrl + tab : switch applications in split screen

Navpad (Touchpad)

ctrl : accesses secondary (blue) commands

ctrl + ▲ : displays page sorter

ctrl + ◀ : displays previous page

ctrl + ▶ : displays next page


⌂ on : displays icon page to select applications, mode, My Documents and start a new document

menu : options for each application

ctrl + menu : contextual menus (same as right mouse click)

(mouse pointer key) : mouse pointer (cursor). Selects items.

ctrl + (mouse pointer key) : grab

del : backspace, deletes a character

(catalogue key) : catalogue

(template key) : 2D maths template

ctrl + ÷ : adds fraction template

enter : completes commands and displays results


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# Mode Settings

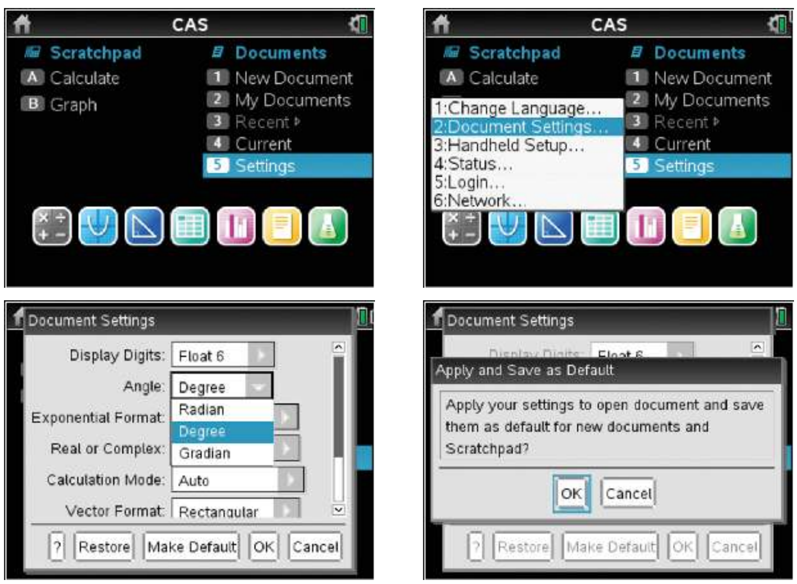
## How to set in Degree mode

For this subject it is necessary to set the calculator to **Degree** mode right from the start. This is very important for the Trigonometry topic. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press ⌂on and move to **Settings>Document Settings**, arrow down to the **Angle** field, press ▶ and select **Degree** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note that there is a separate settings menu for the **Graphs** and **Geometry** pages. These settings are accessed from the relevant pages. For Mathematics it is not necessary for you to change these settings.

Note: When you start your new document you will see **DEG** in the top status line.




## How to set in Approximate (Decimal) mode

For this subject it is useful to set the calculator to **Approximate (Decimal)** mode right from the start. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press ⌂on and move to **Settings>Document Settings**, arrow down to the **Calculation Mode** field, press ▶ and select **Approximate** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note: You can make both the **Degree** and **Approximate Mode** selections at the same time if desired.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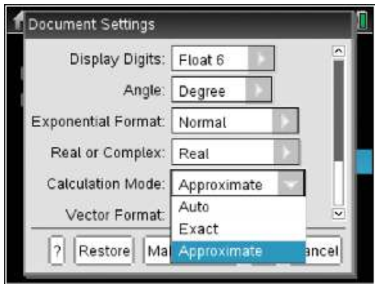


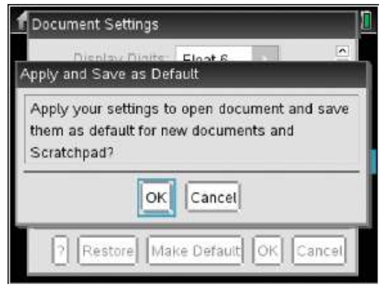


The home screen is divided into two main areas – **Scratchpad** and **Documents**.

All instructions given in the text, and in the Appendix, are based on the **Documents** platform.

## Documents

**Documents** can be used to access all the functionality required for this subject including all calculations, graphing, statistics and geometry.

## Starting a new document

1. To start a new document, press [⌂ on] and select **New Document**.
2. If prompted to save an existing document move the cursor to **No** and press [enter].
   Note: Pressing [ctrl]+[N] will also start a new document.


**A: Calculator page** - this is a fully functional CAS calculation platform that can be used for calculations such as arithmetic, algebra, finance, trigonometry and matrices. When you open a new document select **Add Calculator** from the list.


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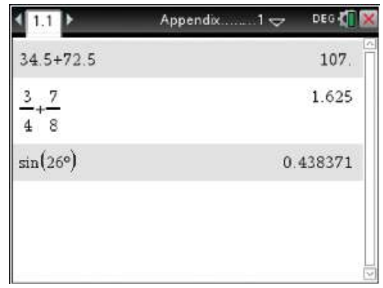
**1** You can enter fractions using the fraction template if you prefer. Press ctrl ÷ to paste the fraction template and enter the values. Use the tab key or arrows to move between boxes. Press enter to display the answer. Note that all answers will be either whole numbers or decimals because the mode was set to approximate (decimal).




**2** For problems that involve angles (e.g. evaluate sin(26°)) it is good practice to include the degree symbol even if the mode is set to degree (DEG) as recommended.

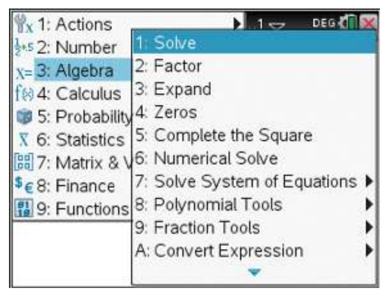
Note: If the calculator is accidentally left in radian (RAD) mode the degree symbol will override this and compute using degree values.

The degree symbol can be accessed using ?!▸. Alternatively select from the **Symbols** palette ctrl.To enter trigonometry functions such as *sin, cos*, press the trig key or just type them in with an opening parenthesis.

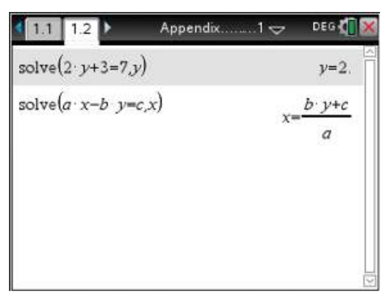
## Solving equations

Using the **Solve command** Solve $2y + 3 = 7$ for $y$.

In a **Calculator** page press menu**>Algebra>Solve** and complete the **Solve** command as shown opposite. You must include the variable you are making the subject at the end of the command line.




Hint: You can also type in **solve(** directly from the keypad but make sure you include the opening bracket.




Literal equations such as $ax - by = c$ can be solved in a similar way.

Note that you must use a multiplication sign between two letters.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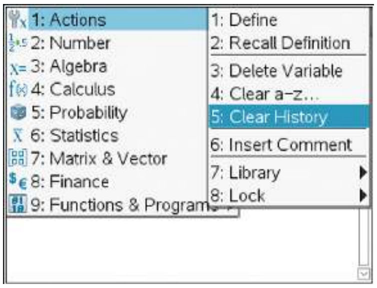
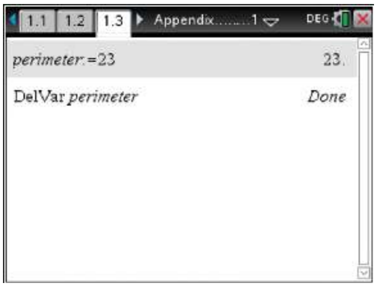
## Clearing the history area




Once you have pressed enter the computation becomes part of the **History** area. To clear a line from the History area, press ▲ repeatedly until the expression is highlighted and press enter. To completely clear the History area, press menu**>Actions>Clear History** and press enter again.

Alternatively press ctrl+menu to access the contextual menu.




It is also useful occasionally to clear any previously stored values. Clearing **History** does not clear stored variables.
Pressing menu**>Actions>Clear a-z. . .** will clear any stored values for single letter variables that have been used.
Use menu**>Actions>Delete Variable** if the variable name is more than one letter. For example, to delete the variable perimeter, then use **DelVar *perimeter***.

Note: When you start a new document any previously stored variables are deleted.

## How to construct parallel boxplots from two data lists

Construct parallel boxplots to display the pulse rates of 23 adult females and 23 adult males.

| Pulse rate (beats per minute) | |
|---|---|
| Females | Males |
| 65 73 74 81 59 64 76 83 95 70 73 79 64 77 80 82 77 87 66 89 68 78 74 | 80 73 73 78 75 65 69 70 70 78 58 77 64 76 67 69 72 71 68 72 67 77 73 |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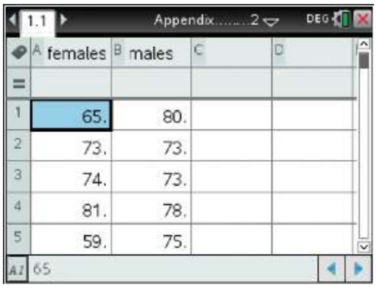
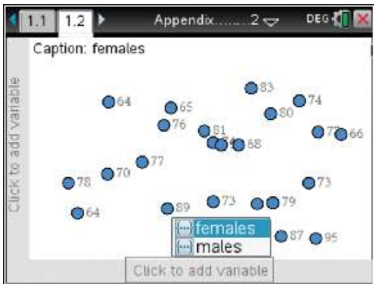
**Steps**

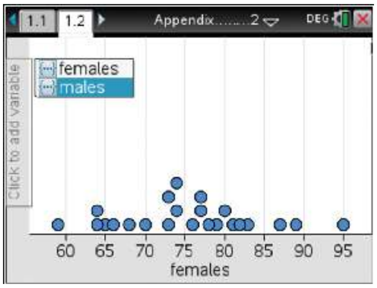
1. Start a new document: ctrl + N.
2. Select **Add Lists & Spreadsheet**. Enter the data into lists called ***females*** and ***males*** as shown.



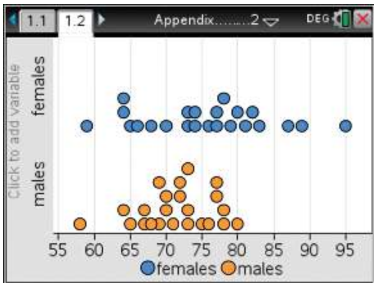

3. Statistical graphing is done through the **Data & Statistics** application.
   Press ctrl + I and select **Add Data & Statistics**
   (or press on and arrow to and press enter).
   Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



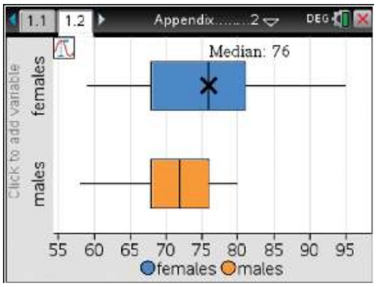

   **a** Press tab, or navigate and click on the "Click to add variable" box to show the list of variables. Select the variable, ***females***. Press enter or to paste the variable to the *x*-axis. A dot plot is displayed by default as shown.




   **b** To add another variable to the *x*-axis press menu**>Plot Properties>Add X Variable**, then enter. Select the variable ***males***. Parallel dot plots are displayed by default.




   **c** To change the plots to box plots press menu**>Plot Type>Box Plot**, then press enter. Your screen should now look like that shown below.




Use ▼ to trace the other plot.

Press esc to exit the **Graph Trace** tool.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**4 Data analysis**

Use [menu]**>Analyze>Graph Trace** and use the cursor arrows to navigate through the key points. Alternatively just move the cursor over the key points. Starting at the far left of the plots, we see that, for females, the

- minimum value is 59: **MinX** = **59**
- first quartile is 68: **Q1**= **68**
- median is 76: **Median** = **76**
- third quartile is 81: **Q3** = **81**
- maximum value is 95: **MaxX** = **95**

and for males, the

- minimum value is 58: **MinX** = **58**
- first quartile is 68: **Q1** = **68**
- median is 72: **Median** = **72**
- third quartile is 76: **Q3** = **76**
- maximum value is 80: **MaxX** = **80**


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Appendix

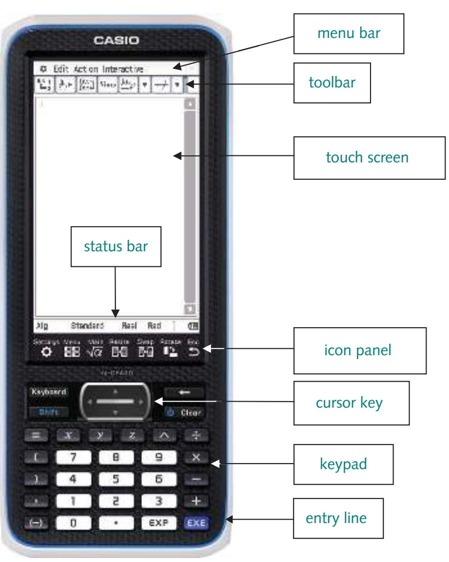
# B Casio ClassPad II

## Operating system

Written for operating system 2.0 or above.

## Terminology

Some of the common terms used with the ClassPad are:




**The menu bar**

**The toolbar**

**The touch screen** contains the work area where the input is displayed on the left and the output is displayed on the right. Use your finger or stylus to tap and perform calculations.

**The icon panel** contains seven permanent icons that access settings, applications and different view settings. Press **escape** to cancel a calculation that causes the calculator to freeze.

**The cursor key** works in a similar way to a computer cursor keys.

**The keypad** refers to the hard keyboard.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# Calculating

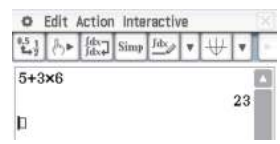
Tap Menu from the **icon panel** to display the application menu if it is not already visible.

Tap Main to open the **Main** application.

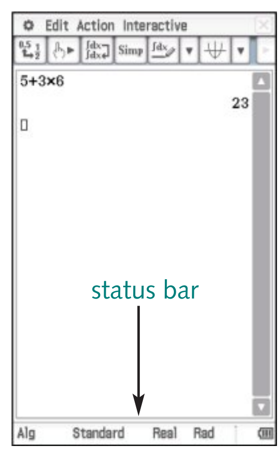
Note: There are two application menus. Alternate between the two by tapping on the screen selector at the bottom of the screen.

**1** The main screen consists of an entry line which is recognised by a flashing vertical line (cursor) inside a small square. The history area, showing previous calculations, is above the entry line.


**2** To calculate, enter the required expression in the entry line and press EXE. For example, if we wish to evaluate $5 + 3 \times 6$, type the expression in the entry line and press EXE.
You can move between the entry line and the history area by tapping or using the cursor keys (i.e. ▼, ◀, ▲, ▶).




**3** The ClassPad gives answers in either exact form or as a decimal approximation. Tapping settings in the **status bar** will toggle between the available options.





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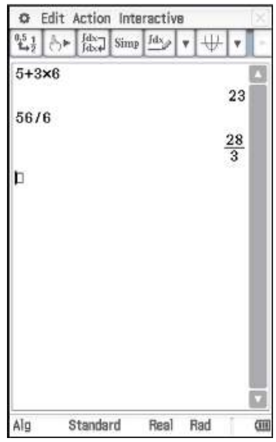
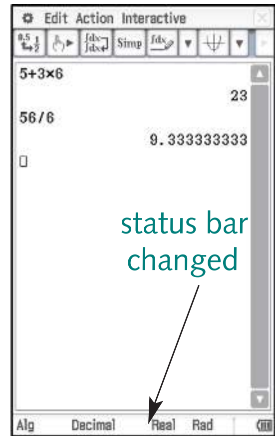
**4** For example, if an exact answer is required for the calculation 56 ÷ 6, the **Standard** setting must be selected.

**5** If a decimal approximation is required, change the **Standard** setting to **Decimal** by tapping it and press EXE.

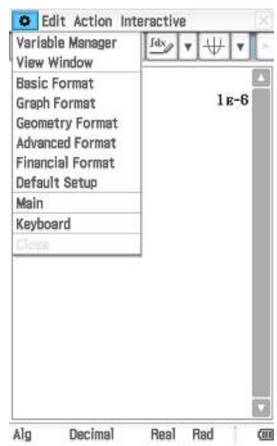



## Extremely large and extremely small numbers

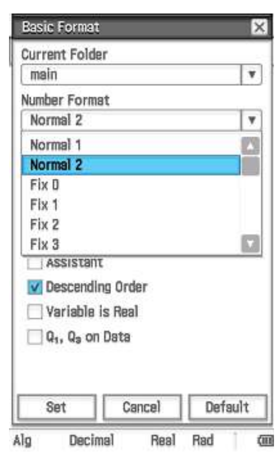
When solving problems that involve large or small numbers the calculator's default setting will give answers in scientific form.

For example, one millionth, or 1/1000000, in scientific form is written as $1 \times 10^{-6}$ and the calculator will present this as 1E-6.




To change this setting, tap on the settings icon and select **Basic Format.**

Under the Number Format select **Normal 2** and tap SET.





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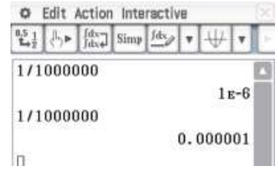
In the Main screen type 1/1000000 and press **EXE**.

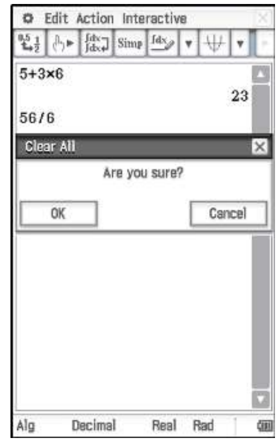
The answer will now be presented in decimal form 0.000001

This setting will remain until the calculator is reset.




## Clearing the history screen

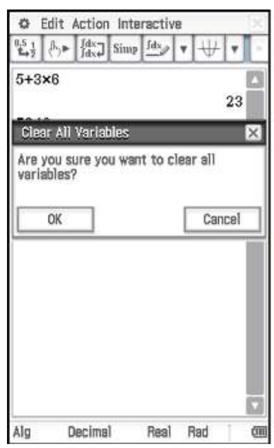
To clear the **Main** application screen, select **Edit** from the menu bar and then tap **Clear All**. Confirm your selection by tapping **OK**. The entire screen is now cleared. To clear the entry line only, press **Clear** on the front calculator.

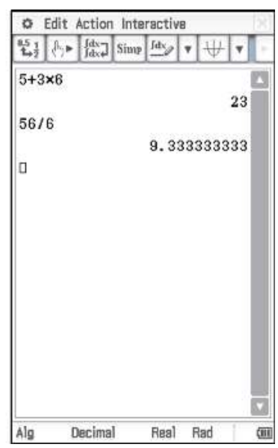




## Clearing variables

To clear stored variable values, select **Edit** from the menu bar and then tap **Clear All Variables**. Confirm your selection by tapping OK.

The variables are cleared but the history created on the main screen is kept.





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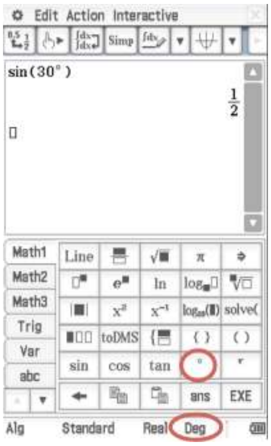
## Degree mode


When solving problems in trigonometry, your calculator should be kept in **Degree** mode. In the main screen, the status bar displays the angle mode.

To change the angle mode, tap on the angle unit in the status bar until **Deg** is displayed.

In addition, it is recommended that you always insert the degree symbol after any angle. This overrides any mode changes and reminds you that you should be entering an angle, not a length.

The degree symbol is found in the **Math1** keyboard.





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